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LIFTS FOR SEMIGROUPS OF MONOMORPHISMS OF AN INDEPENDENCE ALGEBRA

BY

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Abstract. For a universal algebra $\mathcal{A}$, let $\operatorname{End}(\mathcal{A})$ and $\operatorname{Aut}(\mathcal{A})$ denote, respectively, the endomorphism monoid and the automorphism group of $\mathcal{A}$. Let S be a semigroup and let T be a characteristic subsemigroup of S. We say that $\phi \in \operatorname{Aut}(S)$ is a *lift* for $\psi \in \operatorname{Aut}(T)$ if $\phi|T = \psi$. For $\psi \in \operatorname{Aut}(T)$ we denote by $L(\psi)$ the set of lifts of ψ, that is, $L(\psi) = \{\phi \in \operatorname{Aut}(S) \mid \phi|T = \psi\}$. Let $\mathcal{A}$ be an independence algebra of infinite rank and let S be a monoid of monomorphisms such that $G = \operatorname{Aut}(\mathcal{A}) \le S \le \operatorname{End}(\mathcal{A})$. In [2] it is proved that if $\mathcal{A}$ is a set (that is, an algebra without operations), then $|L(\phi)| = 1$. The analogous result for vector spaces does not hold. Thus the natural question is: Characterize the independence algebras in which $|L(\phi)| = 1$. The aim of this note is to answer this question.

1. Introduction. We assume the reader to be familiar with both semigroup theory and universal algebra. We recommend as references [7] and [9]. Also we assume the reader to have a basic knowledge of the theory of independence algebras. We recommend [3], [4] and [6] as references. Independence algebras, inspired by Marczewski, were introduced as v^*-algebras by Narkiewicz [10]. For an excellent survey paper see [11].

The first step in the definition of independence algebras is the introduction of a notion of independence valid for universal algebras. Let $\mathcal{A}$ be an algebra of universe A and let X be a set contained in A. Then we denote by $\langle X\rangle$ the algebra generated by X. Now, a subset X of an algebra is said to be *independent* if $X = \emptyset$ or if, for every element $x \in X$, we have $x \notin \langle X \setminus \{x\}\rangle$; a set is *dependent* if it is not independent.

LEMMA 1.1. *For an algebra $\mathcal{A}$, the following conditions are equivalent*:

(1) *for every subset X of A and all elements u, v of A, if $u \in \langle X \cup \{v\}\rangle$ and $u \notin \langle X\rangle$, then $v \in \langle X \cup \{u\}\rangle$*;

(2) *for every subset X of A and every element $u \in A$, if X is independent and $u \notin \langle X\rangle$, then $X \cup \{u\}$ is independent*;

(3) *for every subset X of A, if Y is a maximal independent subset of X, then $\langle X\rangle = \langle Y\rangle$*;

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(4) *for subsets X, Y of A with $Y \subseteq X$, if Y is independent, then there is an independent set Z with $Y \subseteq Z \subseteq X$ and $\langle Z \rangle = \langle X \rangle$.*

Proof. For a proof see [9, p. 50, Exercise 6]. ∎

Let $\mathcal{A}$ be an algebra with universe A. Then $\mathcal{A}$ is said to have the *exchange property* or to satisfy [EP] if it satisfies the equivalent conditions of Lemma 1.1. A *basis* for $\mathcal{A}$ is a subset of A which generates A and is independent. It is clear from Lemma 1.1 that any algebra with [EP] has a basis. Furthermore, for such an algebra, bases may be characterized as minimal generating sets or maximal independent sets, and all bases for $\mathcal{A}$ have the same cardinality ([6, Proposition 3.3]). This cardinality is called the *rank* of $\mathcal{A}$ and is written $\operatorname{rank}(\mathcal{A})$.

We say that $\mathcal{A}$ is an *independence algebra* if $\mathcal{A}$ satisfies [EP] and the following property:

[F] for every basis X of $\mathcal{A}$ and mapping $f : X \to A$, there exists a morphism $F : \langle X \rangle \to \mathcal{A}$ extending f.

Let f be an endomorphism of an independence algebra $\mathcal{A}$. Then B is a *preimage basis* for f if Bf is a basis for the image of f and $f|B$ is injective.

If $\mathcal{A}$ is a universal algebra, denote by $\operatorname{End}(\mathcal{A})$ and $\operatorname{Aut}(\mathcal{A})$, respectively, the endomorphism monoid and the automorphism group of $\mathcal{A}$. By $\operatorname{PEnd}(\mathcal{A})$ we denote the monoid of partial endomorphisms of $\mathcal{A}$.

When $\mathcal{A} = X$, where X is a set (that is, when the algebra has no operations), $\operatorname{End}(\mathcal{A}) = T(X)$ and $\operatorname{Aut}(\mathcal{A}) = \operatorname{Sym}(X)$, that is, respectively, the monoid of all transformations on X and the symmetric group on X.

Let S be a semigroup and let T be a subsemigroup of S. We say that T is *characteristic* in S if, for every $\phi \in \operatorname{Aut}(S)$, we have $\phi|T \in \operatorname{Aut}(T)$, that is, if the restriction to T of every automorphism of S is an automorphism of T. Now suppose that T is a characteristic subsemigroup of S. We say that an automorphism $\phi \in \operatorname{Aut}(S)$ is a *lift* for $\psi \in \operatorname{Aut}(T)$ if $\phi|T = \psi$. For $\phi \in \operatorname{Aut}(T)$ we denote by $L(\phi)$ the set of lifts of ϕ, that is,

$$L(\phi) = \{\psi \in \operatorname{Aut}(S) \mid \psi|T = \phi\}.$$

It is well known that lifts play a crucial role in the description of the automorphism group of a semigroup. The general scheme goes as follows: if we want to describe the automorphism group of S, a good idea is to try to find a subsemigroup $T \leq S$ such that:

(1) T is characteristic in S;

(2) we have (or can find) a description of $\operatorname{Aut}(T)$, the automorphism group of T;

(3) for every $g \in \operatorname{Aut}(T)$ we can describe $L(g)$.

Since (1) implies that $\mathrm{Aut}(S) = \bigcup_{g\in \mathrm{Aut}(T)} L(g)$, from (3) we get a description of $\mathrm{Aut}(S)$. (Usually it is necessary to have (2) in order to find (3).)

For example, in [8] Mal'tsev described $\mathrm{Aut}(T(X))$. He considered the semigroup

$$T_1(X) = \{f \in T(X) : |(X)f| = 1\},$$

proved that $T_1(X)$ is characteristic in $T(X)$, described the automorphisms of $T_1(X)$ (which are the mappings $\tau^g : a \mapsto g^{-1}ag$, where $g \in \mathrm{Sym}(X)$ and $a \in T_1(X)$) and proved that $|L(\tau^g)| = 1$ for all $g \in \mathrm{Aut}(T_1(X))$. Thus $\mathrm{Aut}(T(X)) = \mathrm{Inn}(T(X))$, the inner automorphisms, that is, the automorphisms induced by conjugation by elements $g \in \mathrm{Sym}(X)$.

Another example: in [2] Fitzpatrick and Symons considered semigroups S of injective mappings such that $\mathrm{Sym}(X) \le S \le T(X)$ (where X is an infinite set). In the most delicate part of their proof they show that, given an $h \in \mathrm{Aut}(\mathrm{Sym}(X))$, we have $|L(h)| = 1$. Thus, since $\mathrm{Sym}(X)$ is characteristic in S (and all automorphisms of $\mathrm{Sym}(X)$ are inner), we have $\mathrm{Aut}(S) = \bigcup_{\tau^g\in \mathrm{Aut}(\mathrm{Sym}(X))} L(\tau^g)$ and hence for every semigroup S of one-one mappings such that $\mathrm{Sym}(X) \le S \le T(X)$ we have

$$\mathrm{Aut}(S) = \{\tau^g : s \mapsto g^{-1}sg \mid g \in \mathrm{Sym}(X)\}.$$

In [5] Gluskin used the same general scheme to describe the automorphism group of $\mathrm{End}(V)$, where V is a vector space, thus proving the vector space analogue of the result proved by Mal'tsev for sets, as referred to above.

The linear analogue of the result of Fitzpatrick and Symons referred to above would read as follows: given a vector space V of infinite dimension and a semigroup S of injective linear transformations such that $\mathrm{Aut}(V) \le S \le \mathrm{End}(V)$, prove that for all $h \in \mathrm{Aut}(\mathrm{Aut}(V))$ we have

$$|L(h)| = |\{g \in \mathrm{Aut}(S) \mid g|\mathrm{Aut}(\mathrm{Aut}(V)) = h\}| = 1.$$

We prove that this is not true and then characterize the independence algebras in which the analogue of this result holds.

Let $\mathcal{A}$ be an algebra. We denote by Con the constants of $\mathcal{A}$. Throughout this paper $\mathcal{A}$ will be an independence algebra of infinite rank such that $\mathrm{Con} \subseteq \{0\}$ (that is, Con has at most one element), and S is a semigroup of monomorphisms of an independence algebra $\mathcal{A}$ of infinite rank such that $\mathrm{Aut}(\mathcal{A}) < S < \mathrm{End}(\mathcal{A})$.

Suppose a is a partial endomorphism of $\mathcal{A}$. We denote the domain of a by Δa and the image of a by ∇a. Observe that both Δa and ∇a are subalgebras of $\mathcal{A}$.

2. A counterexample. The aim of this section is to prove that the analogue of the Fitzpatrick and Symons result referred to above is not true. Therefore we want to prove the following. Let V be a vector space and S be a semigroup of monomorphisms such that $\mathrm{Aut}(V) \le S \le \mathrm{End}(V)$. Then it is not necessarily true that for all $h \in \mathrm{Aut}(\mathrm{Aut}(V))$ we have

$$|L(h)| = |\{g \in \mathrm{Aut}(S) \mid g|\mathrm{Aut}(\mathrm{Aut}(V)) = h\}| = 1.$$

LEMMA 2.1. *Let V be an infinite-dimensional vector space over the field of real numbers F and let S be the following semigroup*:

$$\{\alpha \in \mathrm{End}(V) \mid \mathrm{corank}(\nabla\alpha) < \aleph_0 \text{ and } \alpha \text{ is one-one}\}.$$

Moreover, *let λ be a real number different from* 0 *and* 1. *Then the mapping $\tau_\lambda : S \to S$ defined by $a\tau_\lambda = \lambda^{\mathrm{corank}(\nabla\alpha)}a$ is an automorphism of S.*

Proof. The key observation is that for the semigroup under consideration we have, for all $\alpha, \beta \in S$,

$$\mathrm{corank}(\nabla\alpha) + \mathrm{corank}(\nabla\beta) = \mathrm{corank}(\nabla\alpha\beta).$$

To simplify notation we write $\mathrm{corank}(\nabla\alpha)$ as $\mathrm{corank}(\alpha)$. Now, if B is a basis for V then $B\alpha$ is a basis for $\nabla\alpha$ and it can be extended to $B\alpha \cup Y$, a basis of V. Moreover, $B\alpha\beta$ is a basis for $\nabla(\alpha\beta)$ and $B\alpha\beta \cup Y\beta$ is a basis for $\nabla\beta$, which can be extended to a basis of V, say $B\alpha\beta \cup Y\beta \cup W$. Hence $|W| = \mathrm{corank}(\beta)$ and $|Y\beta| = |Y| = \mathrm{corank}(\alpha)$. Thus,

$$\mathrm{corank}(\beta) + \mathrm{corank}(\alpha) = |W| + |Y| = |W| + |Y\beta| = \mathrm{corank}(\alpha\beta).$$

Now, the proof of the lemma is straightforward. In fact, for every $a, b \in S$, we have

$$\begin{aligned}(ab)\tau_\lambda &= \lambda^{\mathrm{corank}(ab)}ab = \lambda^{\mathrm{corank}(a)+\mathrm{corank}(b)}ab = \lambda^{\mathrm{corank}(a)}\lambda^{\mathrm{corank}(b)}ab \\ &= (\lambda^{\mathrm{corank}(a)}a)(\lambda^{\mathrm{corank}(b)}b) = (a\tau_\lambda)(b\tau_\lambda)\end{aligned}$$

and so τ_λ is a morphism. Moreover, $\mathrm{corank}(\mu a) = \mathrm{corank}(a)$ for every non-zero real number μ and any element $a \in S$. In particular, $\mathrm{corank}(a) = \mathrm{corank}(a\tau_\lambda)$ for all $a \in S$. Thus, if $a\tau_\lambda = b\tau_\lambda$, where $a, b \in S$, then

$$\mathrm{corank}(a) = \mathrm{corank}(a\tau_\lambda) = \mathrm{corank}(b\tau_\lambda) = \mathrm{corank}(b)$$

and so $\lambda^{\mathrm{corank}(a)}a = a\tau_\lambda = b\tau_\lambda = \lambda^{\mathrm{corank}(b)}b = \lambda^{\mathrm{corank}(a)}b$, implying $a = b$. Thus, τ_λ is a monomorphism. Finally, it is obvious that $a/\lambda^{\mathrm{corank}(a)}$ belongs to S and that $(a/\lambda^{\mathrm{corank}(a)})\tau_\lambda = a$. The lemma is proved. ∎

It is obvious, with the notation of the previous lemma, that $\tau_\lambda|\mathrm{Aut}(V)$ is the identity. Therefore $L(\mathrm{id}_{\mathrm{Aut}(V)}) \supseteq \{\tau_\lambda \mid \lambda \in F \setminus \{0,1\}\}$.

3. The main theorem. We start by an observation which will be very useful in the remainder of this paper.

LEMMA 3.1. *Let $T \le S$ be two semigroups with T characteristic in S. Let id_T be the identity automorphism of T. Then the following are equivalent*:

(1) $|L(h)| = 1$ *for all* $h \in \mathrm{Aut}(T)$;
(2) $|L(\mathrm{id}_T)| = 1$.

Proof. It is obvious that (1) implies (2). Conversely, suppose that $g_1, g_2 \in L(h)$ for some $h \in \mathrm{Aut}(T)$. Then $g_1|T = h = g_2|T$ so that $(g_1 g_2^{-1})|T = \mathrm{id}_T$. This shows that $g_1 g_2^{-1} \in L(\mathrm{id}_T)$. However, by (2), $|L(\mathrm{id}_T)| = 1$ and it is obvious that $\mathrm{id}_S \in L(\mathrm{id}_T)$. Therefore $g_1 g_2^{-1} = \mathrm{id}_S$ and hence $g_1 = g_2$. ∎

Let $T \le S$ be two semigroups with T characteristic in S. Then, in view of the previous lemma, to prove that $|L(h)| = 1$ for all $h \in \mathrm{Aut}(T)$, we only have to check if $|L(\mathrm{id}_T)| = 1$.

We now introduce a definition and the main theorem of this paper. Let $\alpha \in \mathrm{PEnd}(\mathcal{A})$ be one-one. Moreover, let B and C be bases of $\Delta\alpha$ and $\nabla\alpha$, respectively, such that $B\alpha = C$. The pair (B, C) is said to be a *basic pair* for α and the set of all basic pairs is denoted by $\mathcal{B}(\alpha)$.

The aim of this section is to prove the following theorem. (We recall that $G = \mathrm{Aut}(\mathcal{A})$.)

THEOREM 3.2. *Let S be a semigroup of monomorphisms such that $G < S < \mathrm{End}(\mathcal{A})$ and let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. Then, for all $\alpha \in S$, the following are equivalent*:

(1) $\alpha\tau = \alpha$;
(2) $\mathcal{B}(\alpha) \cap \mathcal{B}(\alpha\tau) \ne \emptyset$;
(3) $\mathcal{B}(\alpha) = \mathcal{B}(\alpha\tau)$;
(4) $(\exists_{b \in \mathcal{A} \setminus \mathrm{Con}}) b\alpha = b(\alpha\tau)$;
(5) $(\exists_{(B,C) \in \mathcal{B}(\alpha)}) B\alpha \cap B(\alpha\tau) \ne \emptyset$;
(6) $(\exists_{(B,C) \in \mathcal{B}(\alpha)})\ B\alpha = B(\alpha\tau)$.

If B is an independent set and $x, y \in B$, we denote by $(xy)_B$ the (unique) automorphism $h \in \mathrm{Aut}(\langle B \rangle)$ such that $xh = y$, $yh = x$, and $zh = z$ for all $z \in B \setminus \{x, y\}$.

LEMMA 3.3. *Let $\alpha \in \mathrm{End}(\mathcal{A})$ be one-one and let $(B, C) \in \mathcal{B}(\alpha)$. Then, for every $x, y \in B$, we have $(xy)_B \alpha (x\alpha\ y\alpha)_C = \alpha$.*

Proof. Clearly, $a(xy)_B\alpha(x\alpha\ y\alpha)_C = a\alpha$ for $a \in \{x, y\}$. Thus let $a \in B \setminus \{x, y\}$. Then we have $a(xy)_B = a$. Similarly, since $a\alpha \in C \setminus \{x\alpha, y\alpha\}$, we obtain $a\alpha(x\alpha\ y\alpha)_C = a\alpha$. ∎

LEMMA 3.4. *Let S be a semigroup of monomorphisms such that $G < S < \mathrm{End}(\mathcal{A})$ and let $\tau \in \mathrm{Aut}(S)$ be such that $\tau|_G = \mathrm{id}_G$. Let $b \in A \setminus \mathrm{Con}$. If $b\alpha = b(\alpha\tau)$, then $\alpha\tau = \alpha$.*

Proof. Let $b \in A \setminus \mathrm{Con}$ be such that $b\alpha = b(\alpha\tau)$. As α is one-one it follows that $b\alpha \notin \mathrm{Con}$. Hence the set $\{b\alpha\}$ can be extended to a basis, C, of $\nabla\alpha$. Let B be a preimage basis such that $(B, C) \in \mathcal{B}(\alpha)$. Now, by Lemma 3.3, for all $c \in B \setminus \{b\}$ we have the following

$$\begin{aligned}(cb)_B\alpha(b\alpha\ c\alpha)_C = \alpha \ &\Rightarrow\ (cb)_B(\alpha\tau)(b\alpha\ c\alpha)_C = \alpha\tau \\ &\Rightarrow\ c(cb)_B(\alpha\tau)(b(\alpha\tau)\ c\alpha)_C = c(\alpha\tau) \quad (\text{as } b(\alpha\tau) = b\alpha)\end{aligned}$$

$$\Rightarrow\ b(\alpha\tau)(b(\alpha\tau)\ c\alpha)_C = c(\alpha\tau)$$
$$\Rightarrow\ c\alpha = c(\alpha\tau).$$

Thus $a\alpha = a(\alpha\tau)$ for all $a \in B$. The lemma is proved. ∎

Proof of Theorem 3.2. We are going to prove that (1)⇒(3)⇒(2)⇒(6) ⇒(5)⇒(4)⇒(1). The non-trivial implications are (2)⇒(6), (5)⇒(4) and (4)⇒(1).

We start with the proof that (2)⇒(6). If $\mathcal{B}(\alpha) \cap \mathcal{B}(\alpha\tau) \neq \emptyset$ then there is $(B, C) \in \mathcal{B}(\alpha) \cap \mathcal{B}(\alpha\tau)$ and hence $B\alpha = C = B(\alpha\tau)$. The implication is proved.

Now we prove that (5)⇒(4). Let $(B, C) \in \mathcal{B}(\alpha)$ and suppose that $(B)\alpha \cap (B)\alpha\tau \neq \emptyset$. Moreover, let $b, d \in B$ be such that $b\alpha = d(\alpha\tau)$. We claim that $b = d$. In fact, suppose that $b \neq d$. Then $\{b, d\} \subseteq B$ is an independent set. Moreover, $\{b\alpha, d\alpha\} \subseteq C$ is an independent set, as α is one-one. Now, let $a \in B \setminus \{b, d\}$. Such an a exists because rank($\mathcal{A}$) is infinite. Then

$$\begin{aligned}
(da)_B\alpha(d\alpha\ a\alpha)_C = \alpha\ &\Rightarrow\ (da)_B(\alpha\tau)(d\alpha\ a\alpha)_C = \alpha\tau\\
&\Rightarrow\ a(da)_B(\alpha\tau)(d\alpha\ a\alpha)_C = a(\alpha\tau)\\
&\Rightarrow\ d(\alpha\tau)(d\alpha\ a\alpha)_C = a(\alpha\tau)\\
&\Rightarrow\ b\alpha(d\alpha\ a\alpha)_C = a(\alpha\tau) \quad (\text{as } d(\alpha\tau) = b\alpha)\\
&\Rightarrow\ b\alpha = a(\alpha\tau).
\end{aligned}$$

But a was arbitrary in $B \setminus \{b, d\}$. Thus, for all $x, y \in B \setminus \{b, d\}$, $x \neq y$, we have $x(\alpha\tau) = b\alpha = y(\alpha\tau)$. Let D be a basis for $\mathcal{A}$ such that $\{x, y\} \subseteq D$. Then $(xy)_D(\alpha\tau) = \alpha\tau$, which implies $(xy)_D\alpha = \alpha$ and hence $x\alpha = y\alpha$. This is a contradiction since α is one-one. This shows that if $b\alpha = d(\alpha\tau)$, then $b = d$.

That (4)⇒(1) follows from Lemma 3.4. The result is proved. ∎

We now give an alternative proof that (2)⇒(1). This new proof contains some ideas which are very instructive.

Lemma 3.5. *Let $(B, C) \in \mathcal{B}(\alpha)$. Moreover, let $x, y \in B$ and $u, v \in C$. Then*

$$(xy)_B\alpha(uv)_C = \alpha\ \Rightarrow\ \{x\alpha, y\alpha\} = \{u, v\}.$$

Proof. From $(xy)_B\alpha(uv)_C = \alpha$ it follows $x(xy)_B\alpha(uv)_C = x\alpha$ and hence $x\alpha = y\alpha(uv)_C$. If $y\alpha \notin \{u, v\}$, then $y\alpha(uv)_C = y\alpha$, which is different from $x\alpha$ and hence we have a contradiction. Thus $y\alpha \in \{u, v\}$ and, by symmetry, $\{x\alpha, y\alpha\} = \{u, v\}$. ∎

Lemma 3.6. *Let $(B, C) \in \mathcal{B}(\alpha) \cap \mathcal{B}(\alpha\tau)$. Then $\alpha\tau = \alpha$.*

Proof. Let x, y, z be three different elements of B. Using the previous lemma and Lemma 3.3, we get

$$\begin{aligned}(xy)_B\alpha(x\alpha\ y\alpha)_C = \alpha\ &\Rightarrow\ (xy)_B(\alpha\tau)(x\alpha\ y\alpha)_C = \alpha\tau\\ &\Rightarrow\ \{x\alpha, y\alpha\} = \{x(\alpha\tau), y(\alpha\tau)\}.\end{aligned}$$

Now, repeating the procedure with z and x we get $\{x\alpha, z\alpha\} = \{x(\alpha\tau), z(\alpha\tau)\}$ and hence

$$\{x\alpha\} = \{x\alpha, y\alpha\} \cap \{x\alpha, z\alpha\} = \{x(\alpha\tau), y(\alpha\tau)\} \cap \{x(\alpha\tau), z(\alpha\tau)\} = \{x(\alpha\tau)\}.$$

Thus $\alpha = \alpha\tau$. The lemma follows. ∎

3.1. *Applications.* Let X be an infinite set and consider a semigroup S of one-one transformations such that $\mathrm{Sym}(X) \leq S \leq T(X)$. Moreover, let $\alpha \in S$. Then, for every $x, y \in \nabla\alpha$, $x \neq y$, we have $\alpha(xy) \neq \alpha$. Thus $(\alpha\tau)(xy) \neq (\alpha\tau)$ and hence $\nabla(\alpha\tau) \cap \{x, y\} \neq \emptyset$. This shows that $(X)\alpha \cap (X)(\alpha\tau) \neq \emptyset$ and hence, by (5) of Theorem 3.2, we have $\alpha = \alpha\tau$. Thus if two automorphisms of S coincide on $\mathrm{Sym}(X)$ they are equal. Now, since the automorphisms of $\mathrm{Sym}(X)$ are inner automorphisms and admit an obvious extension to an automorphism of S, it follows that each automorphism of S is the unique and obvious extension of one (inner) automorphism of $\mathrm{Sym}(X)$.

Chantip and Wood [1] proved this result for the semigroup of all one-one transformations on a set X. Fitzpatrick and Symons [2] proved it for any semigroup S of one-one transformations such that $\mathrm{Sym}(X) \leq S \leq T(X)$.

In the second section of this paper we saw a semigroup S and an automorphism τ such that $\nabla\alpha = \nabla(\alpha\tau)$ but $b\alpha \neq b(\alpha\tau)$ for all $b \in \mathcal{A} \setminus \mathrm{Con}$. Such a situation is impossible for transformations on a set but possible for linear transformations of a vector space. This explains why the analogue of the result proved by Fitzpatrick and Symons does not hold for independence algebras.

Finally, we state two open problems related with the questions above. When $\mathcal{A}$ is a set X the group $\mathrm{Aut}(\mathrm{Aut}(\mathcal{A}))$, that is, the group $\mathrm{Aut}(\mathrm{Sym}(X))$ is well known. Also the group $\mathrm{Aut}(\mathrm{Aut}(\mathcal{A}))$ is well known when $\mathcal{A}$ is a vector space. However a description of $\mathrm{Aut}(\mathrm{Aut}(\mathcal{A}))$ when $\mathcal{A}$ is a general independence algebra is not known.

Another open problem is the following. Let $\mathcal{A}$ be an independence algebra of infinite rank and let S be a semigroup of monomorphisms such that $\mathrm{Aut}(\mathcal{A}) < S < \mathrm{End}(\mathcal{A})$. For every $\psi \in \mathrm{Aut}(\mathcal{A})$ describe the set

$$L(\psi) = \{\phi \in \mathrm{Aut}(S) \mid \phi|\mathrm{Aut}(\mathcal{A}) = \psi\}.$$

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REFERENCES

[1] S. Chantip and G. R. Wood, *Automorphisms of the semigroup of all onto mappings of a set*, Bull. Austral. Math. Soc. 14 (1976), 399–403.

[2] S. P. Fitzpatrick and J. S. Symons, *Automorphisms of transformation semigroups*, Proc. Edinburgh Math. Soc. 19 (1974/75), 327–329.

[3] J. Fountain and A. Lewin, *Products of idempotent endomorphisms of an independence algebra of finite rank*, ibid. 35 (1992), 493–500.

[4] —, —, *Products of idempotent endomorphisms of an independence algebra of infinite rank*, Math. Proc. Cambridge Philos. Soc. 114 (1993), 303–319.

[5] L. M. Gluskin, *Semigroups and rings of endomorphisms of linear spaces I*, Amer. Math. Soc. Transl. 45 (1965), 105–137.

[6] V. A. R. Gould, *Independence algebras*, Algebra Universalis 33 (1995), 294–318.

[7] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford Univ. Press, New York, 1995.

[8] A. I. Mal'cev [A. I. Mal'tsev], *Symmetric groupoids*, Mat. Sb. 31 (1952), 136–151.

[9] R. N. McKenzie, G. F. McNulty and W. F. Taylor, *Algebra*, *Lattices*, *Varieties*, Vol. I, Wadsworth, Monterey, 1987.

[10] W. Narkiewicz, *Independence in a certain class of abstract algebras*, Fund. Math. 50 (1962), 333–340.

[11] K. Urbanik, *Linear independence in abstract algebras*, Colloq. Mat. 14 (1966), 233–255.

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