COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 1

HANKEL OPERATORS AND WEAK FACTORIZATION FOR HARDY–ORLICZ SPACES

BY

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This paper is dedicated to the memory of Andrzej Hulanicki,
a colleague and friend we will never forget

Abstract. We study the holomorphic Hardy–Orlicz spaces $\mathcal{H}^{\Phi}(\Omega)$, where Ω is the unit ball or, more generally, a convex domain of finite type or a strictly pseudoconvex domain in $\mathbb{C}^n$. The function Φ is in particular such that $\mathcal{H}^1(\Omega) \subset \mathcal{H}^{\Phi}(\Omega) \subset \mathcal{H}^p(\Omega)$ for some $p > 0$. We develop maximal characterizations, atomic and molecular decompositions. We then prove weak factorization theorems involving the space $BMOA(\Omega)$. As a consequence, we characterize those Hankel operators which are bounded from $\mathcal{H}^{\Phi}(\Omega)$ into $\mathcal{H}^1(\Omega)$.

Introduction. This work has been motivated by a new kind of factorization in the unit disc, obtained in [BIJZ]. Namely, the product of a function in $BMOA$ with a function in the Hardy space $\mathcal{H}^1$ of holomorphic functions lies in some Hardy–Orlicz space defined in terms of the function $\Phi_1(t) := t/\log(e+t)$. Conversely, every holomorphic function in this Hardy–Orlicz space can be written as the product of a function in $BMOA$ and a function in $\mathcal{H}^1$. This exact factorization relies heavily on the classical factorization theorem through Blaschke products and does not generalize to higher dimensions. Nevertheless, it was proven by Coifman, Rochberg and Weiss in the seventies [CRW] that $\mathcal{H}^p$, for $p \leq 1$, admits weak factorization, namely, $F = \sum_j G_j H_j$ with $\sum_j \|G_j\|_q^p \|H_j\|_r^p \leq C_{pq} \|F\|_p^p$ when $1/q + 1/r = 1/p$. This was extended later on by Krantz and Li to strictly pseudoconvex domains [KL], then by Peloso, Symesak and the present authors to convex domains of finite type [BPS1], [GP]. We rely on the methods of these two last papers, which are somewhat simpler, to obtain the weak factorization of Hardy–Orlicz spaces under consideration. Note that such a weak factorization for $\mathcal{H}^p$ is typical of the case $p \leq 1$, in contrast to the case of the unit disc where factorization is valid for all $p > 0$.

2010 *Mathematics Subject Classification*: 32A35, 32A37, 47B35.
Key words and phrases: Hardy–Orlicz spaces, atomic decomposition, finite type domains, weak factorization, logarithmic mean oscillation, BMO with weights, Hankel operator.

DOI: 10.4064/cm118-1-5

A natural application of such factorizations is to characterize classes of symbols of Hankel operators. We are able to characterize the symbols of Hankel operators mapping continuously Hardy–Orlicz spaces into $\mathcal{H}^1$ for a large class of Hardy–Orlicz spaces containing $\mathcal{H}^1$. We do this for all domains for which we have weak factorization. However, weak factorization is a stronger property, since the Hardy–Orlicz spaces under consideration are not Banach spaces. We have given in [BGS] a direct proof of the fact that Hankel operators are bounded on $\mathcal{H}^1$ of the unit ball if and only if their symbol is in the space $LMOA$, without involving Hardy–Orlicz spaces, even if the idea of weak factorization is indirectly present in that note.

Let us mention, in the same direction, the factorization obtained by Cohn and Verbitsky [CV] in the disc, which allows characterizing those symbols for which the corresponding Hankel operator is bounded from $\mathcal{H}^2$ into some Hardy–Sobolev space. A generalization to higher dimensions of their factorization seems much more difficult than ours.

At the end of this paper, we state the same theorems for a class of domains in $\mathbb{C}^n$ which includes the strictly pseudoconvex domains and the convex domains of finite type. We explain briefly how to modify the proofs.

Let us give some notations and describe the results more precisely. Let $\mathbb{B}^n$ be the unit ball and $\mathbb{S}^n$ be the unit sphere in $\mathbb{C}^n$. Let Φ be a continuous, positive and non-decreasing function on $[0,\infty)$. The Hardy–Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$ is defined as the space of holomorphic functions f such that

$$\sup_{0<r<1} \int_{\mathbb{S}^n} \Phi(|f(rw)|)\, d\sigma(w) < \infty \tag{1}$$

where $d\sigma$ denotes the surface measure on $\mathbb{S}^n$. We recover the Hardy spaces $\mathcal{H}^p(\mathbb{B}^n)$ when $\Phi(t) = t^p$. We are especially interested in the case $\Phi_p(t) = t^p/\log(e+t)^p$, $0 < p \le 1$, since the space $\mathcal{H}^{\Phi_p}(\mathbb{B}^n)$ arises naturally in the study of pointwise products of functions in $\mathcal{H}^p(\mathbb{B}^n)$ with functions in $BMOA(\mathbb{B}^n)$ inside the unit ball. Indeed, we prove that the product of an $\mathcal{H}^p(\mathbb{B}^n)$-function and a $BMOA(\mathbb{B}^n)$-function belongs to $\mathcal{H}^{\Phi_p}(\mathbb{B}^n)$, and conversely, there is weak factorization.

We will restrict ourselves to concave functions Φ which satisfy an additional assumption so that $\mathcal{H}^1(\mathbb{B}^n) \subset \mathcal{H}^\Phi(\mathbb{B}^n) \subset \mathcal{H}^p(\mathbb{B}^n)$ for some $0 < p \le 1$. In particular, any function f in the Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$ admits a unique boundary function still denoted by f which, by the Fatou Theorem, satisfies $\int_{\mathbb{S}^n} \Phi(|f|)\, d\sigma < \infty$.

Before going on, we need some basic notations for the geometry of $\mathbb{S}^n$. We recall that the *Korányi metric* on $\mathbb{S}^n$ is given by $d(z,w) := |1 - z.\overline{w}|$ (see [R]). All along the paper, except when speaking of the unit ball itself, *balls* (or *Korányi balls*) will be open subsets of $\mathbb{S}^n$ that are balls for the Korányi metric.

We also consider the real Hardy–Orlicz space $H^{\Phi}(\mathbb{S}^n)$ defined as the space of distributions on $\mathbb{S}^n$ which have an atomic decomposition defined in terms of Korányi balls. More precisely, $H^{\Phi}(\mathbb{S}^n)$ is the space of distributions f which can be written as $\sum_{j=0}^{\infty} a_j$, where the a_j's satisfy adapted cancellation properties, are supported in some ball B_j and are such that $\sum_j \sigma(B_j)\Phi(\|a_j\|_2\sigma(B_j)^{-1/2}) < \infty$.

We first prove maximal characterizations of Hardy–Orlicz spaces. As a corollary, we deduce that the Hardy–Orlicz space $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ continuously embeds into $H^{\Phi}(\mathbb{S}^n)$, while the Szegö projection is a projector onto $\mathcal{H}^{\Phi}(\mathbb{B}^n)$. In particular, every $f \in \mathcal{H}^{\Phi}(\mathbb{B}^n)$ has boundary values that belong to $H^{\Phi}(\mathbb{S}^n)$, and f may be written in terms of the Szegö projection of its atomic decomposition. The work of Viviani [V] plays a central role: atomic decomposition is proved there for Hardy–Orlicz spaces in the context of spaces of homogeneous type with a restriction on the lower type p of Φ, which, in the case of the unit ball, is the condition $p > 2n/(2n+1)$. We prove the atomic decomposition for all values of p, with the same kind of control of the norm as obtained by Viviani.

Since the Szegö projection of an atom is a molecule, we also get a molecular decomposition as in the classical Hardy spaces (see [TW] for instance).

The atomic decomposition allows us to prove a (weak) factorization theorem on $\mathcal{H}^{\Psi}(\mathbb{B}^n)$. In particular, we generalize the factorization theorem proved in the disc for $\mathcal{H}^{\Phi_1}$ in [BIJZ]. More precisely, we prove that, given any $f \in \mathcal{H}^{\Psi}(\mathbb{B}^n)$, there exist $f_j \in \mathcal{H}^{\Phi}(\mathbb{B}^n)$, $g_j \in BMOA(\mathbb{B}^n)$ such that $f = \sum_{j=0}^{\infty} f_j g_j$ where Ψ and Φ are linked by the relation $\Psi(t) = \Phi(t/\log(e+t))$.

As a consequence, we characterize the class of symbols for which the Hankel operators are bounded from $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$. The symbols belong to the dual space of $\mathcal{H}^{\Psi}(\mathbb{B}^n)$, which can be identified with the $BMOA$-space with weight ρ_Ψ where $\rho_\Psi(t) = 1/t\Psi^{-1}(1/t)$. Weighted $BMOA$-spaces have been considered by Janson in the Euclidean space [J]. Here they are defined by

$$BMOA(\rho_\Psi) := \Big\{ f \in \mathcal{H}^2(\mathbb{B}^n) : \sup_B \inf_{P \in \mathcal{P}_N(B)} \frac{1}{\sigma(B)\rho_\Psi(\sigma(B))^2} \int_B |f-P|^2\, d\sigma < \infty \Big\}.$$

where the integral is taken on the unit sphere, f stands for the boundary values of the function, and balls are with respect to the Korányi metric. Moreover, $\mathcal{P}_N(B)$ denotes the set of polynomials of degree $\le N = N_\Psi$ in an appropriate basis, with N large enough.

When $\Psi = \Phi_1$, this space is usually referred to as the space $LMOA$ of functions of logarithmic mean oscillation. Duality has been proven in $\mathbb{R}^n$ by Janson [J]. Viviani proves it as a consequence of atomic decomposition. In the context of holomorphic functions, this is also a consequence of atomic decomposition and continuity of the Szegö projection.

Our method allows us to characterize $BMOA(\rho_\Phi)$ as the class of symbols of Hankel operators which map $\mathcal{H}^\Phi$ into $\mathcal{H}^1_{\mathrm{weak}}$.

As pointed out before, we have chosen to allow the lower type of Φ to be arbitrarily small, and not only the upper type to be larger than $2n/(2n+1)$ (for the unit ball of $\mathbb{C}^n$, or for a strictly pseudoconvex domain; for a general convex domain of finite type, the critical index is different). This raises many technical difficulties: for instance, it is not sufficient to deal with atoms with mean 0 and we need extra moment conditions; in parallel, one has to deal with polynomials of positive degree to define the dual space BMO, and not only with constants.

Here and in what follows, $\mathcal{H}(\mathbb{B}^n)$ denotes the space of holomorphic functions in $\mathbb{B}^n$. For two positive functions f and g, we use the notation $f \lesssim g$ when there is some constant c such that $f(w) \leq c\, g(w)$, where w stands for the parameters that we are interested in (typically, the constant c will only depend on the geometry of the domain under consideration). We define the symbols $\gtrsim$ and $\simeq$ analogously.

1. Statements of results

1.1. Growth functions and Orlicz spaces. Let us give a precise definition for the *growth functions* which are used in the definition of Hardy–Orlicz spaces (see also [V]).

DEFINITION 1.1. Let $0 < p \leq 1$. A function Φ is called a *growth function of order* p if it has the following properties:

(G1) Φ is a homeomorphism of $[0,\infty)$ such that $\Phi(0) = 0$. Moreover, the function $t \mapsto \Phi(t)/t$ is non-increasing.

(G2) Φ is of lower type p, that is, there exists a constant $c > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq ct^p\Phi(s). \tag{2}$$

We will also say that Φ is a *growth function* whenever it is a growth function of some order $p < 1$. Two growth functions Φ_1 and Φ_2 define the same Hardy–Orlicz spaces when $\Phi_1 \simeq \Phi_2$. In particular, the growth function Φ of order p is equivalent to the function $\int_0^t (\Phi(s)/s)\, ds$, which is also a growth function of the same order and has the following additional property.

(G3) Φ is concave. In particular, it is subadditive.

Our typical example $\Phi_p(t) = t^p/\log(e+t)^p$ satisfies (G1) and (G2) for $p \leq 1$. The same is valid for the function $\Phi_{p,\alpha}(t) = t^p(\log(C+t))^{\alpha p}$, provided that C is large enough, for $p < 1$ and any α, or for $p = 1$ and $\alpha < 0$. In the following, we still denote by Φ_p (or $\Phi_{p,\alpha}$) the modified but equivalent functions which satisfy (G3) as well.

REMARK 1.2. If Φ and Ψ are two growth functions, then so is $\Phi \circ \Psi$.

Observe also that Φ is doubling: more precisely,

$$\Phi(2t) \leq 2\Phi(t), \tag{3}$$

a property that will be widely used.

For $(X, d\mu)$ a measure space, we denote by L^{Φ} the corresponding Orlicz space, that is, the space of functions f such that

$$\|f\|_{L^{\Phi}} := \int_X \Phi(|f|)\, d\mu < \infty.$$

The quantity $\|\cdot\|_{L^{\Phi}}$ is subadditive, but is not homogeneous. One may prefer the Luxemburg quasi-norm, which is homogeneous but not subadditive. It is defined as

$$\|f\|_{L^{\Phi}}^{\mathrm{lux}} = \inf\Big\{\lambda > 0 : \int_X \Phi(|f(x)|/\lambda)\, d\mu(x) \leq 1\Big\}.$$

It is easily seen that, when Φ is of lower type p,

$$\|f\|_{L^{\Phi}}^{\mathrm{lux}} \lesssim \max\{\|f\|_{L^{\Phi}}, \|f\|_{L^{\Phi}}^{1/p}\},$$

while

$$\|f\|_{L^{\Phi}} \lesssim \max\{\|f\|_{L^{\Phi}}^{\mathrm{lux}}, (\|f\|_{L^{\Phi}}^{\mathrm{lux}})^p\}.$$

Endowed with the distance $\|f - g\|_{L^{\Phi}}$, L^{Φ} is a metric space. When T is a continuous linear operator from L^{Φ} into a Banach space $\mathcal{B}$, there exists a constant C such that

$$\|Tf\|_{\mathcal{B}} \leq C\|f\|_{L^{\Phi}}^{\mathrm{lux}}.$$

Conversely, a bounded operator is continuous.

1.2. Adapted geometry on the unit ball. Let us recall here some geometric notions (see [R]) that will be necessary for the description of spaces of holomorphic functions.

For $\zeta \in \mathbb{S}^n$ and $w \in \overline{\mathbb{B}^n}$, let

$$d(\zeta, w) := |1 - \langle \zeta, w\rangle|.$$

We recall that, when restricted to $\mathbb{S}^n \times \mathbb{S}^n$, this is a quasi-distance. For $\zeta_0 \in \mathbb{S}^n$ and $0 < r < 1$, we denote by $B(\zeta_0, r)$ the ball on $\mathbb{S}^n$ of center ζ_0 and radius r for the distance d. Recall that $\sigma(B(\zeta_0, r)) \simeq r^n$. In particular,

$$\sigma(B(\zeta_0, \lambda r)) \simeq \lambda^n \sigma(B(\zeta_0, r)), \tag{4}$$

with constants that do not depend on ζ_0 and r.

For each $\zeta_0 \in \mathbb{S}^n$, we choose an orthonormal basis $v^{(1)}, \ldots, v^{(n)}$ in $\mathbb{C}^n$ such that $v^{(1)}$ is the outward normal vector to the unit sphere. In particular, we can choose the canonical basis for the point with coordinates $(1, 0, \ldots, 0)$. Let $x_j + iy_j$ be the coordinates of $z - \zeta_0$ in this basis. Then $y_1, \ldots, y_n, x_2, \ldots, x_n$ can be used as coordinates on $\mathbb{S}^n$ in a neighborhood of ζ_0, say in the ball $B(\zeta_0, \delta)$. We can take δ uniformly for all points ζ_0. We will speak of the *special coordinates related to* ζ_0.

Given $\zeta \in \mathbb{S}^n$ we define the *admissible approach region* $\mathcal{A}_\alpha(\zeta)$ by

$$\mathcal{A}_\alpha(\zeta) = \{z = rw \in \mathbb{B}^n : d(\zeta, w) = |1 - \langle \zeta, w\rangle| < \alpha(1-r)\}.$$

We then define the *admissible maximal function* of a holomorphic function f by

$$\mathcal{M}_\alpha(f)(\zeta) = \sup_{z \in \mathcal{A}_\alpha(\zeta)} |f(z)|. \tag{5}$$

1.3. Hardy–Orlicz spaces. Hardy–Orlicz spaces $\mathcal{H}^\Phi(\mathbb{B}^n)$ have been defined in (1). We define on $\mathcal{H}^\Phi(\mathbb{B}^n)$ the (quasi-)norms by

$$\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} := \sup_{0<r<1} \int_{\mathbb{S}^n} \Phi(|f(rw)|)\, d\sigma(w),$$

$$\|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi(\mathbb{B}^n)} = \inf\left\{\lambda > 0 : \sup_{0<r<1} \int_{\mathbb{S}^n} \Phi\left(\frac{|f(rw)|}{\lambda}\right) d\sigma(w) \le 1\right\},$$

which are finite for $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$ and define the same topology.

The assumptions on the growth function Φ give the inclusions

$$\mathcal{H}^1(\mathbb{B}^n) \subset \mathcal{H}^\Phi(\mathbb{B}^n) \subset \mathcal{H}^p(\mathbb{B}^n). \tag{6}$$

A basic property of Hardy spaces is that they can be equivalently defined in terms of maximal functions, which generalizes to our setting.

THEOREM 1.3. *Let $\alpha > 0$. There exists a constant $C > 0$ such that, for any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$,*

$$\|\Phi(\mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \le C\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{7}$$

So the two quantities are equivalent.

Next we define the real Hardy–Orlicz spaces on the unit sphere in terms of atoms.

For $\zeta_0 \in \mathbb{S}^n$, we denote by $\mathcal{P}_N(\zeta_0)$ the set of functions on $B(\zeta_0, \delta)$ which are polynomials of degree $\le N$ in the $2n-1$ special coordinates related to ζ_0. Notice that $\mathcal{P}_N(\zeta_0)$ does not depend on the choice of $v^{(2)}, \ldots, v^{(n)}$.

DEFINITION 1.4. A square integrable function a on $\mathbb{S}^n$ is called an *atom* of order $N \in \mathbb{N}$ associated to the ball $B := B(\zeta_0, r_0)$, for some $\zeta_0 \in \mathbb{S}^n$, if the following conditions are satisfied:

(A1) $\operatorname{supp} a \subseteq B$;
(A2) when $r_0 < \delta$, $\int_{\mathbb{S}^n} a(\zeta)P(\zeta)\,d\sigma(\zeta) = 0$ for every $P \in \mathcal{P}_N(\zeta_0)$.

The second condition is also called the *moment condition*.

We can now define the real Hardy–Orlicz spaces. Recall that the term "real" is related to the fact that the definition makes sense for real functions, and does not require any assumption of holomorphy. Here we consider complex-valued functions, since in particular we are interested in the fact that these spaces contain boundary values (in the distribution sense) of holomorphic functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$.

Definition 1.5. The *real Hardy–Orlicz space* $H^{\Phi}(\mathbb{S}^n)$ is the space of distributions f on $\mathbb{S}^n$ which can be written as the limit, in the distribution sense, of series

$$f = \sum_j a_j, \qquad \sum_j \sigma(B_j)\Phi(\|a_j\|_2\sigma(B_j)^{-1/2}) < \infty, \tag{8}$$

where the a_j's are atoms of order N, associated to the balls B_j. Here N is an integer chosen so that $N > N_p := 2n(1/p - 1) - 1$.

The (quasi-)norm on $H^{\Phi}(\mathbb{S}^n)$ is defined by

$$\|f\|_{H^{\Phi}} = \inf\Big\{\sum_j \sigma(B_j)\Phi(\|a_j\|_2\sigma(B_j)^{-1/2}) : f = \sum_j a_j\Big\}. \tag{9}$$

It is also subadditive. In particular, with the distance between f and g given by $\|f-g\|_{H^{\Phi}}$, $H^{\Phi}(\mathbb{S}^n)$ is a complete metric space. It is easy to verify that the series in (8) converges in the *metric*. Observe that convergence in $H^{\Phi}(\mathbb{S}^n)$ implies convergence in the sense of distributions.

We will see that the Szegö projection P_S is a bounded operator from $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ into itself.

Remark 1.6. The condition on N guarantees that the Szegö projection of the atom a (or its maximal function) is well defined with L^{Φ} norm uniformly bounded in terms of $\Phi(\|a\|_2\sigma(B)^{-1/2})\sigma(B)$. It follows from the theorems below that the space $H^{\Phi}(\mathbb{S}^n)$ does not depend on $N > N_p$.

We have the following atomic decomposition.

Theorem 1.7. *Let $N \in \mathbb{N}$ be larger than N_p. Given any $f \in \mathcal{H}^{\Phi}(\mathbb{B}^n)$ there exist atoms a_j of order N such that $\sum_{j=0}^{\infty} a_j \in H^{\Phi}(\mathbb{S}^n)$ and*

$$f = P_S\Big(\sum_{j=0}^{\infty} a_j\Big) = \sum_{j=0}^{\infty} P_S(a_j).$$

Moreover,

$$\sum_{j=0}^{\infty} \sigma(B_j)\Phi(\|a_j\|_2\sigma(B_j)^{-1/2}) \simeq \|f\|_{\mathcal{H}^{\Phi}(\mathbb{B}^n)}.$$

As in the atomic decomposition of Hardy spaces on $\mathbb{R}^n$, the order of the moment conditions on the atoms can be chosen arbitrarily large. Having optimal values has no importance later on, which allows an easy adaptation of the proofs to a class of domains including convex domains of finite type and strictly pseudoconvex domains, for which the optimal values of N_p are different. The fact that atoms may satisfy moment conditions up to an arbitrarily large order will play a crucial role for the factorization.

Szegö projections of atoms are best described in terms of molecules, which we introduce now.

Definition 1.8. A holomorphic function $A \in \mathcal{H}^2(\mathbb{B}^n)$ is called a *molecule* of order L, associated to the ball $B := B(z_0, r_0) \subset \mathbb{S}^n$, if it satisfies

$$(10)\quad \|A\|_{\mathrm{mol}(B,L)} := \left(\sup_{r<1} \int_{\mathbb{S}^n} \left(1 + \frac{d(z_0, r\xi)^{L+n}}{r_0^{L+n}}\right) |A(r\xi)|^2 \frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2} < \infty.$$

Proposition 1.9. *For an atom a of order N associated to a ball $B \subset \mathbb{S}^n$, its Szegö projection $P_S(a)$ is a molecule associated to $\tilde{B}$ of double radius, of any order $L < N+1$. It satisfies*

$$\|A\|_{\mathrm{mol}(B,L)} \lesssim \|a\|_2 \sigma(B)^{-1/2}.$$

Proposition 1.10. *Any molecule A of order L associated to a ball B, such that $L > L_p := 2n(1/p - 1)$, belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ with*

$$\|A\|_{\mathcal{H}^\Phi} \lesssim \Phi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B).$$

The atomic decomposition and the previous propositions have, as corollaries, the molecular decomposition of functions in $\mathcal{H}^\Phi(\mathbb{B}^n)$, the continuity of the Szegö projection, and the identification of the dual space. Let us state first the molecular decomposition.

Theorem 1.11. *For any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$, there exist molecules A_j of order $L > L_p$, associated to balls B_j, such that f may be written as*

$$f = \sum_j A_j$$

with $\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} \simeq \sum_j \Phi(\|A_j\|_{\mathrm{mol}(B_j,L)})\sigma(B_j)$.

The continuity of the Szegö projection is also a direct consequence of the atomic decomposition and the fact that an atom is projected to a molecule.

Theorem 1.12. *The Szegö projection extends to a continuous operator*

$$P_S : H^\Phi(\mathbb{S}^n) \to \mathcal{H}^\Phi(\mathbb{B}^n).$$

Before giving the duality statement, let us define generalized $BMO(\varrho)$-spaces as follows. We assume that ϱ is a continuous increasing function from

$[0,1]$ onto $[0,1]$, which is of *upper type* α, that is,

$$\varrho(st) \le s^{\alpha}\varrho(t) \tag{11}$$

for $t \in [0,1]$ and $s > 1$ with $st \le 1$. We then define

$$BMO(\varrho) = \left\{ f \in L^2(\mathbb{S}^n) : \sup_B \inf_{P \in \mathcal{P}_N(B)} \frac{1}{\varrho(\sigma(B))^2 \sigma(B)} \int_B |f - P|^2 \, d\sigma < \infty \right\}.$$

Here, for B a ball of center ζ_B, assumed to be of radius $r < \delta$, we denote by $\mathcal{P}_N(B)$ the set $\mathcal{P}_N(\zeta_B)$. The integer N is taken large enough, say $N > 2n\alpha - 1$. Before going on, let us make some remarks.

Remark 1.13. The definition does not depend on $N > 2n\alpha - 1$. We will not prove this and refer to [BPS2] for a proof for $\alpha < 1/2$. It is a consequence of duality and atomic decomposition.

Remark 1.14. One may prove that, as in the Euclidean case (see [J]), if ϱ is of upper type less than $1/2n$ and satisfies the Dini condition

$$\int_r^1 \frac{\varrho(s)}{s^2} \, ds \lesssim \varrho(r),$$

then $BMO(\varrho)$ coincides with the Lipschitz space $\Lambda(\varrho)$, defined as the space of bounded functions such that

$$|f(z) - f(\zeta)| \le \varrho(d(z,\zeta)^n).$$

Spaces $BMO(\varrho)$ have been introduced by Janson [J] in $\mathbb{R}^n$, and proved to be the dual spaces of maximal Hardy–Orlicz spaces related to the growth function Φ when $\varrho(t) = \varrho_\Phi(t) := 1/t\Phi^{-1}(1/t)$. With our definition of $H^\Phi(\mathbb{S}^n)$ in terms of atoms, this duality is straightforward, as pointed out by Viviani [V]. For holomorphic Hardy–Orlicz spaces, we also have

Theorem 1.15. *The dual space of* $\mathcal{H}^\Phi(\mathbb{B}^n)$ *is*

$$BMOA(\varrho) = \left\{ f \in \mathcal{H}^2(\mathbb{B}^n) : \sup_B \inf_{P \in \mathcal{P}_N(B)} \frac{1}{\varrho(\sigma(B))^2 \sigma(B)} \int_B |f - P|^2 \, d\sigma < \infty \right\}$$

where $\varrho(t) = \varrho_\Phi(t) := 1/t\Phi^{-1}(1/t)$. *The duality is given by the limit as* r *tends to* 1, $r < 1$, *of scalar products on spheres of radius* r.

In other terms, $BMOA(\varrho)$ is the space of holomorphic functions in the Hardy space $\mathcal{H}^2(\mathbb{B}^n)$ whose boundary values belong to $BMO(\varrho)$.

1.4. Products of functions and Hankel operators. We now have all prerequisites to study the product of a function $h \in \mathcal{H}^\Phi(\mathbb{B}^n)$ with a function $b \in BMOA(\mathbb{B}^n)$. Observe that, using (6), we already know that the product is well defined as the product of an $\mathcal{H}^p(\mathbb{B}^n)$-function and an $\mathcal{H}^s(\mathbb{B}^n)$-function for all $1 < s < \infty$. So it is in $\mathcal{H}^q(\mathbb{B}^n)$ for $q < p$. We want to replace this inclusion by a sharp statement.

Proposition 1.16. *The product maps continuously $\mathcal{H}^{\Phi}(\mathbb{B}^n)\times BMOA(\mathbb{B}^n)$ into $\mathcal{H}^{\Psi}(\mathbb{B}^n)$, where $\Psi(t)=\Phi(t/\log(e+t))$.*

Proof. We know that Ψ is also a growth function by Remark 1.2. We prove more: using the John–Nirenberg inequality, we know that a function b in BMO is also in the exponential class. More precisely, we only use the fact that $b(r\cdot)$ is uniformly in the exponential class, and prove that, for such a function b and for a function $h\in\mathcal{H}^{\Phi}(\mathbb{B}^n)$, the product bh is continuously embedded in $\mathcal{H}^{\Psi}(\mathbb{B}^n)$. We start from the following elementary inequality (see [BIJZ]): for any $u,v>0$,

$$\frac{uv}{\log(e+uv)}\leq u+e^v-1.$$

It follows that

$$\Psi(uv)\lesssim\Phi(u+e^v-1)\lesssim\Phi(u)+e^v-1.$$

When u and v are replaced by measurable positive functions on the measure space $(X,d\mu)$, we have, by homogeneity of the Luxemburg norms,

$$\|fg\|^{\mathrm{lux}}_{L^{\Psi}}\lesssim\|f\|^{\mathrm{lux}}_{L^{\Phi}}\|g\|^{\mathrm{lux}}_{\exp L}.$$

We refer to [VT] for a more general Hölder inequality on Orlicz spaces.

Let us come back to Hardy spaces. Applying this inequality on each sphere of radius less than 1, we conclude that

$$\|fg\|^{\mathrm{lux}}_{\mathcal{H}^{\Psi}}\lesssim\|f\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi}}\|g\|^{\mathrm{lux}}_{\exp L}\lesssim\|f\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi}}\|g\|_{BMOA}. \quad\blacksquare \tag{12}$$

We are going to prove converse statements.

Theorem 1.17. *Let A be a molecule associated to the ball B. Then A may be written as fg, where f is a molecule and g is in $BMOA(\mathbb{B}^n)$. Moreover, f and g may be chosen such that*

$$\|g\|_{BMOA(\mathbb{B}^n)}\lesssim 1,\qquad \|f\|_{\mathrm{mol}(B,L')}\lesssim\frac{\|A\|_{\mathrm{mol}(B,L)}}{\log(e+\sigma(B)^{-1})}$$

when $L'<L$. In particular, if $\Psi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B)\leq 1$, then

$$\Phi(\|f\|_{\mathrm{mol}(B,L')})\lesssim\Psi(\|A\|_{\mathrm{mol}(B,L)}).$$

Theorem 1.18. *Given any $f\in\mathcal{H}^{\Psi}(\mathbb{B}^n)$ there exist $f_j\in\mathcal{H}^{\Phi}(\mathbb{B}^n)$, $g_j\in BMOA(\mathbb{B}^n)$, $j\in\mathbb{N}$, with the norm of g_j bounded by 1, such that*

$$f=\sum_{j=0}^{\infty}f_jg_j.$$

Moreover, we can take for f_j a molecule and, if $\|f\|_{\mathcal{H}^{\Psi}}\leq 1$, then

$$\sum_j\Phi(\|f_j\|_{\mathrm{mol}(B_j,L)})\sigma(B_j)\lesssim\|f\|_{\mathcal{H}^{\Psi}}.$$

In particular,

$$\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^\Phi} \|g_j\|_{BMOA} \lesssim \|f\|_{\mathcal{H}^\Psi}.$$

Notice that the quantity $\sum_j \Phi(\|f_j\|_{\mathrm{mol}(B_j,L)})\sigma(B_j)$ is not equivalent to the norm of f. When the dimension is equal to 1, one can proceed as in [BIJZ], and prove that there is an exact factorization.

As a corollary, we obtain the following characterization of bounded Hankel operators. Recall that, for $b \in \mathcal{H}^2(\mathbb{B}^n)$, the (small) Hankel operator h_b of symbol b is given, for functions $f \in \mathcal{H}^2(\mathbb{B}^n)$, by $h_b(f) = P_S(b\bar{f})$.

COROLLARY 1.19. *A Hankel operator h_b extends to a continuous operator from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$ if and only if $b \in (\mathcal{H}^\Psi(\mathbb{B}^n))' = BMOA(\varrho_\Psi)$.*

All these results may be extended to the more general setting of strictly pseudoconvex domains or of convex domains of finite type in $\mathbb{C}^n$. We give a sketch of the proofs in Section 6.

2. Maximal characterizations of Hardy–Orlicz spaces. Let us prove the equivalent characterization of $\mathcal{H}^\Phi$-spaces, given in Theorem 1.3. In order to adapt the proofs given for usual Hardy spaces, we need the lemma below, where $\mathcal{M}^{HL}$ denotes the Hardy–Littlewood maximal operator related to the distance on the unit sphere. In fact, the statement is valid in the general context of spaces of homogeneous type. In particular, we will also use it for the maximal operator on the sphere related to the Euclidean distance.

LEMMA 2.1. *Let Φ be a growth function of order p, and $\beta < p$. There exists a constant $C > 0$ such that, for any measurable function f,*

$$\int_{\mathbb{S}^n} \Phi(\mathcal{M}^{HL}(|f|^\beta)^{1/\beta})\, d\sigma \le C \int_{\mathbb{S}^n} \Phi(|f|)\, d\sigma.$$

Proof. Set $g := |f|^\beta$. We only use the fact that

$$t\sigma(\mathcal{M}^{HL}(g) \ge t) \lesssim \int_{\{g \ge t/2\}} g\, d\sigma,$$

which is a consequence of the weak $(1,1)$ boundedness of $\mathcal{M}^{HL}$.

Define $\Psi(t) := \Phi(t^{1/\beta})$, which is a function of lower type $p/\beta > 1$. In particular,

$$\int_0^s \frac{\Psi(t)}{t^2}\, dt = s^{-1} \int_0^1 \frac{\Psi(st)}{t^2}\, dt \lesssim \frac{\Psi(s)}{s} \tag{13}$$

since $\int_0^1 t^{p/\beta-2}\, dt$ is finite. It follows, by splitting this integral into intervals

$(2^k, 2^{k+1})$, that

$$\sum_{k;\, s>2^k} 2^{-k}\Psi(2^k) \lesssim \frac{\Psi(s)}{s}. \tag{14}$$

Now, we have to estimate

$$\begin{aligned}\int_{\mathbb{S}^n} \Psi(\mathcal{M}^{HL}(g))\, d\sigma &\le \sum_k \Psi(2^k)\sigma(\mathcal{M}^{HL}(g) \ge 2^{k-1}) \\ &\lesssim \sum_k 2^{-k}\Psi(2^k) \int_{\{g\ge 2^{k-2}\}} g\, d\sigma.\end{aligned}$$

Interchanging the integral and the sum and using (14), we conclude that the left hand side is bounded by $C\int_{\mathbb{S}^n} \Psi(g)\, d\sigma$, which we wanted to prove. ∎

Proof of Theorem 1.3. We proceed in two steps, as is classical. Let

$$\mathcal{M}_0(f)(\zeta) = \sup_{0<r<1} |f(r\zeta)|$$

be the radial maximal function. We first prove that

$$\|\Phi(\mathcal{M}_0(f))\|_{L^1(\mathbb{S}^n)} \le C\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{15}$$

Let $\beta < p$, Ψ and $g = |f|^\beta$ be as before. The function g is subharmonic, and satisfies the condition

$$\sup_{0<r<1} \int_{\mathbb{S}^n} \Psi(g(r\zeta))\, d\sigma(\zeta) < \infty.$$

We claim that there exists some constant C, independent of g, such that

$$\int_{\mathbb{S}^n} \Psi(\sup_{0<r<1} g(r\zeta))\, d\sigma(\zeta) \le C \sup_{0<r<1} \int_{\mathbb{S}^n} \Psi(g(r\zeta))\, d\sigma(\zeta), \tag{16}$$

which will immediately imply (15). The proof of (16) follows the same lines as in the unit disc. Assume first that g extends to a continuous function on the closed ball and let $\tilde{g}$ be the function on the unit sphere that coincides with this extension. With this assumption, the right hand side is the integral of $\Psi(\tilde{g})$. Then it follows from the maximum principle that $g \le G$, where G is the Poisson integral of $\tilde{g}$. Moreover, we know that $\sup_{0<r<1} g(r\zeta)$ is bounded by the Hardy–Littlewood maximal function (for the Euclidean metric on the unit sphere) of $\tilde{g}$. We deduce the inequality (16) by using the previous lemma, or its proof, in the context of this Hardy–Littlewood maximal function. To handle general g, it is sufficient to see that inequality (15) is valid for g once it is valid for all $g(r\,\cdot)$ with $0 < r < 1$.

Let $\tilde{f}$ be the a.e. boundary values of f, which we know to exist since f belongs to $\mathcal{H}^p(\mathbb{B}^n)$ by (6). Observe that once we have done this first step, we also know, using Fatou's lemma, that $\|\Phi(|\tilde{f}|)\|_{L^1(\mathbb{S}^n)} \le \|f\|_{\mathcal{H}^\Phi}$.

To deal with $\mathcal{M}_\alpha$, we use the known inequality (see for instance [St])

$$\mathcal{M}_\alpha(f)^\beta \le C_\alpha \mathcal{M}^{HL}(\mathcal{M}_0(f)^\beta). \tag{17}$$

We then use Lemma 2.1 to conclude the proof of Theorem 1.3. ∎

We need stronger characterizations of $\mathcal{H}^\Phi(\mathbb{B}^n)$ to obtain atomic decomposition. First, looking at the proof of (17), one observes that the constant C_α has a polynomial behavior when α tends to ∞. In the Euclidean case, details are given in [St]. This means in particular, using the fact that Φ is doubling, that for some large n_0 and all $\alpha > 0$, we have the inequality

$$\|\Phi(\mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \le C(1+\alpha)^{n_0}\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{18}$$

Let us now consider the tangential variant of admissible maximal operators, defined by

$$\mathcal{N}_M(f)(\zeta) = \sup_{rw\in\mathbb{B}^n}\left(\frac{1-r}{(1-r)+d(\zeta,w)}\right)^M |f(rw)|. \tag{19}$$

Here $d(\zeta,w)$ denotes the pseudodistance on $\mathbb{S}^n$, given as before by $d(\zeta,w) := |1-\langle\zeta,w\rangle|$. We claim that

$$\|\Phi(\mathcal{N}_M(f))\|_{L^1(\mathbb{S}^n)} \le C\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{20}$$

Using the definition, we have

$$\begin{aligned}\mathcal{N}_M(f)(\zeta) &= \sup_{k\in\mathbb{N}}\sup_{rw\in\mathcal{A}_{2^k}(\zeta)}\left(\frac{1-r}{(1-r)+d(\zeta,w)}\right)^M |f(rw)|\\ &\lesssim \sup_{k\in\mathbb{N}} 2^{-kM}\mathcal{M}_{2^k}f(\zeta).\end{aligned}$$

It then follows that

$$\begin{aligned}\|\Phi(\mathcal{N}_M(f))\|_{L^1(\mathbb{S}^n)} &\le \sum_{k\in\mathbb{N}}\|\Phi(2^{-kM}\mathcal{M}_{2^k}f)\|_{L^1(\mathbb{S}^n)}\\ &\le \sum_{k\in\mathbb{N}} 2^{-kMp}\|\Phi(\mathcal{M}_{2^k}f)\|_{L^1(\mathbb{S}^n)}.\end{aligned}$$

For $Mp > n_0$ we can conclude the proof by using (18).

Let us now introduce the grand maximal function. Firstly, we define the set of *smooth bump functions at* ζ, which we denote by $\mathcal{K}_\alpha^N(\zeta)$, as the set of smooth functions φ supported in $B(\zeta_0, r_0)$ for some $\zeta_0 \in \mathcal{A}_\alpha(\zeta)$ and normalized in the following way. In the neighborhood of ζ_0, when we use special coordinates related to ζ_0, the unit sphere coincides with the graph $\Re w_1 = h(\Im w_1, w')$, with $w' = (w_2,\dots,w_n)$ and h a smooth function. We write $w_j = x_j + y_j$, and consider all derivatives $D^{(k,l)}\varphi$, where $D^{(k,l)}$ consists of k derivatives in x' or y', and l derivatives in y_1. We assume that bump functions $\varphi \in \mathcal{K}_\alpha^N(\zeta)$ satisfy the inequality

$$\sum_{k+l\le N,} \|D^{(k,l)}\varphi\|_{L^\infty(B(\zeta_0,r_0))} r_0^{k/2+l} \le \sigma(B)^{-1}.$$

The *grand maximal function* is defined as

$$K_{\alpha,N}(f)(\zeta) = \sup_{\varphi\in\mathcal{K}_\alpha^N(\zeta)} \Big| \lim_{r\to 1} \int_{S^n} f(r\zeta)\varphi(\zeta)\, d\sigma(\zeta)\Big|. \tag{21}$$

The limit exists for $f \in \mathcal{H}^\Phi(\mathbb{B}^n) \subset \mathcal{H}^p(\mathbb{B}^n)$ since holomorphic functions in Hardy spaces have distributional boundary values.

We use the following inequality (see [GP], and [St] for the Euclidean case).

LEMMA 2.2. *With the definitions above, there exist $c = c(\mathbb{B}^n)$ and $\tilde{N} = \tilde{N}(\alpha, N)$ such that*

$$K_{\alpha,N} f(\zeta) \lesssim \mathcal{M}_{c\alpha}(f)(\zeta) + \mathcal{N}_{\tilde{N}}(f)(\zeta).$$

We now turn to atomic decomposition. We first prove in the next section that holomorphic extensions of $H^\Phi(\mathbb{S}^n)$-functions are in the Hardy–Orlicz space.

3. Atoms and molecules: proof of Theorem 1.12. We first consider the Szegö projection of atoms and prove the following lemma.

LEMMA 3.1. *Let a be an atom of order N associated to the ball $B = B(\zeta_0, r_0)$, and let $A = P_S(a)$. Then*

$$\sup_{0<r<1} \int_{B(\zeta_0,2r_0)} \Phi(|A(rw)|)\, \frac{d\sigma(w)}{\sigma(B)} \lesssim \Phi(\|a\|_2 \sigma(B)^{-1/2}), \tag{22}$$

$$|A(r\zeta)| \lesssim \left(\frac{r_0}{d(r\zeta,\zeta_0)}\right)^{n+(N+1)/2} \|a\|_2\sigma(B)^{-1/2} \quad \textit{for } d(\zeta,\zeta_0) \ge 2r_0. \tag{23}$$

Proof. Let us prove (22). We have assumed that Φ is concave. In particular, if $d\mu$ is a probability measure and f a positive function on the measure space $(X, d\mu)$, then we have the Jensen inequality

$$\int_X \Phi(f)\, d\mu \le \Phi\Big(\int_X f\, d\mu\Big) \le \Phi(\|f\|_{L^2(X,d\mu)}). \tag{24}$$

If we use it for the measure $d\sigma$ on $B(z_0, 2r_0)$ after normalization, we find that

$$\sup_{0<r<1} \int_{B(\zeta_0,2r_0)} \Phi(|A(rw)|)\, \frac{d\sigma(w)}{\sigma(B)} \lesssim \Phi\Big(\frac{\|A\|_{\mathcal{H}^2}}{\sigma(B)^{1/2}}\Big). \tag{25}$$

Since the Szegö projection is bounded in L^2, we have the inequality

$$\|A\|_{\mathcal{H}^2} \le \|a\|_{L^2},$$

and this proves (22).

The inequality (23) is classical and used for classical Hardy spaces. It is a consequence of the estimates of the Szëgo kernel, which are explicit for the unit ball. Without loss of generality we can assume that $\zeta_0 = (1, 0, \dots, 0)$, so that the coordinates related to ζ_0 may be taken as the ordinary ones. Otherwise we use the action of the unitary group. In the neighborhood of ζ_0, the unit sphere coincides with the graph $\Re w_1 = h(\Im w_1, w')$, with $w' = (w_2, \dots, w_n)$. We recall that $S(\zeta, w) = c_n(1 - \langle\zeta, w\rangle)^{-n}$. In the following, we are interested in estimates on $D_w^{(k,l)} S(r\zeta, (h(t_1, s' + it') + it_1, w'))$, where $D^{(k,l)}$ consists of k derivatives in s' or t', and l derivatives in t_1. It follows from elementary computations that

$$|D_w^{(k,l)} S(r\zeta, (h(t_1, s' + it') + it_1, w'))| \le C(|\zeta'|^k + |w'|^k)|1 - r\langle\zeta, w\rangle|^{-(n+k+l)}.$$

For $d(w, \zeta_0) < r_0$ and $\zeta \notin B(\zeta_0, 2r_0)$, we know that $|1 - r\langle\zeta, w\rangle| \gtrsim |1 - \langle\zeta, \zeta_0\rangle| \gtrsim r_0$. In particular, $|w'| \lesssim |1 - r\langle\zeta, \zeta_0\rangle|^{1/2}$, and the same for $|\zeta'|$. So,

$$|D_w^{(k,l)} S(r\zeta, (h(t_1, s' + it') + it_1, w'))| \le C|1 - r\langle\zeta, \zeta_0\rangle|^{-(n+k/2+l)}. \tag{26}$$

We use the vanishing moment condition, in the computation of

$$P_S a(r\zeta) = \int S(r\zeta, w) a(w)\, d\sigma(w),$$

to replace $S(r\zeta, \cdot)$ by $S(r\zeta, \cdot) - P$, where P is its Taylor polynomial of order N. By Taylor's formula, the rest may be bounded by the sum, for $k + l = N + 1$, of the quantities $|t_1|^l |w'|^k |1 - \langle\zeta, \zeta_0\rangle|^{-(n+k/2+l)}$. Using the fact that $|t_1|^l |w'|^k \lesssim r_0^{k/2+l}$, we have

$$|S(r\zeta, w) - P(w)| \le C \frac{r_0^{(N+1)/2}}{d(r\zeta, \zeta_0)^{n+(N+1)/2}}.$$

This gives the result, as $\sigma(B) \lesssim r_0^n$. ∎

Proof of Proposition 1.9. The fact that $P_S(a)$ is a molecule is classical. We give the proof for completeness. Coming back to the definition of $\|P_S(a)\|^2_{\mathrm{mol}(B,L)}$ given in (10), we split the integral involved into two pieces. We already know that the integral on $B(\zeta_0, 2r_0)$ satisfies the right estimate. So it is sufficient to show that

$$\int_{\mathbb{S}^n \setminus B(\zeta_0, 2r_0)} \left(\frac{d(r\xi, \zeta_0)}{r_0}\right)^{L+n} |P_S a(r\xi)|^2 \frac{d\sigma(\xi)}{\sigma(B)} \le C\|a\|_2^2,$$

with C independent of $r < 1$. By (23), this is a consequence of the estimate

$$\int_{\mathbb{S}^n \setminus B(\zeta_0, 2r_0)} \left(\frac{r_0}{d(\xi, \zeta_0)}\right)^M \frac{d\sigma(\xi)}{\sigma(B)} \le C \tag{27}$$

for some constant C that does not depend on ζ_0 and r_0, when $M > n$ (see for instance [R]). ∎

Proof of Proposition 1.10. Let A be a molecule of order L associated to $B := B(z,r)$. We want to prove that A belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ for L large enough, with the estimate

$$\|A\|_{\mathcal{H}^\Phi} \lesssim \Phi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B).$$

Let us denote $B_k := B(z, 2^k r)$. It is sufficient to prove that, for g a positive function on the unit sphere,

$$\int_{\mathbb{S}^n} \Phi(g)\,\frac{d\sigma}{\sigma(B)} \lesssim \Phi\biggl(\biggl(\int_{\mathbb{S}^n} \Bigl(\frac{d(z,\xi)}{r}\Bigr)^{L+n} g(\xi)^2\,\frac{d\sigma(\xi)}{\sigma(B)}\biggr)^{1/2}\biggr).$$

Splitting the integral into pieces, it is sufficient to prove that

$$\int_B \Phi(g)\,\frac{d\sigma}{\sigma(B)} \lesssim \Phi\biggl(\biggl(\int_B g(\xi)^2\,\frac{d\sigma(\xi)}{\sigma(B)}\biggr)^{1/2}\biggr),$$

which is a direct consequence of the Jensen inequality (24) as before, and, for $k \geq 1$,

$$\int_{B_k\setminus B_{k-1}} \Phi(g)\,\frac{d\sigma}{\sigma(B)} \lesssim 2^{-k\varepsilon}\Phi\biggl(\biggl(2^{k(L+n)}\int_{B_k\setminus B_{k-1}} g(\xi)^2\,\frac{d\sigma(\xi)}{\sigma(B)}\biggr)^{1/2}\biggr)$$

for some $\varepsilon > 0$. To prove this last inequality, we use again the Jensen inequality (24) for the measure $d\sigma$ on $B_k \setminus B_{k-1}$, divided by its total mass $\sigma(B_k \setminus B_{k-1}) \simeq 2^{kn}\sigma(B)$. This gives

$$\int_{B_k\setminus B_{k-1}} \Phi(g)\,\frac{d\sigma}{\sigma(B)} \lesssim 2^{kn}\Phi\biggl(\biggl(2^{-kn}\int_{B_k\setminus B_{k-1}} g(\xi)^2\,\frac{d\sigma(\xi)}{\sigma(B)}\biggr)^{1/2}\biggr).$$

We conclude by using the fact that Φ is of lower type p, which yields $2^{kn}\Phi(t) \lesssim \Phi(2^{kn/p}t)$. It is sufficient to choose $L > 2n(1/p-1)$. ∎

4. Proof of the atomic decomposition Theorem 1.7. Let f be a fixed function in $\mathcal{H}^\Phi$. As noticed before, f admits boundary values defined a.e. on $\mathbb{S}^n$, which we still denote by f.

We also fix an integer N larger than N_p.

Let k_0 be the least integer such that

$$\|\Phi(K_{\alpha,M}(f) + \mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \leq 2^{k_0}. \tag{28}$$

For a positive integer k, we define

$$\mathcal{O}_k = \{z \in \mathbb{S}^n : K_{\alpha,M}f(z) + \mathcal{M}_\alpha(f)(z) > 2^{k_0+k}\}. \tag{29}$$

For each k, we then fix a Whitney covering $\{B_i^k\}$ of $\mathcal{O}_k$. As usual, one can associate to f an atomic decomposition (see [GL] for a proof for Hardy spaces in the unit ball; we also refer to [GP] for a proof in the general context considered in the last section).

Namely, there exist a function h_0 and atoms b_i^k corresponding to the Whitney covering $\{B_i^k\}$ such that the following equality holds in the distribution sense and almost everywhere:

$$f = h_0 + \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} b_i^k. \tag{30}$$

Here, h_0 is a so called "junk atom" bounded by $c2^{k_0}$ while the b_i^k's are atoms supported in the B_i^k's, bounded by $c2^{k+k_0}$, with moment conditions of order N.

Since $\|b_i^k\|_2\sigma(B_i^k)^{-1/2} \le \|b_i^k\|_\infty$, it is sufficient to prove that

$$\sum_{i,k}\sigma(B_i^k)\Phi(\|b_i^k\|_\infty) < \infty.$$

We have

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sigma(B_i^k)\Phi(\|b_i^k\|_\infty) &\le \sum_{k=0}^{\infty}\Phi(2^{k+k_0})\sigma(\mathcal{O}_k)\\ &\le c\int_1^\infty \frac{\Phi(t)}{t}\,\sigma(\{\zeta\in\mathbb{S}^n : K_\alpha^M f(\zeta)+\mathcal{M}_\alpha(f)(\zeta)\ge t\})\,dt\\ &\lesssim c\int_1^\infty \Phi'(t)\sigma(\{\zeta\in\mathbb{S}^n : K_\alpha^M f(\zeta)+\mathcal{M}_\alpha(f)(\zeta)\ge t\})\,dt\\ &\le \|\Phi(K_\alpha^M f)\|_{L^1(\mathbb{S}^n)} + \|\Phi(\mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \le c\|f\|_{\mathcal{H}^\Phi}.\end{aligned}$$

To obtain the converse inequality, we use the decomposition $f=\sum A_{j,k}$ where the $A_{j,k}=P(b_{j,k})$'s are molecules and satisfy, by Proposition 1.9.

$$\|A_{j,k}\|_{\mathcal{H}^\Phi} \le \sigma(B_i^k)\Phi(\|b_i^k\|_2\sigma(B_i^k)^{-1/2}) \le \sigma(B_i^k)\Phi(\|b_i^k\|_\infty) \ \blacksquare$$

As pointed out before, atomic decomposition allows us to obtain a lot of results such as molecular decomposition that we are going to consider now.

5. Factorization theorem and Hankel operators. Now, we prove the factorization theorem.

Proof of Theorem 1.17. Let A be a molecule associated to the ball $B = B(\zeta_0, r_0)$ with $r_0 < 1$. We write $A = fg$ with

$$g(z) := \log\biggl(\frac{4}{1-\langle z,\zeta\rangle}\biggr),$$

where $\zeta := (1-r_0)\zeta_0$. The constant 4 has been chosen in such a way that g, which is holomorphic on $\mathbb{B}^n$, does not vanish. We first observe that we have the required inequality for f, that is,

$$\|f\|_{\mathrm{mol}(B,L')} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\log(e+\sigma(B)^{-1})} \tag{31}$$

for $L' < L$. For this, we have the two inequalities

$$|g(\xi)| \gtrsim \log(e+r_0^{-1}) \simeq \log(e+\sigma(B)^{-1}), \qquad \xi \in B(\zeta_0, r_0) \subset \mathbb{S}^n, \tag{32}$$

$$|g(r\xi)| \gtrsim \log(e+\sigma(B)^{-1})\left(\frac{r_0}{d(\zeta_0, r\xi)}\right)^{\varepsilon}, \qquad \xi \notin B(\zeta_0, r_0), \tag{33}$$

for all $r < 1$ and some $\varepsilon > 0$. We have used the fact that, for $u > 1$ and $v > e$,

$$\log(uv) \leq C_\varepsilon u^\varepsilon \log v.$$

In view of (31), we can use (33) for the computation of the integral outside B. Inside B, we observe that it is sufficient to prove that

$$\|f\|_{\mathcal{H}^2} \lesssim \frac{\|A\|_{\mathcal{H}^2}}{\log(e+\sigma(B)^{-1})},$$

which uses the boundary values of f and estimate (32).

We now prove that g belongs uniformly to $BMOA(\mathbb{B}^n)$, or equivalently, that $(1-|z|^2)|\nabla g|^2 \simeq (1-|z|^2)/|1-\langle z, \zeta\rangle|^2$ is a Carleson measure with uniform bound. Let $B_\rho = B(x_0, \rho)$ be a ball on the boundary of $\mathbb{B}^n$ and $T(B_\rho)$ be the tent over this ball. We have to prove that

$$\int_{T(B_\rho)} \frac{1-|z|^2}{|1-\langle z, \zeta\rangle|^2}\, dV(z) \lesssim \sigma(B_\rho)$$

with constants that are independent of B_ρ, r and ζ_0, or, equivalently,

$$\int_0^\rho \int_{B_\rho} \frac{t}{(d(w, \zeta_0)+t)^2}\, dt\, d\sigma(w) \lesssim \sigma(B_\rho).$$

If $d(x_0, \zeta_0) \geq 2\rho$ then, for $w \in B_\rho$, we have $d(w, \zeta_0) \geq \rho$ and the denominator is bounded below by ρ, which allows us to conclude. If $d(x_0, \zeta_0) \leq 2\rho$, then B_ρ is included in $\widetilde{B}_\rho := B(\zeta_0, 3\rho)$ which has a measure comparable to B_ρ. Integrating first in t, we have to prove that

$$\int_{\widetilde{B}_\rho} \log\left(\frac{\rho}{d(\zeta_0, w)}\right) d\sigma(w) \lesssim \sigma(\widetilde{B}_\rho).$$

To prove this last inequality, we cut the ball $\widetilde{B}_\rho$ into dyadic shells. We conclude by using the inequality

$$\sum_{j>0} j\sigma(B(\zeta_0, 2^{-j}\rho)) \lesssim \sigma(B_\rho),$$

which is a consequence of the fact that

$$\sigma(B(z,2^{-j}\rho)) \lesssim 2^{-jn}\sigma(B(z,\rho)).$$

We have recalled this classical inequality in (4).

Assume now that $\Psi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B) \le 1$. We use the fact that $\log t \simeq \log \Psi(t)$ to get

$$\log(e + \|A\|_{\mathrm{mol}(B,L)}) \lesssim \log(e + \sigma(B)^{-1})$$

and (31) to conclude that

$$\Phi(\|f\|_{\mathrm{mol}(B,L')}) \lesssim \Psi(\|A\|_{\mathrm{mol}(B,L)}). \ \blacksquare$$

Observe that we have as well the following.

PROPOSITION 5.1. *There exists a constant C such that every molecule A can be written as fg with $\|g\|_{BMOA} \le 1$ and*

$$\|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi} \le C\|A\|_{\mathrm{mol}(B,L)}\sigma(B)\varrho_\Psi(\sigma(B)).$$

Proof. Indeed, by homogeneity of both sides of the last inequality, it is sufficient to prove it when the right hand side is equal to 1. Equivalently, this means that

$$\|A\|_{\mathrm{mol}(B,L)} = \Psi^{-1}(1/\sigma(B)),$$

or $\sigma(B)\Psi(\|A\|_{\mathrm{mol}(B,L)}) = 1$. So we know that we can find f and g such that $\|g\|_{BMOA} \le 1$ and

$$\|f\|_{\mathcal{H}^\Phi} \le \sigma(B)\Phi(\|f\|_{\mathrm{mol}(B,L')}) \lesssim 1.$$

This means that $\|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi} \lesssim 1$, which we wanted to prove. $\blacksquare$

Weak factorization, that is, Theorem 1.18, follows directly from Theorem 1.11 (molecular decomposition) and Theorem 1.17 (factorization of molecules), with the bound below for the quasi-norm of f in the Hardy–Orlicz space.

Proof of Corollary 1.19. Let h_b be a Hankel operator of symbol b. Let us first assume that $b \in BMOA(\varrho_\Psi)$. Then, for any g in $BMOA$,

$$\begin{aligned}|\langle h_b(f), g\rangle| &= |\langle P_S(b\bar{f}), g\rangle| = |\langle b, fg\rangle| \\ &\lesssim \|b\|_{BMOA(\varrho_\Psi)}\|fg\|^{\mathrm{lux}}_{\mathcal{H}^\Psi} \lesssim \|b\|_{BMOA(\varrho_\Psi)}\|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi}\|g\|_{BMOA}.\end{aligned}$$

It follows that h_b is bounded from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$, which we wanted to prove.

Conversely, assume that h_b is bounded from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$. To prove that $b \in BMOA(\varrho_\Psi)$, it is sufficient to show that there exists a positive constant C such that for each ball B we can find a polynomial $R \in \mathcal{P}_N(B)$ such that

$$\int_B |b - R|^2 d\sigma \le C\varrho_\Psi(\sigma(B))^2\sigma(B).$$

We take for R the orthogonal projection of b onto $\mathcal{P}_N(B)$, so that the function $a := \chi_B(b-R)$ on $\mathbb{S}^n$ is an atom. We know that $A := Pa$ is a molecule associated to B, with $\|A\|_{\mathrm{mol}(B,L)} \lesssim \|a\|_2\sigma(B)^{-1/2}$. From Proposition 5.1, we know that A may be written as fg with

$$\|g\|_{BMOA} \le 1, \qquad \|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi} \lesssim \|A\|_{\mathrm{mol}(B,L)}\sigma(B)\varrho_\Psi(\sigma(B)).$$

Next, we write

$$\int_B |b-R|^2\,d\sigma = \langle b,a\rangle = \langle b,Pa\rangle = \langle h_b(f),g\rangle,$$

so that

$$\int_B |b-R|^2\,d\sigma \le \|h_b\|\,\|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi}\|g\|_{BMOA} \lesssim \|h_b\|\,\|a\|_2\sigma(B)^{1/2}\varrho_\Psi(\sigma(B)).$$

We divide both sides by $\|a\|_2 = (\int_B |b-R|^2 d\sigma)^{1/2}$ to conclude the proof. ∎

We will give some complements to the characterization of symbols of bounded Hankel operators. If $\exp\mathcal{H}$ denotes the class of holomorphic functions f such that $f(r\,\cdot)$ is uniformly in the exponential class $\exp L$, then Proposition 1.16 is still valid with $BMOA$ replaced by $\exp\mathcal{H}$. Let us point out that $\exp\mathcal{H}$ is the dual of $P_S(L\log L)$, the space of functions that may be written as $P_S g$ with $g \in L\log L$, equipped with the norm

$$\|h\|_{P_S(L\log L)} := \inf\{\|g\|_{L\log L} : h = P_S g\}.$$

Then, looking at the proof of Corollary 1.19, we see that we have as well the following improvement, since $P_S(L\log L)$ is contained in $\mathcal{H}^1$.

Proposition 5.2. *If b belongs to $BMOA(\varrho_\Psi)$, then h_b extends to a continuous operator from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $P_S(L\log L)$.*

This has been proven by different methods in [BM].

The same reasoning allows us to characterize as well the Hankel operators which map $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1_{\mathrm{weak}}$.

Proposition 5.3. *h_b extends to a continuous operator from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1_{\mathrm{weak}}$ if and only if b belongs to $BMOA(\varrho_\Phi)$.*

The necessity of the condition follows from the fact that

$$|\langle b,f\rangle| = |h_b(f)(0)| \lesssim \|h_b\|\,\|f\|_{\mathcal{H}^\Phi},$$

so that b defines a continuous linear form on $\mathcal{H}^\Phi$. We have used the fact that $\mathcal{H}^1_{\mathrm{weak}}$ is continuously contained in $\mathcal{H}^{1/2}$, and evaluation at 0 is bounded on this space. For the sufficiency, we prove that h_b maps $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $P_S(L^1)$ when $b \in BMOA(\varrho_\Phi)$. But the dual of $P_S(L^1)$ identifies with $\mathcal{H}^\infty$. So, using duality, it is sufficient to prove that multiplication by an element of the dual, that is, $\mathcal{H}^\infty$, maps $\mathcal{H}_\Phi$ into itself. This is straightforward.

6. Extension of the results to a general setting. We are now going to give the main points which allow extending our results to a larger class of domains including strictly pseudoconvex domains and convex domains of finite type. Let Ω be a smooth bounded domain in $\mathbb{C}^n$. Define the Hardy–Orlicz space on Ω as the space of holomorphic functions f such that

$$\sup_{0<\varepsilon<\varepsilon_0} \int_{\delta(w)=\varepsilon} \Phi(|f|)(w)\, d\sigma_\varepsilon(w) < \infty$$

where Φ is as before of lower type p, $\delta(w)$ is the distance from w to $\partial\Omega$, and $d\sigma_\varepsilon$ the Euclidean measure on the level set $\delta(w) = \varepsilon$. Recall that the usual Hardy space $\mathcal{H}^p(\Omega)$ of holomorphic functions on Ω corresponds to the case $\Phi(t) = t^p$.

6.1. Geometry of H-domains

Definition 6.1. We say that Ω is an *H-domain* if it is a smoothly bounded pseudoconvex domain of finite type and if, moreover, for each $\zeta \in \partial\Omega$ there exist a neighborhood V_ζ and a biholomorphic map Φ_ζ defined on V_ζ such that $\Phi_\zeta(\Omega \cap V_\zeta)$ is geometrically convex.

We recall that a point $\zeta \in \partial\Omega$ is said to be *of finite type* if the (normalized) order of contact with $\partial\Omega$ of complex varieties at ζ is finite. By [BS] and our assumption it suffices to consider the order of contact of $\partial\Omega$ at ζ with 1-dimensional complex manifolds (see [BS] and references therein). The domain Ω is said to be of finite type if every point on $\partial\Omega$ is of finite type. We denote by M_Ω the maximum of the types of points on $\partial\Omega$. Notice that the class of H-domains contains both the convex domains of finite type and the strictly pseudoconvex domains.

We describe the geometry of an H-domain Ω. This is done locally, using a partition of unity. Moreover, in a neighborhood of a point $\zeta \in \partial\Omega$, using local coordinates, we may assume that Ω is geometrically convex. Thus, we do not lose generality if we assume that it is globally convex. Then, there exist an $\varepsilon_0 > 0$ and a defining function ϱ for Ω such that for $-\varepsilon_0 < \varepsilon < \varepsilon_0$ the sets $\Omega_\varepsilon := \{z \in \mathbb{C}^n : \varrho(z) < \varepsilon\}$ are all convex. Moreover, denote by $U = U_{\varepsilon_0}$ the tubular neighborhood of $\partial\Omega$ given by $\{z \in \mathbb{C}^n : -\varepsilon_0 < \varrho(z) < \varepsilon_0\}$. By taking $\varepsilon_0 > 0$ sufficiently small, we may assume that on $\overline{U}$ the normal projection π of U onto $\partial\Omega$ is uniquely defined. Let $z \in U$ and let v be a unit vector in $\mathbb{C}^n$. We denote by $\tau(z, v, r)$ the distance from z to the surface $\{z' : \varrho(z') = \varrho(z) + r\}$ along the complex line determined by v. One of the basic relations among the quantities defined above is the following. There exists a constant C depending only on the geometry of the domain such that, given $z \in U$, any unit vector $v \in \mathbb{C}^n$ orthogonal to the level set of the function ϱ, and $r \le r_0$ and $\eta < 1$, we have

$$C^{-1}\eta^{1/2}\tau(z,v,r) \le \tau(z,v,\eta r) \le C\eta^{1/M_\Omega}\tau(z,v,r). \tag{34}$$

We next define the *r-extremal orthonormal basis* $\{v^{(1)},\dots,v^{(n)}\}$ at z, which generalizes the choices that we have made for the unit ball. The first vector is given by the direction transversal to the level set of ϱ containing z, pointing outward. Among the complex directions orthogonal to $v^{(1)}$ we choose $v^{(2)}$ in such a way that $\tau(z,v^{(2)},r)$ is maximum. We repeat the same procedure to determine the remaining elements of the basis. We set

$$\tau_j(z,r) = \tau(z,v^{(j)},r).$$

By definition, $\tau_1(z,r) \simeq r$. The polydisc $Q(z,r)$ is now given as

$$Q(z,r) = \{w : |w_k| \le \tau_k(z,r),\ k = 1,\dots,n\}.$$

Here $(w_1,\dots,w_n)$ are the coordinates determined by the r-extremal orthonormal basis $\{v^{(1)},\dots,v^{(n)}\}$ at z. Note that these coordinates $(w_1,\dots,w_n)$ $= (w_1^{z,r},\dots,w_n^{z,r})$ depend on z and r. They are called *special coordinates at the point z and at scale r*. A quasi-distance is defined by setting

$$d_b(z,w) = \inf\{r : w \in Q(z,r)\}. \tag{35}$$

Notice that by the above properties the sets $Q(z,r)$ are in fact equivalent to balls in the quasi-distance d_b. We also consider balls on the boundary $\partial\Omega$ defined in terms of d_b. For $\zeta \in \partial\Omega$ and $r > 0$ we set

$$B(\zeta,r) = \{z \in \partial\Omega : d_b(z,\zeta) < r\}.$$

These balls are equivalent to the sets $Q(\zeta,r)\cap\partial\Omega$: there exist $c, C > 0$ so that

$$Q(\zeta,cr)\cap\partial\Omega \subset B(\zeta,r) \subset Q(\zeta,Cr)\cap\partial\Omega.$$

We define the function d on $\overline{\Omega}\times\overline{\Omega}$ by setting

$$d(z,w) = \delta(z) + \delta(w) + d_b(\pi(z),\pi(w)), \tag{36}$$

where π is the normal projection of a point z onto the boundary. We set

$$\tau(z,r) = (\tau_1(z,r),\dots,\tau_n(z,r)).$$

and, for α a multiindex,

$$\tau^\alpha(z,r) = \prod_{j=1}^{n} \tau_j^{\alpha_j}(z,r).$$

When Ω is strictly pseudoconvex, we have simply $\tau^\alpha(z,r) \simeq r^{(|\alpha|+\alpha_1)/2}$. Let σ denote the surface measure on $\partial\Omega$. Then

$$\sigma(B(w,r)) \simeq \tau^{(1,2,\dots,2)}(w,r).$$

Moreover, the property (4) is replaced by the double inequality

$$\lambda^n\sigma(\zeta_0,r)\lesssim\sigma(B(\zeta_0,\lambda r))\lesssim\lambda^{1+(2n-2)/M_\Omega}\sigma(B(\zeta_0,\lambda r)), \tag{37}$$

As we said before, all these definitions are local, and may be given in the context of H-domains.

As in the case of the unit ball, if w_j are the coordinates on $w-z$ in the basis $\{v^{(1)},\dots,v^{(n)}\}$ and if $w_j=s_j+it_j$, then s_j for $j\ge 2$ and t_j for $j\ge 1$ define $2n-1$ local coordinates on $\partial\Omega$ in a neighborhood of z. We will still speak of *special coordinates at the point z and at scale r*.

In the neighborhood of $z\in\partial\Omega$, the hypersurface $\partial\Omega$ coincides with the graph $\Re w_1=h(\Im w_1,w')$, with $w'=(w_2,\dots,w_n)$. As in the case of the unit ball, we are interested in estimates on $D_w^{(\alpha,\beta)}S(\,(h(t_1,s'+it')+it_1,w'))$, where α is an $n-1$-index of derivation in the variable s', while β is an n-index of derivation in t. The equivalent of (26) is given by the estimates of McNeal and Stein [McS1] and [McS2] (see also [BPS1, Lemma 4.7] for an analogous context). For $d(w,z)<r$ and $\zeta\notin B(z,Cr)$, we have

$$\begin{aligned}&|D_w^{(\alpha,\beta)}S(\zeta,(h(t_1,s'+it')+it_1,w'))|\\&\qquad\lesssim\tau^{-(1+\beta_1,2+\alpha_2+\beta_2,\dots,2+\alpha_n+\beta_n)}(z,d(w,z)).\end{aligned}\tag{38}$$

As in [BPS1], we will also use the existence of a support function given by Diederich and Fornæss [DFo].

THEOREM 6.2. *Let Ω be a smoothly bounded pseudoconvex H-domain of finite type in $\mathbb{C}^n$. Then there exist a neighborhood U of the boundary $\partial\Omega$ and a function $H\in\mathcal{C}^\infty(\mathbb{C}^n\times U)$ such that the following conditions hold:*

(i) *$H(\cdot,w)$ is holomorphic on Ω for all $\zeta\in U$;*
(ii) *there exists a constant $c_1>1$ such that*

$$\frac{1}{c_1}\,d(z,w)\le|H(z,w)|\le c_1 d(z,w).$$

With all these definitions, we claim the following.

STATEMENT OF RESULTS FOR H-DOMAINS. *The analogues of Theorems* 1.3 *to Corollary* 1.19 *are valid for the H-domain Ω with the following modifications: $N_p:=(1/p-1)(M_\Omega+2n-2)-1$ in Definition* 1.5*; in Proposition* 1.9*, the condition is $L<(2N+2)/M_\Omega$, while in Proposition* 1.10*, we have $L_p:=2(1/p-1)(1+(2n-2)/M_\Omega)$. Finally, for the definition of $BMO(\varrho)$, we have to take $N+1>\alpha(M_\Omega+2n-2)$.*

Let us sketch the modifications. Atoms adapted to a ball $B:=B(\zeta_0,r_0)$ are defined as before, using special coordinates at ζ_0 and at scale r_0 to define the vanishing moment conditions. Notice that the coordinates depend on r_0, but the space $\mathcal{P}_N(\zeta_0)$ does not.

Then, in Lemma 3.1, the second estimate has to be replaced by

$$(39)\qquad |A(\zeta)| \lesssim \left(\frac{r_0}{d(\zeta,\zeta_0)}\right)^{(N+1)/M_\Omega} \frac{\|a\|_2\sigma(B)^{1/2}}{\sigma(\zeta_0, d(\zeta,\zeta_0))} \qquad \text{for } d(\zeta,\zeta_0) \geq Cr_0.$$

The proof is the same, using the estimates (38) in place of (26).

Next, molecules are defined as follows.

Definition 6.3. A holomorphic function $A \in \mathcal{H}^2(\Omega)$ is called a *molecule* of order L, associated to the ball $B := B(z_0, r_0) \subset \partial\Omega$, if it satisfies

(40)

$$\sup_{\varepsilon<\varepsilon_0} \int_{\partial\Omega} \left(1 + \frac{(\varepsilon + d(z_0,\xi))^L}{r_0^L} \frac{\sigma(B(z_0, d(z_0,\xi)))}{\sigma(B(z_0,r_0))}\right) |A(\xi - \varepsilon\nu(\xi))|^2 \,\frac{d\sigma(\xi)}{\sigma(B)} < \infty,$$

with ν the outward normal vector. In this case, the left side is $\|A\|^2_{\mathrm{mol}(B,L)}$.

It follows from (39), by splitting the integral into dyadic balls, that the projection of an atom associated to the ball $B := B(z_0, r_0) \subset \partial\Omega$ is a molecule of order $L < (2N+2)/M_\Omega$.

Finally, to see that a molecule of order L is in the Hardy space $\mathcal{H}^\Phi$, we prove that, with $B_k := B(z_0, 2^k r_0)$,

$$\int_{B_k\setminus B_{k-1}} \Phi(g) \frac{d\sigma}{\sigma(B)} \lesssim 2^{-k\varepsilon} \Phi\left(\left(2^{kL} \frac{\sigma(B_k)}{\sigma(B)} \int_{B_k\setminus B_{k-1}} g(\xi)^2 \,\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2}\right)$$

for some $\varepsilon > 0$. To do this, we use again the Jensen inequality (24) for the measure $d\sigma$ on B_k, divided by its total mass $\sigma(B_k)$. This gives

$$\int_{B_k\setminus B_{k-1}} \Phi(g) \,\frac{d\sigma}{\sigma(B)} \lesssim \frac{\sigma(B_k)}{\sigma(B)} \,\Phi\left(\left(\int_{B_k\setminus B_{k-1}} g(\xi)^2 \,\frac{d\sigma(\xi)}{\sigma(B_k)}\right)^{1/2}\right).$$

We conclude by using the fact that Φ is of lower type p, which yields

$$\frac{\sigma(B_k)}{\sigma(B)} \,\Phi(t) \lesssim \Phi\left(\left(\frac{\sigma(B_k)}{\sigma(B)}\right)^{1/p} t\right).$$

Using (37) shows that it is sufficient to choose $L > L_p := 2(1/p - 1)(1 + (2n-2)/M_\Omega)$.

Up to now, we have given the modifications to obtain atomic decomposition, the continuity of the Szegö projection, and duality. It remains to indicate the modifications in the proof of the factorization theorem. As at the beginning of Section 5, we factorize each molecule A associated to a ball $B := B(\zeta_0, r)$ as $B = fg$, with f a molecule and g a *BMOA*-function.

For this factorization, we use the support function given in Theorem 6.2. We set $H_0 = H(\cdot, \tilde{\zeta}_0)$, where $\tilde{\zeta}_0 = \zeta_0 - r\nu(\zeta_0)$. We choose $g = \log(cH_0^{-1})$ with c such that g does not vanish in Ω.

We have, as before, the inequality

$$\|f\|_{\mathrm{mol}(B,L')} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\log(e+\sigma(B)^{-1})} \tag{41}$$

for $L' < L$ (just use Theorem 6.2(ii)).

We now prove that $\log(cH_0^{-1})$ belongs to $BMOA$ with bounds independent of ζ_0 and r. The proof follows the same lines as the one in the unit ball, using that $|H_0|$ and $|\nabla H_0|$ are uniformly bounded in Ω.

These are the principal points to modify in the proofs.

Acknowledgments. The authors are partially supported by the project ANR AHPI number ANR-07-BLAN-0247-01.

REFERENCES

[BS] H. P. Boas and E. J. Straube, *On equality of line type and variety type of real hypersurfaces in* $\mathbb{C}^n$, J. Geom. Anal. 2 (1992), 95–98.

[BGS] A. Bonami, S. Grellier, and B. Sehba, *Boundedness of Hankel operators on* $\mathcal{H}^1$, C. R. Math. Acad. Sci. Paris 344 (2007), 749–752.

[BIJZ] A. Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister, *On the product of functions in* BMO *and* $\mathcal{H}^1$, Ann. Inst. Fourier (Grenoble) 57 (2007), 1405–1439.

[BM] A. Bonami and S. Madan, *Balayage of Carleson measures and Hankel operators on generalized Hardy spaces*, Math. Nachr. 153 (1991), 237–245.

[BPS1] A. Bonami, M. M. Peloso, and F. Symesak, *Powers of the Szegö kernel and Hankel operators on Hardy spaces*, Michigan Math. J. 46 (1999), 225–250.

[BPS2] —, —, —, *Factorization of Hardy spaces and Hankel operators on convex domains in* $\mathbb{C}^n$, J. Geom. Anal. 11 (2001), 363–397.

[CV] W. S. Cohn and I. E. Verbitsky, *Factorization of tent spaces and Hankel operators*, J. Funct. Anal. 175 (2000), 308–329.

[CRW] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 611–635.

[DFo] K. D. Diederich and J. E. Fornæss, *Support functions for convex domains of finite type*, Math. Z. 230 (1999), 145–164.

[GL] J. B. Garnett and R. H. Latter, *The atomic decomposition for Hardy spaces in several complex variables*, Duke Math. J. 45 (1978), 815–845.

[GP] S. Grellier and M. Peloso, *Decomposition theorems for Hardy spaces on convex domains of finite type*, Illinois J. Math. 46 (2002), 207–232.

[J] S. Janson, *Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation*, Duke Math. J. 47 (1980), 959–982.

[KL] S. G. Krantz and S.-Y. Li, *On decomposition theorems for Hardy spaces on domains in* $\mathbb{C}^n$ *and applications*, J. Fourier Anal. Appl. 2 (1995), 65–107.

[McS1] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke J. Math. 73 (1994), 177–199.

[McS2] —, —, *The Szegö projection on convex domains*, Math. Z. 224 (1997), 519–553.

[R] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, New York, 1980.

[St] E. M. Stein, *Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, 1993.

[TW] M. H. Taibleson and G. Weiss, *The molecular characterization of Hardy spaces*, in: Harmonic Analysis in Euclidean Spaces (Williamstown, MA, 1978), Part 1, Proc. Sympos. Pure Math. 35, Amer. Math. Soc., Providence, RI, 1979, 281–287.

[V] B. E. Viviani, *An atomic decomposition of the predual of* BMO(ρ), Rev. Mat. Iberoamer. 3 (1987), 401–425.

[VT] A. L. Volberg and V. A. Tolokonnikov, *Hankel operators and problems of best approximation of unbounded functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 141 (1985), 5–17, 188 (in Russian).

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Received 10 February 2009;
revised 3 November 2009 (5159)