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UNIFORMLY CYCLIC VECTORS

BY

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Abstract. A group acting on a measure space (X, β, λ) may or may not admit a cyclic vector in $L_\infty(X)$. This can occur when the acting group is as big as the group of all measure-preserving transformations. But it does not occur, even though there is no cardinality obstruction to it, for the regular action of a group on itself. The connection of cyclic vectors to the uniqueness of invariant means is also discussed.

1. Introduction. Cyclic vectors are important in the representation theory of locally compact groups. The traditional context is where the vector is in a Hilbert space on which the group G acts by unitary transformations. This has led to some interest in the existence of cyclic vectors for natural actions of the group on other Banach spaces. Here we discuss one particular situation that does not seem to have been discussed before, the case where the Banach space is L_∞ of a measure space.

2. Uniformly cyclic vectors. Throughout this note, a (locally) compact group G will be a Hausdorff (locally) compact group with a fixed left invariant Haar measure λ_G. Assume G acts as a group of linear operators on a Banach space X over the real or complex scalar field. A vector $x \in X$ will be called a *cyclic vector* if the linear span of $\{gx : g \in G\}$ is norm-dense in X. We are interested here in the existence of cyclic vectors for actions on L_∞ spaces. For example, suppose (X, β, λ) is a positive measure space and the group G acts as measure-preserving invertible measurable transformations of (X, β, λ). With the action of G on $L_\infty(X)$ being the associated regular action, we ask: under what conditions does a cyclic vector exist?

It is clear that there is a fundamental issue of cardinality that impinges on the existence of cyclic vectors in the case of $L_\infty(X)$. For example, consider the case of a countably infinite discrete group G acting on itself by left translations, i.e. $X = G$ in the counting measure and the action is left multiplication in the group. Then the smallest cardinality for a dense set in $L_\infty(X)$ is the cardinality c of the continuum, which is the cardinality of

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the power set of G in this case. But continuity of scalar multiplication, and countability of the group G itself, guarantee that the L_∞-norm closure of the span of any orbit would have a countable dense subset. So there certainly is no cyclic vector in this case.

However, in the case of G being the unit circle $\mathbb{T}$, the cardinality of the group is the same as the smallest cardinality for a dense subset of $L_\infty(\mathbb{T})$, both being the continuum. Moreover, translation in $L_\infty(G)$ is highly discontinuous in both the norm and weak topologies, although it is continuous in the weak* topology. For example, see Rosenblatt [4] and Rudin [5]. This fact of extreme discontinuity of the translation makes it quite reasonable to ask: can there be a cyclic vector in $L_\infty(\mathbb{T})$?

It turns out that the question of existence of cyclic vectors for L_∞ is not just one of cardinality. To see this, we first consider a situation where such cyclic vectors do exist. Let IM be the group of invertible measure-preserving transformations of $\mathbb{T}$ with respect to the usual Lebesgue measure $\lambda_{\mathbb{T}}$ on Lebesgue measurable sets in $\mathbb{T}$. The group IM becomes a topological group in the weak topology. This topology is the one with a basis for the open sets consisting of all sets

$$N(\sigma, E_1, \ldots, E_n, \varepsilon) = \{\tau \in \mathrm{IM} : \max_{1\le i\le n} \lambda_{\mathbb{T}}(\tau E_i \triangle \sigma E_i) < \varepsilon\}.$$

This topological group is actually a complete pseudo-metric group. One can obtain a suitable pseudo-metric for the weak topology as follows. Let $(D_i : i \ge 1)$ be an enumeration of all finite unions of arcs whose endpoints are rational multiples of π. Define a pseudo-metric ϱ_{w} on IM by

$$\varrho_{\mathrm{w}}(\sigma, \tau) = \sum_{i=1}^{\infty} \frac{\lambda_{\mathbb{T}}(\sigma D_i \triangle \tau D_i)}{2^i}.$$

It is left to the reader to see that the topology associated with ϱ_{w} is the weak topology on IM and that IM is a complete pseudo-metric topological group with respect to ϱ_{w}. Moreover, because of the separability of the underlying σ-algebra of Lebesgue measurable sets, the topological group IM is second countable and has a countable dense set. Therefore, IM has cardinality c, after identifying any $\sigma_1, \sigma_2 \in \mathrm{IM}$ such that $\sigma_1 x = \sigma_2 x$ for a.e. x.

PROPOSITION 2.1. *Let D be any measurable set with $\lambda_{\mathbb{T}}(D) = 1/2$. Then 1_D is L_∞-norm cyclic for the natural action by* IM *on $L_\infty(\mathbb{T})$.*

Proof. We use the Hahn–Banach Theorem to prove this. Suppose μ is an element of the dual $L_\infty^*(\mathbb{T})$, considered as a finitely additive scalar-valued set function on the $\lambda_{\mathbb{T}}$-measurable sets, such that $\mu(N) = 0$ for any $\lambda_{\mathbb{T}}$-null set. Assume $\mu(\sigma D) = 0$ for all $\sigma \in \mathrm{IM}$. We want to prove that $\mu = 0$.

Now if F is another measurable set with $\lambda_{\mathbb{T}}(F) = 1/2$, then there exists $\sigma \in \mathrm{IM}$ such that $\sigma D = F$ a.e. Therefore, for all measurable sets F with

$\lambda_{\mathbb{T}}(F) = 1/2$, we have $\mu(F) = 0$. It follows by the finite additivity of μ that also $\mu(\mathbb{T}) = 0$.

Let $\alpha \in [0, 1/4]$ and suppose F_1, F_2 are measurable sets with $\lambda_{\mathbb{T}}(F_1) = \lambda_{\mathbb{T}}(F_2) = \alpha$. Then let U be a measurable subset of $\mathbb{T} \setminus (F_1 \cup F_2)$ such that $\lambda_{\mathbb{T}}(U \cup F_1) = \lambda_{\mathbb{T}}(U \cup F_2) = 1/2$. Hence, both $\mu(U \cup F_1)$ and $\mu(U \cup F_2)$ are 0. This means that $\mu(U) + \mu(F_1) = \mu(U) + \mu(F_2)$. Hence, $\mu(F_1) = \mu(F_2)$ for any $\alpha \in [0, 1/4]$ and any two measurable sets with $\lambda_{\mathbb{T}}(F_1) = \lambda_{\mathbb{T}}(F_2) = \alpha$. It follows by finite additivity of μ that the same thing holds no matter what the value of $\alpha \in [0, 1]$.

Now take a measurable set F with $n\lambda_{\mathbb{T}}(F) \leq 1$. Let $F_1 = F$, and take pairwise disjoint measurable sets $F_2, \dots, F_n$ that are disjoint from F_1, and such that all the F_i have the same measure with respect to $\lambda_{\mathbb{T}}$. Then by the above, $\mu(F_i) = \mu(F)$ for all $i = 1, \dots, n$. By the finite additivity of μ, this gives

$$\mu(F_1 \cup \cdots \cup F_n) = n\mu(F_1).$$

Since $|\mu(F_1 \cup \cdots \cup F_n)| \leq \|\mu\|_\infty$, this shows that $|\mu(F)| \leq \|\mu\|_\infty / n$. Hence, if (E_s) is a sequence of measurable sets with $\lim_{s\to\infty} \lambda_{\mathbb{T}}(E_s) = 0$, then it is also the case that $\lim_{s\to\infty} \mu(E_s) = 0$.

We claim now that $\mu(F) = 0$ for all measurable sets F. To see this, we first claim that $\mu(F) = 0$ for all measurable sets with $\lambda_{\mathbb{T}}(F) = i/2^n$ for some n and i, $0 \leq i \leq 2^n$. The case that $i = 2^n$ was remarked already. For other values of i, it suffices to prove this with $i = 1$ by the additivity of μ. But if $\lambda_{\mathbb{T}}(F) = 1/2^n$, then there are pairwise disjoint measurable sets F_i, $i = 1, \dots, 2^n$, such that $F_1 = F$ and $\lambda_{\mathbb{T}}(F_i) = 1/2^n$ for all i. Hence,

$$0 = \mu(\mathbb{T}) = \mu(F_1 \cup \cdots \cup F_{2^n}) = 2^n \mu(F_1),$$

and so $\mu(F) = 0$. Therefore $\mu(F) = 0$ whenever the measure $\lambda_{\mathbb{T}}(F)$ is a dyadic rational. But any measurable set F contains measurable sets F_s such that $\lambda_{\mathbb{T}}(F_s)$ is a dyadic rational and such that $\lim_{s\to\infty} \lambda_{\mathbb{T}}(F \setminus F_s) = 0$. Thus, also $\lim_{s\to\infty} \mu(F \setminus F_s) = 0$. So by the finite additivity of μ,

$$\mu(F) = \mu(F_s) + \mu(F \setminus F_s) = \mu(F \setminus F_s).$$

Therefore, $\mu(F) = 0$ for all measurable sets and so $\mu = 0$. ∎

Remark 2.2. We can give a constructive proof of this theorem. The argument was suggested by Bill Johnson. For simplicity, identify $\mathbb{T}$ with the interval $[0, 1]$ and let λ be the usual Lebesgue measure on $[0, 1]$. We fix a measurable set $D \subset [0, 1]$ such that $\lambda(D) = 1/2$. Then let $\mathcal{T}$ be the linear span of $\{1_{gD} : g \in \mathrm{IM}\}$. We claim that the L_∞-norm closure of $\mathcal{T}$ is all of $L_\infty(0, 1)$.

First, it is clear that $\mathcal{T}$ contains differences $1_E - 1_F$ where $\lambda(E) = \lambda(F) \leq 1/2$, and it contains constants. So if $\lambda(E) = 1/2^n$, let

$$S = (2^n - 1)1_E - 1_{E_1} - 1_{E_2} - \cdots - 1_{E_{2^n - 1}}$$

with E and the E_i pairwise disjoint and all E_i having measure $1/2^n$. The support of this sum is the whole interval and $1 + S = 2^n 1_E$. Since this is in $\mathcal{T}$, when we divide by 2^n, we then see that 1_E is also in $\mathcal{T}$. Also, by adding characteristic functions of this type, we get $1_E \in \mathcal{T}$ if $\lambda(E)$ is a dyadic rational.

Then consider a general E and take sets $F(s)$ with dyadic rational measure such that $E \subset F(s)$ and such that $A(s) = F(s) \setminus E$ has λ-measure going to 0 as $s \to \infty$. Choose as many images $\sigma_k F(s)$ as possible under the constraints that $\sigma_k E = E$ and the images $\sigma_k A(s)$ are disjoint from E and pairwise disjoint from each other. Let $L(s)$ denote the number of these images. Since $\lambda(A(s))$ tends to 0 as $s \to \infty$, the number $L(s)$ tends to ∞ as $s \to \infty$. Now consider

$$S = \sum_{k=1}^{L(s)} 1_{\sigma_k F(s)}.$$

This sum is in $\mathcal{T}$ and is equal to $L(s)1_E + \sum_{k=1}^{L(s)} 1_{\sigma_k A(s)}$. So, $(1/L(s))S$ is in $\mathcal{T}$ and

$$\left\| 1_E - \frac{1}{L(s)} S \right\|_\infty = \frac{1}{L(s)}.$$

Thus, 1_E is in the L_∞-norm closure of $\mathcal{T}$ for any measurable set E. But then by approximating a bounded measurable function by dissection of the range, we see that the L_∞-norm closure of $\mathcal{T}$ is all of L_∞.

Proposition 2.1 shows that, if the cardinality of the acting group is not an obstruction, then a cyclic vector in L_∞ can exist. We will see below situations in which the cardinality is not an obstruction, but nonetheless there are no cyclic vectors in L_∞. Hence, the existence of a cyclic vector becomes a matter of the structural nature of the action.

Before moving on to results that give us a counterpoint to Proposition 2.1, there is another observation about cyclic vectors in this context that is worth making. Consider IM more generally to be the invertible measure-preserving transformations of a non-atomic probability space (X, β, λ). Then IM acts in the regular fashion on $L_\infty(X)$. Now restrict to the action by a subgroup H of IM, and assume there is a function in $L_\infty(X)$ that is cyclic under H. It follows that there must be a unique H-invariant mean. Indeed, suppose f is the cyclic vector. Let m be an H-invariant mean on $L_\infty(X)$. Choose a sequence of finite linear combinations

$$c_s = \sum_{h \in H} c_s(h) f \circ h$$

such that $1 = \lim_{s \to \infty} c_s$ in L_∞-norm. Then

$$1 = m(1) = \lim_{s \to \infty} \sum_{h \in H} c_s(h) m(f).$$

Since we could use the invariant mean given by the integral with respect to λ, it follows that $m(f) = \int f\,d\lambda$. But then $m(F) = \int F\,d\lambda$ for all F in the linear span of $\{f \circ h : h \in H\}$; and therefore since f is cyclic, $m(F) = \int F\,d\lambda$ for all $F \in L_\infty(X)$.

This argument shows that there cannot be a uniformly cyclic vector in $L_\infty(\mathbb{T})$ under the regular action of $\mathbb{T}$ because there are many different invariant means on $L_\infty(\mathbb{T})$. The same would hold for any compact group that was amenable as a discrete group because there are many different invariant means. The argument can also be used to show that if G is amenable as a locally compact group and not compact, then there cannot be a uniformly cyclic vector in $L_\infty(G)$ under the regular action of G (because again there are many different invariant means on $L_\infty(G)$ in this case too). See Rosenblatt [3] for a discussion of the existence of invariant means and references for the proofs of the statements above.

However, some compact groups have a unique invariant mean on $L_\infty(G)$ under the regular action of the group. See Paterson [2] for a survey of this uniqueness question. So the argument above does not work for all compact groups, and another method is needed. We will actually give two arguments because they prove different things in different ways. Here is the first line of reasoning.

THEOREM 2.3. *Let G be a non-discrete compact group and let E be a measurable set. Let $\mathcal{T}_E$ be the linear span of $\{1_{gE} : g \in G\}$. Then the generic characteristic function 1_A is not in the L_∞-norm closure of $\mathcal{T}_E$.*

COROLLARY 2.4. *Let G be a non-discrete compact group and let E be a measurable set. The L_∞-norm closed linear span of $\{1_{gE} : g \in G\}$ is a proper subspace of $L_\infty(G)$.*

Proof of Theorem 2.3. We will use the notation E^e, where $e = 1, \mathrm{c}$, with E^1 denoting E and E^c denoting the complement of E, i.e. $E^\mathrm{c} = G \setminus E$. Fix $n \ge 1$. Fix functions $e : \{1, \ldots, n\} \to \{1, \mathrm{c}\}$ and $\gamma : \{1, \ldots, n\} \to G$. Any atom $P = P(e, \gamma)$ in the finite σ-algebra generated by $\gamma(1)E, \ldots, \gamma(n)E$ is of the form

$$P = \bigcap_{k=1}^{n} \gamma(k) E^{e(k)}.$$

The atom depends on the choice of e and γ. But with e fixed, the atom depends continuously on the choice of $\gamma(1), \ldots, \gamma(n)$. That is, if $\gamma_s(k)$ converges to $\gamma(k)$ as $s \to \infty$, for all $k = 1, \ldots, n$, then as $s \to \infty$,

$$\lambda_G(P(e, \gamma_s) \triangle P(e, \gamma)) \to 0.$$

Consider β in the symmetry pseudo-metric ϱ, $\varrho(U, V) = \lambda_G(U \triangle V)$ for $U, V \in \beta$. This is a complete pseudo-metric space. Fix n, m and a finite number of distinct functions $e_i : \{1, \dots, n\} \to \{1, \mathrm{c}\}$ with $i = 1, \dots, m$. Let $\mathcal{R}(n, e_1, \dots, e_m)$ consist of all sets $S \in \beta$ such that for some γ, we have $S = \bigcup_{i=1}^m P(e_i, \gamma)$. Because of the continuity remarked above, and the compactness of G, the set $\mathcal{R}(n, e_1, \dots, e_m)$ is compact, and hence closed, in (β, ϱ).

In addition, $\mathcal{R}(n, e_1, \dots, e_m)$ has no interior. This is a result of a proper understanding of why (β, ϱ) is not locally compact. Because G is non-discrete, for any $\varepsilon > 0$, and any measurable set E with $\lambda_G(E) \geq \varepsilon$, we can construct a sequence $(A_j : j \geq 1)$ of subsets of E with $\lambda_G(A_j) = \varepsilon/2$ for all n such that the $(A_j : j \geq 1)$ are mutually independent. Indeed, if G were a compact, metric group, then any such set E would be bimeasurably isomorphic to $[0, \lambda_G(E)]$ with Lebesgue measure. In $[0, \lambda_G(E)]$, we can take the contraction by ε of the sets $\{x \in [0,1] : r_j(x) = 1\}$, using the Rademacher functions (r_j), to get such mutually independent measurable sets. Then take their isomorphic image in E to get the mutually independent measurable sets $(A_j : j \geq 1)$ in E. For the general compact group, one just takes a quotient G/N by a compact, normal subgroup N such that G/N is a non-discrete, metric compact group and applies the construction above to E/N. Then the inverse images of these sets in G/N under the canonical projection from G to G/N will give us mutually independent sets $(A_j : j \geq 1)$ in E as desired.

So now also suppose $\varepsilon \leq 1$ and $A_0 \in \beta$ with $\lambda_G(A_0^{\mathrm{c}}) \geq \varepsilon$. As remarked above, in $E = A_0^{\mathrm{c}}$, we can construct mutually independent sets $(A_j : j \geq 1)$ with $\lambda_G(A_j) = \varepsilon/2$. Then the sets $B_j = A_0 \cup A_j$ have $\varrho(A_0, B_j) < \varepsilon$ for all j. By the independence, $\lambda_G(A_k \cap A_j) = \lambda_G(A_k)\lambda_G(A_j)$ for all k, j. So, for any distinct k, j,

$$\varrho(B_k, B_j) = 2(\varepsilon/2 - (\varepsilon/2)^2) = \varepsilon(1 - \varepsilon/2)$$

by the pairwise independence of the sets $(A_j : j \geq 1)$. Hence, for any distinct k, j, $\varrho(B_k, B_j) \geq \varepsilon/2$. Therefore, the sequence (B_j) has no convergent subsequences. This shows that no closed neighborhood of N of any $A \in \beta$ can be compact. Indeed, the interior of N would contain a measurable set A_0 with $\lambda_G(A_0^{\mathrm{c}}) > 0$. So we can proceed as above with a suitably small value of ε and obtain sets (B_j) as above that are also all in the original neighborhood N. Hence $\mathcal{R}(n, e_1, \dots, e_m)$, being itself compact, cannot have a non-empty interior.

Hence, by the Baire Category Theorem, the set $\mathcal{R}$ described by

$$\bigcap \{\mathcal{R}(n, e_1, \dots, e_m)^{\mathrm{c}} : n \geq 1,\ e_i : \{1, \dots, n\} \to \{1, \mathrm{c}\},\ i = 1, \dots, m\}$$

is a dense G_δ subset of β and we can choose a set A that is not in any of the sets $\mathcal{R}(n, e_1, \dots, e_m)$. But then 1_A cannot be in the L_∞-closed span of $\{1_{gE} : g \in G\}$. Indeed, suppose there exist scalars $a_1, \dots, a_n$ and elements

$\gamma(1),\dots,\gamma(n)\in G$ such that

$$\Big\|1_A-\sum_{k=1}^{n}a_k1_{\gamma(k)E}\Big\|_\infty<\frac{1}{3}.$$

We can rewrite

$$\sum_{k=1}^{n}a_k1_{\gamma(k)E}=\sum\{c_P1_P: P=P(e,\gamma),\ e:\{1,\dots,n\}\to\{1,c\}\}.$$

Then either $|c_P-1|\le 1/3$ or $|c_P|\le 1/3$, and so each atom P is either a subset of A or a subset of A^{c} a.e. Indeed, $|c_P-1|\le 1/3$ if and only if $P\subset A$, and $|c_P|\le 1/3$ if and only if $P\subset A^{\mathrm{c}}$ a.e. But then A must be the union of the atoms contained in it. Suppose these atoms P are given by $P=P(e_i,\gamma)$, $i=1,\dots,m$. Then $A\in\mathcal{R}(n,e_1,\dots,e_m)$, contrary to the choice of A. ∎

The method of proof above shows that this same argument will work with a slight modification if the single characteristic function is replaced by an L_∞ function or a suitably small set of such functions. Here is the result that can be proved using this line of reasoning.

THEOREM 2.5. *Let G be a non-discrete, compact group and let $\mathcal{F}$ be a subset of $L_\infty(G)$ that has a countable, dense subset in the L_∞-norm topology. Then the generic characteristic function 1_A is not in the L_∞-norm closed span of $\{_gf: g\in G,\ f\in\mathcal{F}\}$.*

Proof. Since $\mathcal{F}$ has a countable, dense subset, using approximation by linear combinations of characteristic functions via dissection of the range, it is clear that there is a sequence $(E_i: i=1,2,\dots)$ of measurable sets such that the L_∞-norm closed span of $\{_gf: g\in G,\ f\in\mathcal{F}\}$ is contained in the L_∞-norm closed span of $\{_g1_{E_i}: g\in G,\ i\ge 1\}$.

We now consider any atom

$$P=P(e,\iota,\gamma)=\bigcap_{k=1}^{n}\gamma(k)E_{\iota(k)}^{e(k)}$$

in the finite σ-algebra generated by $\gamma(1)E_{\iota(1)},\dots,\gamma(n)E_{\iota(n)}$ using any $\iota(k)\ge 1$ and any $\gamma(k)\in G$. Let $\mathcal{R}(I,n,e_1,\dots,e_m)$ consist of all sets $S\in\beta$ such that $S=\bigcup_{s=1}^{m}P(e_s,\iota,\gamma)$, where $e_1,\dots,e_m$ are fixed, $\iota(k)$ is restricted so that $1\le\iota(k)\le I$, but the elements $\gamma(1),\dots,\gamma(n)\in G$ may vary. The same argument as the one in Theorem 2.3 shows that the sets $\mathcal{R}(I,n,e_1,\dots,e_m)$ are closed and nowhere dense in (β,ϱ). It follows in the same way as in the proof of this Theorem 2.3 that the generic set A is not in any of the sets $\mathcal{R}(I,n,e_1,\dots,e_m)$ and, hence, that 1_A is not in the L_∞-norm closed span of $\{_g1_{E_i}: g\in G,\ i\ge 1\}$. ∎

REMARK 2.6. For example, suppose that $\mathcal{F}$ is the closure in the L_∞-norm of a set $\mathcal{F}_0$ that is σ-compact in the L_∞-norm topology. The reader should compare this hypothesis with the one in Theorem 2.8. If we write $\mathcal{F}_0$ as a union of L_∞-norm compact sets $\mathcal{F}_0(n)$, $n \geq 1$, then for each n there is a countable dense subset D_n in $\mathcal{F}_0(n)$. So it follows that the union $\bigcup_{n=1}^\infty D_n$ will be a countable dense subset of $\mathcal{F}$. Hence, $\mathcal{F}$ satisfies the hypotheses of Theorem 2.5.

In addition, by localizing the proof, the same type of result holds for a non-discrete, σ-compact, locally compact group acting on itself.

THEOREM 2.7. *Let G be a non-discrete, σ-compact, locally compact group and let $\mathcal{F}$ be a subset of $L_\infty(G)$ that has a countable dense subset in the L_∞-norm topology. Then the generic characteristic function 1_A is not in the L_∞-norm closed span of $\{_g f : g \in G, f \in \mathcal{F}\}$.*

Proof. Let K be any compact subset of G with positive measure. It is not hard to see that a local measure of relative difference is continuous as a function. That is, for any $E \in \beta$, the function $T(g) = \lambda_G(K \cap (E \bigtriangleup gE))$ is continuous on G. Moreover, we can restrict the translations to a given compact subset of G, taken from an increasing sequence of such sets whose union is all of G. This means that we can localize the definition of $\mathcal{R}$ in the proof of Theorem 2.5 and show that a generic, measurable function 1_A with $A \subset K$ will not be in the L_∞-norm closed span of $\{_g f : g \in G, f \in \mathcal{F}\}$. ∎

Here is another approach to proving results like the previous ones. This method does give a new way of proving at least the main conclusion of Theorem 2.5, that the subspace in question is not the whole space. Thanks are due to Bob Kaufman who provided this formulation of the theorem and a variant of this proof.

THEOREM 2.8. *Let G be a non-discrete compact group. Suppose $\mathcal{F}$ is a subset of $L_\infty(G)$ whose closure in the L_1-norm is a σ-compact set in $L_1(G)$. Then the L_∞-norm closed span of $\{f : f \in \mathcal{F}\}$ is a proper subspace of $L_\infty(G)$.*

REMARK 2.9. The hypotheses in Theorem 2.8 and Theorem 2.5 are different, neither being stronger than the other. Indeed, let G be the circle $\mathbb{T}$ and let $\mathcal{F} = D$ consist of all simple functions constructed from rational linear combinations of arcs with rational endpoints. This countable set is L_1-norm dense in all of $L_1(G)$, and hence its closure in the L_1-norm does not satisfy the hypothesis of Theorem 2.8 even though it does meet the conditions needed to apply Theorem 2.5. On the other hand, if we take $\mathcal{F}$ to consist of the characteristic functions of all arcs in $\mathbb{T}$, then this set is compact in the L_1-norm topology and satisfies the conditions of Theorem 2.8.

However, it cannot satisfy the conditions of Theorem 2.5 because for any two distinct arcs A_1 and A_2, we have $\|1_{A_1} - 1_{A_2}\|_\infty = 2$.

Nonetheless, Theorem 2.8 can be used to give the basic conclusion in Theorem 2.5. Indeed, suppose $\mathcal{F}$ satisfies the assumption in Theorem 2.5, and let $D = \{f_1, f_2, \dots\}$ be a countable set that is dense in $\mathcal{F}$. Then $g_n = f_n/n(\|f_n\|_\infty + 1)$ defines a sequence (g_n) that is converging to 0 in the L_1-norm, and hence whose closure in L_1-norm is σ-compact. Also, the linear span of $\{g_n : n \geq 1\}$ is the linear span of $\{f_n : n \geq 1\}$. Hence, Theorem 2.8 says that the L_∞-norm closed span of $\mathcal{F}$ is a proper subspace of $L_\infty(G)$, the main part of the conclusion of Theorem 2.5. So, Theorem 2.8 is the stronger result, at least with regard to whether the subspace in question is all of $L_\infty(G)$ or not. Inherently, Theorem 2.8 says that the generic function in $L_\infty(G)$ is not in the L_∞-norm closed span of $\mathcal{F}$. But we do not know if this translates into saying that the generic characteristic function is also not in this closed span (although of course there will be some characteristic function that is not in this closed span because it is not the whole space). It should be possible to prove Theorem 2.8 along the lines of the proof of Theorem 2.5, and so obtain the additional information about the generic characteristic function, but we have not been able to see how to achieve this.

Proof of Theorem 2.8. We need to use a suitable operator sequence in this proof. There is some latitude in what we choose, but in this case the simplest one to use is an approximate identity. To make this possible, we first reduce the theorem to the case where G is a compact, metric group. This can be arranged without loss of generality since $\mathcal{F}$ has a σ-compact L_1-norm closure in $L_1(G)$, and hence there exists a compact, normal subgroup N of G such that G/N is a compact, metric group and all the functions in $\mathcal{F}$ are constant on the cosets of N in G.

We may also assume that $\mathcal{F}$ has a closure in the L_1-norm that is compact in $L_1(G)$, an assumption that will make the proof below a little easier. Indeed, we can write $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n$ where each $\mathcal{F}_n$ has a closure in the L_1-norm that is compact in $L_1(G)$. But then $N_n = \sup_{f \in \mathcal{F}_n} \|f\|_1 < \infty$. So there exists (δ_n), $\delta_n > 0$, such that $\lim_{n\to\infty} \delta_n N_n = 0$. It follows that $\widetilde{\mathcal{F}} = \bigcup_{n=1}^\infty \delta_n F_n$ has a closure in the L_1-norm that is compact in $L_1(G)$. But also, $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ have the same linear span in $L_\infty(G)$.

So assume that G is a compact, metric group and that $\mathcal{F}$ has a closure in the L_1-norm in $L_1(G)$ that is compact. We proceed by picking pairwise disjoint open sets (A_k) and an approximate identity D_k with suitable properties. First, fix any sequence of non-empty, pairwise disjoint open sets (A_k). For each A_k, there exists a compact subset $E_k \subset A_k$ such that

$$\lambda_G(A_k \setminus E_k) \leq \frac{\lambda_G(A_k)}{100}.$$

Then there exists a decreasing sequence of symmetric, open neighborhoods W_k of the identity in G such that $W_k E_k \subset A_k$ for all $k \geq 1$. We can make the additional restriction on W_k that for any open neighborhood U of the identity, eventually $W_k \subset U$. Now choose any sequence of continuous functions $(\phi_k : k = 1, 2, \dots)$ in $L_1(G)$ such that $\phi_k \geq 0$, $\|\phi_k\|_1 = 1$ for all $k \geq 1$, and such that ϕ_k is supported in the neighborhood W_k. Let $\phi_k \star f$ denote the convolution given by $\phi_k \star f(x) = \int_G \phi_k(y) f(y^{-1}x)\, d\lambda_G(y)$. It follows that

$$\lim_{k\to\infty} \|\phi_k \star f - f\|_1 = 0$$

for all $f \in L_1(G)$, i.e. (ϕ_k) is an approximate identity in $L_1(G)$. Our choices also tell us that if γ is a function supported in A_k, then we can compute $\phi_k \star \gamma(x)$ for all $x \in E_k$ just in terms of the values of γ on A_k. Thus, for all of A_k, except a subset of measure no more than $\lambda_G(A_k)/100$, we can compute $\phi_k \star \gamma(x)$ just in terms of the values of γ on A_k

Now, choose (A_k), (E_k), (W_k), and (ϕ_k) as described above. Define the operators D_k by $D_k(f) = \phi_k \star f$ for all $f \in L_1(G)$. Because $\mathcal{F}$ has a closure in L_1-norm that is compact in $L_1(G)$, there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that $\|D_n(f) - f\|_1 \leq \varepsilon_n$ for all $f \in \mathcal{F}$. Let (ε_{n_k}) be a subsequence of (ε_n) such that $\varepsilon_{n_k} \leq (1/k)\lambda_G(A_k)$.

We claim there is a function $g \in L_\infty(G)$ such that for all $k \geq 1$,

$$\int_{A_k} |D_{n_k}(g) - g|\, d\lambda_G \geq \frac{1}{2}\, \lambda_G(A_k).$$

Suppose for now that such a function g exists. We claim that then g is not in the L_∞-norm closed span of $\mathcal{F}$. Indeed, we claim that if $\|g - h\|_\infty \leq 1/8$, then h is not in the span of $\mathcal{F}$. To see this, assume that $\|g - h\|_\infty \leq 1/8$ and h is in this span. Then, for some constant C depending only on h, we would have $\|D_{n_k}(h) - h\|_1 \leq C\varepsilon_{n_k} \leq C(1/k)\lambda_G(A_k)$. But also $\|(D_{n_k}(h) - h) - (D_{n_k}(g) - g)\|_\infty \leq 2\|h - g\|_\infty \leq 1/4$. Hence, we would have

$$\begin{aligned} C\,\frac{1}{k}\,\lambda_G(A_k) &\geq \|D_{n_k}(h) - h\|_1 \geq \int_{A_k} |D_{n_k}(h) - h|\, d\lambda_G \\ &\geq \int_{A_k} \left(|D_{n_k}(g) - g| - \frac{1}{4} \right) d\lambda_G \geq \frac{1}{4}\, \lambda_G(A_k). \end{aligned}$$

So $C \geq k/4$ for all k, which is impossible.

So now we only need to construct g. First, we will choose a sequence (g_k) such that each g_k is a bounded measurable function supported on A_k such that $g_k = \pm 1$ everywhere on A_k. We can arrange a sufficiently dense oscillation in the choice of g_k so that $|D_{n_k}(g_k)|$ is uniformly as small as we like on E_k. For example, if we have $|D_{n_k}(g_k)| \leq 1/8$ uniformly on E_k, then we would have

$$\int_{E_k} |D_{n_k}(g_k) - g_k|\, d\lambda_G \geq \frac{7}{8}\,\lambda_G(E_k) \geq \frac{7}{8}\left(\frac{99}{100}\,\lambda_G(A_k)\right) \geq \frac{3}{4}\,\lambda_G(A_k).$$

We then let $g = \sum_{k=1}^{\infty} g_k$. Since the operators D_{n_k} are supported in W_k, the sets W_k are decreasing, and the sets A_k are pairwise disjoint, $D_{n_k}(g_l) = 0$ on E_{n_k} if $l < k$. Hence, we have the following estimate:

$$\begin{aligned}\int_{A_k} |D_{n_k}(g) - g|\, d\lambda_G \geq \int_{E_k} |D_{n_k}(g) - g|\, d\lambda_G &= \int_{E_k} \Big|\sum_{l=k}^{\infty} D_{n_k}(g_l) - g_k\Big|\, d\lambda_G \\ &\geq \int_{E_k} \Big(|D_{n_k}(g_k) - g_k| - \sum_{l=k+1}^{\infty} |D_{n_k}(g_l)|\Big)\, d\lambda_G \\ &\geq \frac{3}{4}\,\lambda_G(A_k) - \int_{E_k} \sum_{l=k+1}^{\infty} |D_{n_k}(g_l)|\, d\lambda_G\end{aligned}$$

for all $k \geq 1$. But as part of the construction of the sequence (g_k), we can arrange for the functions to oscillate between $+1$ and -1 so often that for any $k \geq 1$ and any $l \geq k$, we have $|D_{n_k}(g_l)| \leq \frac{1}{4}\big(\frac{1}{2^l}\big)$ uniformly on E_l. This certainly guarantees the estimate needed above: $|D_{n_k}(g_k)| \leq 1/8$ uniformly on E_k.

The actual construction of (g_k) can proceed inductively as follows. Take g_1 supported on A_1 such that $|D_{n_1}(g_1)| \leq \frac{1}{4}\big(\frac{1}{2}\big)$. Then let $k > 1$ and assume that $g_1, \ldots, g_{k-1}$ have already been chosen appropriately. Then take g_k, supported on A_k, so that for all j, $1 \leq j \leq k$, we have $|D_{n_j}(g_k)| \leq \frac{1}{4}\big(\frac{1}{2^k}\big)$ uniformly on E_k. It follows, by the estimate above, that we have

$$\begin{aligned}\int_{A_k} |D_{n_k}(g) - g|\, d\lambda_G &\geq \frac{3}{4}\,\lambda_G(A_k) - \int_{E_k} \sum_{l=k+1}^{\infty} |D_{n_k}(g_l)|\, d\lambda_G \\ &\geq \frac{3}{4}\,\lambda_G(A_k) - \sum_{l=k+1}^{\infty} \frac{1}{4}\left(\frac{1}{2^l}\right)\lambda_G(E_k) \\ &\geq \frac{3}{4}\,\lambda_G(A_k) - \frac{1}{4}\,\lambda_G(E_k) \geq \frac{1}{2}\,\lambda_G(A_k). \quad ∎\end{aligned}$$

Remark 2.10. This result can also be generalized to the case of a non-discrete, σ-compact, locally compact group in the same manner as we proved Theorem 2.7.

There is another interesting context where the same issues of uniqueness of invariant means and the existence of cyclic vectors arise. With the correct axioms (e.g. CH), one can have amenable groups of permutations of the integers $\mathbb{Z}$ which allow for only one invariant mean on $l_\infty(\mathbb{Z})$; that is, there can be amenable actions with a unique invariant mean. This is not possible with a commutative group of permutations, but it can be done using a locally finite group, and hence an amenable group, for the acting group. See both Yang [6] and Foreman [1] for the construction of such actions by locally finite groups. Given the discussion above, it is natural to ask this question:

QUESTION. Given CH, can one construct a locally finite group of permutations of $\mathbb{Z}$ such that there is a cyclic vector for the regular action of G on $l_\infty(\mathbb{Z})$?

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