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POISSON KERNELS OF DRIFTED LAPLACE OPERATORS ON TREES AND ON THE HALF-PLANE

BY

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Dedicated to the memory of Prof. Andrzej Hulanicki

Abstract. Starting with the computation of the appropriate Poisson kernels, we review, complement, and compare results on drifted Laplace operators in two different contexts: homogeneous trees and the hyperbolic half-plane.

1. Introduction. In harmonic analysis, the analogy between homogeneous trees and the real hyperbolic plane has been much exploited (see, e.g., [C] for a survey in integral geometry). In this paper, as in [BCF, BCFY], we focus on the analogies evidenced by considering a fixed point ω_0 of the boundary of either space and the group G of automorphisms of the space that fix ω_0. In the conformal half-plane model for the real hyperbolic plane the natural choice for ω_0 is the point at infinity ∞; then the complement of ω_0 in the boundary is $\mathbb{R}$, while the full boundary is its one-point compactification, the real projective line. Correspondingly, as shown in [DF], the boundary of a homogeneous tree may be identified with the one-point compactification of one of several non-isomorphic, totally disconnected Abelian groups, among which one of the most natural is $\mathbb{Q}_p$ when the degree of the tree is $p+1$ and p is a prime number; the full boundary may then be identified with the p-adic projective line. The group G is the real affine group in the case of the half-plane, while it contains properly the p-adic affine group in the case of a tree of degree $p+1$. Natural objects of study in both contexts are G-invariant operators, in particular those obtained by adding a G-invariant drift in the direction of ω_0 to the standard isotropic Laplace operator. The resulting drifted Laplace operators were studied in [BCF, CF]. In this paper we proceed to a wider comparison of these operators in the two settings, proving some new results and providing some alternative proofs.

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In Section 2 we give the definitions of drifted Laplace operators. The special undrifted case gives the ordinary Laplace (or Laplace–Beltrami) operators. Section 3 is devoted to the computation of their Poisson kernels as non-negative functions on the Cartesian product of the ambient space and the boundary; the computation of the case of the half-plane is taken from [BCFY]. In Section 4, by means of pointwise multiplication operators we exhibit intertwining relations between drifted Laplace operators having different values of the drift parameter, which translate into relations between their respective eigenfunctions and eigenvalues; this is exploited to provide an alternative method to compute the Poisson kernels. Finally, we recall in Section 5 some relationships between the Poisson kernels and symmetric stable random variables with values in a local field and in $\mathbb{R}$, respectively. Each section is divided into two subsections, one devoted to each of the two settings.

2. Drifted Laplace operators

Trees. Undefined notation and terminology is as in [FN]. Let $\mathfrak{T}$ be a homogeneous tree of degree $q+1>2$, let $\mathfrak{X}$ be the set of its vertices and Ω its *boundary*. Let $[x,y]\subset\mathfrak{X}$ be the finite *chain* linking $x,y\in\mathfrak{X}$; the number $d(x,y)$ of edges in $[x,y]$ gives a natural non-negative integer-valued distance on $\mathfrak{X}$. The isometries with respect to the distance coincide with the *automorphisms* of $\mathfrak{T}$, the one-to-one self-maps of $\mathfrak{X}$ that map edges to edges. Likewise, let $[x,\omega)$ be the one-sided infinite chain between $x\in\mathfrak{X}$ and $\omega\in\Omega$.

Let ω_0 be a fixed element of the boundary Ω. This choice yields a partial order on $\mathfrak{X}$: for $x,y\in\mathfrak{X}$, if $x\in[y,\omega_0)$ we write $x\succ y$ and say that x is an *ancestor* of y. If x,y are also neighbors, we say that x is the unique *parent* y^- of y, and y is one of the q *children* of x. For $0<\theta<1$ a *drifted Laplace operator* on $\mathfrak{X}$ is given by

$$L_\theta f(x)=\theta f(x^-)+\frac{1-\theta}{q}\sum_{y^-=x}f(y)\qquad\text{for every } x\in\mathfrak{X}.$$

This operator is invariant under the group G of automorphisms of $\mathfrak{T}$ that leave ω_0 fixed, and is associated to the nearest-neighbor random walk on $\mathfrak{X}$ such that the probability of moving from any vertex to its parent equals θ, and the probability of moving to each child is $(1-\theta)/q$. (Some authors use $L_\theta-I$ instead of L_θ, with obvious changes in the statement of the results; our choice more directly relates L_θ to the random walk.) In particular,

$$L_{1/(q+1)}=L$$

is the usual isotropic Laplace operator, invariant under every automorphism of $\mathfrak{T}$ and independent of the choice of ω_0.

Half-plane. For real $b < 1/2$ a (vertically) *drifted Laplace operator* on the hyperbolic half-plane $\Pi = \{z = x + iy : y > 0\}$ is

$$L_b = \frac{y^2}{2}\,\Delta + by\,\frac{\partial}{\partial y} \qquad \text{for } x + iy \in \Pi,$$

where Δ is the Euclidean Laplacian. It commutes with the action of the group

$$G = \{z \mapsto \alpha z + \beta,\ z \mapsto -\alpha\bar{z} + \beta : \alpha > 0,\ \beta \in \mathbb{R}\}$$

of all the isometries of Π that fix the point at infinity $\omega_0 = \infty$. On the real line, as a boundary of Π, the group G acts as the real affine group. Observe that

$$L_0 = L$$

is the Laplace–Beltrami operator, which is also invariant under the isometry $z \mapsto -1/z$, and therefore under every isometry of Π.

3. The Poisson kernel

Trees. The *join* $x \vee y$ of $x, y \in \mathfrak{X}$ is the nearest common ancestor of x and y. The relation $x \sim y$ defined by $d(x, x\vee y) = d(y, x\vee y)$ is an equivalence relation in $\mathfrak{X}$. Its equivalence classes are called the *horocycles* of $\mathfrak{T}$ with respect to ω_0. The family of horocycles may be indexed as $\{\mathfrak{H}_n : n \in \mathbb{Z}\}$. This is achieved by choosing $\mathfrak{H}_0$ arbitrarily and stipulating that $x \in \mathfrak{H}_n$ if and only if $x^- \in \mathfrak{H}_{n-1}$. The *horocycle index* is the integer-valued function on $\mathfrak{X}$ given by $h(x) = n$ if $x \in \mathfrak{H}_n$. Consequently, h is decreasing with respect to $\succ$. In particular, $x \vee y$ is the vertex of lowest horocycle index in $[x, y]$. We say that x lies *below* a horocycle $\mathfrak{H}_n$ if an ancestor of x belongs to $\mathfrak{H}_n$, that is, if $h(x) > n$. The join $x \vee \omega$ of $x \in \mathfrak{X}$ and $\omega \in \Omega$, with $\omega \neq \omega_0$, is the vertex of lowest horocycle index in the chain $[x, \omega)$, and is independent of the choice of $\mathfrak{H}_0$. Likewise, for $\omega_0 \neq \omega, \omega' \in \Omega$ let $\omega \vee \omega'$ be the vertex of lowest horocycle index in (ω, ω').

As in [CF], on $\mathfrak{X}$ we define a *bi-distance*, i.e., a distance whose values are pairs of non-negative integers. For $x, y \in \mathfrak{X}$ set

$$\mathbf{d}(x, y) = (d^-(x, y), d^+(x, y)) \quad \text{where} \quad \begin{cases} d^-(x, y) = d(x, x \vee y), \\ d^+(x, y) = d(y, x \vee y). \end{cases}$$

Thus if $\mathbf{d}(x, y) = (j, k)$ then $\mathbf{d}(y, x) = (k, j)$, and $\mathbf{d}(x, y) = \mathbf{d}(gx, gy)$ for every $g \in G$. Moreover,

$$d^-(x, y) + d^+(x, y) = d(x, y), \qquad d^-(x, y) - d^+(x, y) = h(x) - h(y).$$

The distance on $\mathfrak{R} = \Omega \setminus \{\omega_0\}$ given by

$$d(\omega, \omega') = q^{-h(\omega\vee\omega')} \qquad \text{for every } \omega, \omega' \in \mathfrak{R}$$

induces a locally compact topology on $\mathfrak{R}$. This distance takes values in the integer powers of q, and for each $h \in \mathbb{Z}$ the relation $d(\omega, \omega') \leq q^h$ is an equivalence relation. Thus d defines an ultrametric space. A natural Borel measure m on $\mathfrak{R}$ assigns to a metric ball its diameter. Thus, for fixed $\omega' \in \mathfrak{R}$ and $h \in \mathbb{Z}$, the metric ball $\{\omega \in \mathfrak{R} : d(\omega, \omega') \leq q^h\}$ has measure q^h.

It is possible to identify the vertices of the tree with subsets of $\mathfrak{R}$, in such a way that the relation $\succ$ corresponds to $\supset$. It suffices to identify a vertex x with the set of all the infinite chains starting at x^- and containing x. Indeed each such chain corresponds uniquely to an element of $\mathfrak{R}$. With this correspondence the vertex x corresponds to a metric ball of radius (or diameter) $q^{-h(x)}$. In particular, $m(x) = 1$ if $x \in \mathfrak{H}_0$. The union $\mathfrak{X} \cup \mathfrak{R}$ is naturally a locally compact space. A basic open set in $\mathfrak{X} \cup \mathfrak{R}$ is either a single vertex or, for a given vertex x, the union of $\{y \in \mathfrak{X} : y \prec x\}$ and the metric ball of $\mathfrak{R}$ that is identified with the vertex x.

PROPOSITION 3.1 (Maximum principle for L_θ). *A non-constant function f on $\mathfrak{X}$ that satisfies $L_\theta f = f$ cannot attain a maximum in $\mathfrak{X}$.*

Proof. This follows easily from the observation that $L_\theta f(x)$ is a weighted average of the values of f at the neighbors of x. ∎

PROPOSITION 3.2. *If $1/2 \leq \theta < 1$, then every bounded function f on $\mathfrak{X}$ such that $L_\theta f = f$ is constant.*

Proof. If f is not constant, there is an x such that $f(x) \neq f(x^-)$. Taking $-f$ in place of f if necessary, we may assume that $f(x^-) < f(x)$. Label x_1 the child of x where f takes the largest value, and set $x_0 = x$. The condition $L_\theta f = f$ implies

$$0 < \frac{\theta}{1-\theta}\,(f(x) - f(x^-)) = \frac{1}{q} \sum_{y^- = x} f(y) - f(x) \leq f(x_1) - f(x_0).$$

This argument may be repeated inductively. If x_{j+1} is the child of x_j where f takes the largest value,

$$\begin{aligned} f(x_{j+1}) - f(x_j) &\geq \frac{\theta}{1-\theta}\,(f(x_j) - f(x_{j-1})) \\ &\geq \left(\frac{\theta}{1-\theta}\right)^{j+1} (f(x) - f(x^-)) \quad \text{for every } j > 0. \end{aligned}$$

Thus

$$f(x_n) - f(x) = \sum_{j=0}^{n-1} (f(x_{j+1}) - f(x_j)) \geq \sum_{j=0}^{n-1} \left(\frac{\theta}{1-\theta}\right)^{j+1} (f(x) - f(x^-)).$$

The last sum grows indefinitely with n because $\theta/(1-\theta) \geq 1$. It follows that f is unbounded. ∎

In the remainder of this section we assume that $0 < \theta < 1/2$. The *Poisson kernel* of L_θ is the function P_θ on $\mathfrak{X} \times \mathfrak{R}$ defined as follows: for every $x \in \mathfrak{X}$ the function $P_\theta(x, \cdot\,)$ on $\mathfrak{R}$ is the Radon–Nikodým derivative with respect to m of the hitting distribution (at infinite time) on $\mathfrak{R}$ of the random walk associated to L_θ starting at $x \in \mathfrak{X}$. The condition $\theta < 1/2$ implies that the random walk will almost surely keep below any given horocycle. The hitting distribution $P_\theta(x, \omega)\, dm(\omega)$ is invariant under the diagonal action of G on the pair (x, ω).

To compute P_θ we first define, for any two vertices x and y, the probability $\mathrm{Pr}_\theta(x, y)$ that the random walk starting at x keeps eventually below y. This probability, because of the invariance of the random walk with respect to the action of G, depends only on the bi-distance $\mathbf{d}(x, y)$. With this definition, we let

$$P_\theta(x, \omega) = \lim_{y \to \omega} \frac{\mathrm{Pr}_\theta(x, y)}{m(y)} = \lim_{y \to \omega} \mathrm{Pr}_\theta(x, y) q^{h(y)}. \tag{3.1}$$

The limit exists because, as soon as $x \vee y = x \vee \omega$, the product $\mathrm{Pr}_\theta(x, y) q^{h(y)}$ becomes independent of y.

LEMMA 3.3. *For every $x \in \mathfrak{X}$, the probability that starting at x the random walk visits x^- at least once is*

$$p = \frac{\theta}{1 - \theta},$$

and we have

$$\mathrm{Pr}_\theta(x, x) = \frac{q(1 - 2\theta)}{q - (q + 1)\theta},$$

independently of $x \in \mathfrak{X}$.

Proof. At its first step the random walk moves to x^- with probability θ and to one of the children of x with overall probability $1 - \theta$. In the latter case the probability of later visiting x^- is p^2. Thus $p = \theta + p^2(1 - \theta)$. The solution $p = 1$ must be discarded because $\theta < 1/2$. Therefore $p = \theta/(1 - \theta)$. Let $x_0 = x$ and $x_{n+1} = x_n^-$ for every $n > 0$. Almost surely, starting at x, the random walk visits at most a finite number of x_n. For each $n \geq 0$, the probability that it visits x_n, but not x_{n+1}, and stays eventually below x, is $p^n(1 - p)/q^n$. Indeed, the probability that it visits x_n is p^n. Therefore the probability that it visits x_n but not x_{n+1} is $p^n(1 - p)$. Finally, since the tree is homogeneous, the probability that the random walk visits x_n but not x_{n+1} and stays eventually below x_0 is $p^n(1 - p)/q^n$. For different n the events are mutually exclusive. Therefore

$$\mathrm{Pr}_\theta(x, x) = \sum_{n=0}^{\infty} \frac{p^n(1 - p)}{q^n} = q\, \frac{1 - p}{q - p} = \frac{q(1 - 2\theta)}{q - (q + 1)\theta}. \quad \blacksquare$$

THEOREM 3.4. *For every $x, y \in \mathfrak{X}$,*

$$\Pr_\theta(x,y) = \frac{q(1-2\theta)}{q-(q+1)\theta}\left(\frac{\theta}{1-\theta}\right)^{d^-(x,y)} q^{-d^+(x,y)}.$$

Therefore, once the horocycle $\mathfrak{H}_0$ is chosen, the Poisson kernel P_θ of L_θ is given by

$$P_\theta(x,\omega) = \frac{q(1-2\theta)}{q-(q+1)\theta}\left(\frac{\theta}{1-\theta}\right)^{d(x,x\vee\omega)} q^{h(x\vee\omega)}$$

for every $x \in \mathfrak{X}$ and $\omega \in \mathfrak{R}$,

and has the following properties:

(T1) $L_\theta P_\theta(\,\cdot\,,\omega) = P_\theta(\,\cdot\,,\omega)$ *on $\mathfrak{X}$ for every $\omega \in \mathfrak{R}$;*
(T2) $P_\theta(x,\omega) > 0$ *for every $x \in \mathfrak{X}$ and $\omega \in \mathfrak{R}$;*
(T3) $\int_{\mathfrak{R}} P_\theta(x,\omega)\, dm(\omega) = 1$ *for every $x \in \mathfrak{X}$;*
(T4) $\lim_{x\to\omega'} \int_{\mathfrak{R}\setminus U} P_\theta(x,\omega)\, dm(\omega) = 0$ *for every $\omega' \in \mathfrak{R}$ and every neighborhood U of ω' in $\mathfrak{R}$.*

Proof. In order to move from x to y the random walk must visit $x \vee y$, stay eventually below $x \vee y$, and, after visiting $x \vee y$ for the last time, reach y. The first event has probability $p^{d^-(x,y)}$. The second event, conditioned on the first, has probability $\Pr_\theta(x \vee y, x \vee y)$. The third event, conditioned on the first two, has probability $q^{-d(x\vee y,y)}$, because the tree is homogeneous. The product of these three numbers is $\Pr_\theta(x,y)$. As $y \to \omega$ the vertex $x \vee y$ becomes eventually $x \vee \omega$. Therefore the expression for $P_\theta(x,\omega)$ follows from the definition, identity (3.1), the fact that $m(y) = q^{-h(y)}$, and the identity $d(x \vee y, y) = h(y) - h(x \vee y)$.

Property (T1) follows from a simple computation based on the equality

$$P_\theta(x^-,\omega) = \begin{cases} \dfrac{1}{q}\, P_\theta(x,\omega) & \text{if } \omega \in x, \\[2ex] \dfrac{1-\theta}{\theta}\, P_\theta(x,\omega) & \text{if } \omega \notin x. \end{cases}$$

Property (T2) is immediate from the expression for P_θ. Property (T3) follows from the definition of $P_\theta(x,\,\cdot\,)$ as a Radon–Nikodým derivative of a probability distribution. A neighborhood U of ω' in $\mathfrak{R}$ contains a metric ball which may be identified with a vertex x', thus, in order to prove (T4), we may assume that $x \prec x'$ and $\omega \in \mathfrak{R} \setminus x'$. Under these conditions we have $x \prec x' \prec x' \vee \omega = x \vee \omega$, whence $q^{h(x\vee\omega)} = q^{h(x'\vee\omega)}$, independently of x, and $d(x, x \vee \omega) \geq d(x', x' \vee \omega)$, which together imply that $P_\theta(x,\omega) \leq P_\theta(x',\omega)$ because $\theta/(1-\theta) < 1$. We also have $d(x, x \vee \omega) \geq d(x, x')$, which tends to

$+\infty$ as $x \to \omega'$, therefore

$$\left(\frac{\theta}{1-\theta}\right)^{d(x,x\vee\omega)} \le \left(\frac{\theta}{1-\theta}\right)^{d(x,x')} \to 0.$$

Lebesgue's dominated convergence theorem applies to the functions $P_\theta(x, \cdot\,)$ on $\mathfrak{R} \setminus x'$, because they tend to 0 there as $x \to \omega'$ and are dominated by $P_\theta(x', \cdot\,)$, which is integrable on $\mathfrak{R}$ by (T2) and (T3). ∎

The first statement of the next result may be regarded as a particular instance of [CKW, Theorem 3].

THEOREM 3.5. *If ϕ is any bounded continuous function on $\mathfrak{R}$, then the Dirichlet problem*

$$\begin{cases} L_\theta f = f & \text{on } \mathfrak{X}, \\ \lim\limits_{x\to\omega} f(x) = \phi(\omega) & \text{for every } \omega \in \mathfrak{R}, \end{cases}$$

has a unique solution given by

$$f(x) = \int_{\mathfrak{R}} P_\theta(x,\omega)\phi(\omega)\, dm(\omega) \qquad \text{for every } x \in \mathfrak{X},$$

where P_θ is any kernel on $\mathfrak{X} \times \mathfrak{R}$ that enjoys properties (T1)–(T4).

Consequently, properties (T1)–(T4) *characterize the Poisson kernel.*

Proof. For $\epsilon > 0$ let U be a neighborhood of $\omega' \in \mathfrak{R}$ such that $|\phi(\omega) - \phi(\omega')| < \epsilon$ whenever $\omega \in U$, and let M be a bound for $|\phi|$. Then

$$\begin{aligned} |\phi(\omega') - f(x)| &= \Big| \int_{\mathfrak{R}} (\phi(\omega') - \phi(\omega)) P_\theta(x,\omega)\, dm(\omega) \Big| \\ &\le \int_U P_\theta(x,\omega) |\phi(\omega') - \phi(\omega)|\, dm(\omega) + 2M \int_{\mathfrak{R}\setminus U} P_\theta(x,\omega)\, dm(\omega). \end{aligned}$$

The first integral on the right-hand side of this inequality is less than ϵ by (T3), while the second converges to 0 by (T4) if $x \to \omega'$. This proves the boundary condition for f at ω'.

The equality $L_\theta f = f$ follows from (T1). Proposition 3.1 gives the uniqueness of f. For $x \in \mathfrak{X}$ the linear functional $\phi \mapsto f(x)$ defined on $\mathcal{C}_0(\mathfrak{R})$ yields the harmonic measure m_x on $\mathfrak{R}$. Finally, $P_\theta(x, \cdot\,)$ is recovered as the Radon–Nikodým derivative of m_x. ∎

Notice that $P_\theta(x, \cdot\,)$ may be extended to all of Ω by setting $P_\theta(x, \omega_0) = 0$ for every $x \in \mathfrak{X}$.

Half-plane. The operator L_b, with $b < 1/2$, is the infinitesimal generator of a diffusion process on Π. Its Poisson kernel $P_b(z, \xi)$ is the density of the hitting distribution on the boundary $\mathbb{R}$, when $z \in \Pi$ is the starting point [BCFY, (1.3)]. The following properties characterize P_b:

(H1) $L_b P_b(\,\cdot\,,\xi) = 0$ for every $\xi \in \mathbb{R}$;
(H2) $P_b(z,\xi) > 0$ for every $z \in \mathit{\Pi}$ and $\xi \in \mathbb{R}$;
(H3) $\int_{\mathbb{R}} P_b(z,\xi)\,d\xi = 1$ for every $z \in \mathit{\Pi}$;
(H4) $\lim_{z\to\xi'} \int_{\mathbb{R}\setminus U} P_b(z,\xi)\,d\xi = 0$ for every $\xi' \in \mathbb{R}$ and every neighborhood U of ξ' in $\mathbb{R}$.

THEOREM 3.6. *For $z = x + iy$ with $y > 0$ and $b < 1/2$,*

$$P_b(z,\xi) = \frac{\mathit{\Gamma}(1-b)}{\sqrt{\pi}\,\mathit{\Gamma}(1/2-b)}\,\frac{y^{1-2b}}{((x-\xi)^2+y^2))^{1-b}}.$$

Proof. We give a special case of the calculations performed in [BCFY, §2]. The operator L_b is invariant with respect to the group of transformations $\{z \mapsto \alpha z + \beta : \alpha > 0,\ \beta \in \mathbb{R}\}$, which is a subgroup of G. It follows that the measure $P_b(z,\xi)\,d\xi$ is invariant with respect to the diagonal action of this group. In other words, $P_b(z,\xi) = \alpha P_b(\alpha z+\beta, \alpha\xi+\beta)$. If we let $f(\xi) = P_b(i,\xi)$ it follows that

$$P_b(x+iy,\xi) = \frac{1}{y}\,P_b\left(i, \frac{\xi-x}{y}\right) = \frac{1}{y}\,f\left(\frac{\xi-x}{y}\right).$$

We apply (H1), then let $y = 1$ and $\xi = 0$ to obtain an ordinary differential equation of the second order $Mf = 0$, which must be satisfied by f:

$$\begin{aligned} Mf(x) &= \frac{1+x^2}{2}\,f''(x) + (2-b)xf'(x) + (1-b)f(x) \\ &= \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{1+x^2}{2}\,f(x)\right) - bxf(x)\right] = 0. \end{aligned}$$

This means that for some real constant k the function f must satisfy the first order equation

$$\frac{d}{dx}\left(\frac{1+x^2}{2}f(x)\right) - bxf(x) = k.$$

For simplicity we introduce the function $g(x) = f(x)(1+x^2)/2$, which satisfies the equation

$$g'(x) - \frac{2bx}{1+x^2}\,g(x) = k.$$

The solution is

$$g(x) = (1+x^2)^b\Big(c + k\int_0^x (1+t^2)^{-b}\,dt\Big) \quad \text{for } c \in \mathbb{R}. \tag{3.2}$$

Property (H2) implies that f and hence g take only positive values. Yet the condition $b < 1/2$ implies that the integral in (3.2) takes arbitrarily large positive and negative values. This implies that if $k \neq 0$, for any fixed c, the function $g(x)$ will assume negative values. We conclude that $k = 0$ and

$$f(x) = c(1+x^2)^{b-1}.$$

It follows that

$$P_b(x+iy,\xi)=\frac{1}{y}\,f\left(\frac{\xi-x}{y}\right)=c\,\frac{y^{1-2b}}{((x-\xi)^2+y^2))^{1-b}}.$$

The value of c is determined by the condition

$$\int_{\mathbb{R}} f(x)\,dx=1,$$

as shown in [GR, 8.335.1]. ∎

As in the case of trees, the kernel $P_b(z,\cdot\,)$ may be extended to the full boundary of Π by setting $P_b(z,\infty)=0$ for every $z\in\Pi$.

4. Intertwining operators and eigenfunctions

Trees. For $a>0$ denote by M_a the operator of multiplication by $a^{h(x)}$, defined on the space $\ell_2(\mathfrak{X})$. Observe that $M_a^{-1}=M_{a^{-1}}$. The following result, taken from [CF], may be regarded as a direct consequence of the reversibility of the random walk associated to L_θ [W, p. 126], and may be used for an alternative computation of $P_\theta(x,\omega)$.

PROPOSITION 4.1. *For $0<\theta<1$, if*

$$a=\sqrt{\frac{1-\theta}{\theta q}},\qquad \mu=(q+1)\sqrt{\frac{\theta(1-\theta)}{q}},$$

then

$$L_\theta=\mu M_a^{-1}LM_a.$$

Therefore a function f is an eigenfunction of L with eigenvalue λ if and only if $M_{1/a}f$ is an eigenfunction of L_θ with eigenvalue $\mu\lambda$.

Proof. For every $x\in\mathfrak{X}$ we have

$$\begin{aligned}\mu M_a^{-1}LM_af(x)&=\mu a^{-h(x)}L(a^hf)(x)\\&=\mu a^{-h(x)}\,\frac{1}{q+1}\Big(a^{h(x^-)}f(x^-)+\sum_{y^-=x}a^{h(y)}f(y)\Big)\\&=\frac{\mu}{q+1}\left(\frac{f(x^-)}{a}+\sum_{y^-=x}af(y)\right)\\&=\theta f(x^-)+\frac{1-\theta}{q}\sum_{y^-=x}f(y)=L_\theta f(x).\ \blacksquare\end{aligned}$$

In the special case $\theta=1/(q+1)$ the Poisson kernel $P=P_{1/(q+1)}$ of L as computed in [FN] is

$$P(x,\omega)=\frac{q}{q+1}\,q^{h(x\vee\omega)-d(x,x\vee\omega)}.$$

PROPOSITION 4.2. *Let* $0 < \theta < 1/2$ *and*

$$s = \frac{1}{2}\left(1 + \log_q \frac{1-\theta}{\theta}\right).$$

Then

$$P_\theta(x,\omega) = c_s \left(\frac{1-\theta}{q\theta}\right)^{-h(x)/2} P(x,\omega)^s \quad \text{for every } x \in \mathfrak{X} \text{ and } \omega \in \mathfrak{R},$$

where

$$c_s^{-1} = \left(\frac{q}{q+1}\right)^s \int_{\mathfrak{R}} q^{2sh(x\vee\omega)}\, dm(\omega) \quad \text{for any } x \in \mathfrak{H}_0. \tag{4.1}$$

Proof. By [FN, p. 35], for every $\omega \in \mathfrak{R}$, $P(x,\omega)^s$ is an eigenfunction of the undrifted Laplace operator L of eigenvalue $(q^s + q^{1-s})/(q+1)$. From Proposition 4.1 it follows that $a^{-h(x)}P(x,\omega)^s$ is an eigenfunction of L_θ with eigenvalue $\mu(q^s + q^{1-s})/(q+1)$. The latter expression is equal to 1 if $q^s = 1/a$ or if $q^s = qa$. The first alternative must be discarded because (T4) fails, as is easily verified. Thus $s = 1 + \log_q a$, which gives the value for s. The expression (4.1) is a consequence of (T3). ∎

Observe that the expression above for $P_\theta(x,\omega)$ agrees with the expression computed in Theorem 3.4, because using an appropriate partition of $\mathfrak{R}$ we obtain, from (4.1),

$$c_s\left(\frac{q}{q+1}\right)^s = \left(1 + \sum_{k=1}^{\infty}(q-1)q^{k-1}q^{-2sk}\right)^{-1} = \frac{q(1-2\theta)}{q-(q+1)\theta},$$

the same constant appearing in Theorem 3.4.

Half-plane. The analogue of Proposition 4.1 for L_b is the following.

PROPOSITION 4.3. *For* $b < 1/2$, *if* M_b *is the operator of multiplication by* y^b *and* I *is the identity operator, then*

$$L_b = M_b^{-1} L M_b - \frac{b(b-1)}{2}\, I.$$

Therefore f *is an eigenfunction of* L *with eigenvalue* λ *if and only if* $M_{-b}f$ *is an eigenfunction of* L_b *with eigenvalue* $\lambda - b(b-1)/2$.

Proof. We compute directly

$$M_b^{-1} L M_b f = y^{-b}\,\frac{y^2}{2}\,\Delta(y^b f) = \frac{y^2}{2}\,\Delta f + by\,\frac{\partial f}{\partial y} + \frac{b(b-1)}{2}\, f$$
$$= L_b f + \frac{b(b-1)}{2}\, f. \quad ∎$$

As in the case of the tree, the result above allows an alternative computation of the Poisson kernel of L_b starting from the Poisson kernel of the

Laplace–Beltrami operator, which has the following expression:

$$P(x+iy,\xi) = \frac{1}{\pi}\,\frac{y}{(x-\xi)^2+y^2} \qquad \text{for every } y>0 \text{ and } \xi\in\mathbb{R}.$$

PROPOSITION 4.4. *Let $s = 1-b$. Then*

$$P_b(x+iy,\xi) = c_b y^{-b} P(x+iy,\xi)^s \qquad \textit{for every } y>0 \textit{ and } \xi\in\mathbb{R},$$

where

$$c_b^{-1} = \frac{1}{\pi^s}\int_{\mathbb{R}} \frac{d\xi}{(\xi^2+1)^s}.$$

Proof. We must verify that the proposed expression for the Poisson kernel has properties (H1)–(H4). As observed in [E, (14)], a direct computation shows that, for each $\xi\in\mathbb{R}$, $P(\,\cdot\,,\xi)^s$ is an eigenfunction of L with eigenvalue $s(s-1)/2$. By Proposition 4.3 it follows that $y^{-b}P(x+iy,\xi)^s$ is an eigenfunction of L_b with eigenvalue 0, thus fulfilling (H1), provided that $s(s-1)/2-b(b-1)/2=0$. Of the two possible solutions $s=b$ and $s=1-b$, we must choose the latter for (H2) to hold. Condition (H4) is immediate, while (H3) at $x+iy=i$ yields the value of c_b. ∎

5. The Poisson kernel and symmetric stable random variables

Trees. Recall that the Poisson kernel $P_\theta(x,\omega)$, when $\theta<1/2$, may be regarded as the density with respect to the measure m (normalized, so that $m(x)=1$) of the hitting distribution on $\mathfrak{R}$ of the random walk associated to L_θ. A remarkable analogy with the continuous case stems from the observation that in the special case of the Bruhat–Tits tree of a local field, this distribution is related to precisely one α-stable symmetric random variable with values in a local field. This phenomenon is replicated in the continuous case, as we shall see in the second part of this section.

We suppose now that q is a prime power, $\mathfrak{R}=\mathfrak{F}$ is a local field and $\mathfrak{T}$ is the tree naturally associated to $\mathfrak{F}$. We refer to [BCF] and its references for the definition and properties of a local field and its associated tree. We recall only that a local field is endowed with a norm $|\cdot|$, which satisfies $|ab|=|a|\,|b|$ and the *ultrametric inequality* $|a+b|\le\max(|a|,|b|)\le|a|+|b|$. The norm takes only the values 0 and q^n, with $n\in\mathbb{Z}$. If we fix an element $\mathfrak{p}$ of norm q^{-1}, every element of the field will have a polar decomposition $a=\mathfrak{p}^n u$, with $n\in\mathbb{Z}$ and $|u|=1$. With its norm the local field becomes a complete metric space and a locally compact, non-discrete, totally disconnected field. For a general definition of a stable random variable on a local field we refer to [K]. We use here an ad hoc definition, valid only for *symmetric* random variables, that is, for random variables X such that X and uX have the same distribution whenever $u\in\mathfrak{F}$ is such that $|u|=1$.

DEFINITION 5.1. A symmetric random variable with values on a local field is called *α-stable* if its characteristic function is of the form

$$e^{-c|\xi|^\alpha} \quad \text{for some } c > 0 \text{ and } \alpha > 0.$$

If $c = 1$ the random variable is said to be *in standard form.*

PROPOSITION 5.2 (cf. [BCF]). *Let $\theta = 1/(1 + q^\alpha)$ with $\alpha > 0$, and let $X_1, X_2, \dots$ be a sequence of independent random variables all distributed according to the distribution $P_\theta(x_0, \omega)\, dm(\omega)$, with $\omega \in \mathfrak{F}$. Let $j(n) = \lfloor q^{\alpha n} \rfloor$, the integral part of $q^{\alpha n}$. Then the sequence of random variables*

$$\mathfrak{p}^n(X_1 + \cdots + X_{j(n)})$$

converges in distribution to a symmetric α-stable random variable with values in $\mathfrak{F}$ in standard form.

In the terminology used in the theory of stable random variables or processes, this conclusion may be rephrased by saying that the random variable X is in the *domain of attraction* of the α-stable random variable.

Half-plane

DEFINITION 5.3. A real-valued random variable X such that X and $-X$ have the same distribution (i.e., X is *symmetric*) is said to be *α-stable* if its characteristic function is of the form

$$e^{-c|\xi|^\alpha} \quad \text{for some } c > 0 \text{ and } 0 < \alpha \leq 2.$$

If $c = 1$ the variable X is said to be *in standard form.*

We refer to [GK] for a more general definition and a complete treatment of stable random variables. The result that corresponds to Proposition 5.2 for the case of L_b is the following.

PROPOSITION 5.4 (cf. [BCF] or [BCFY]). *Let $X_1, X_2, \dots$ be real-valued independent random variables distributed according to $P_b(i, \xi)\, d\xi$. Let $\alpha = 1 - 2b$. Then the sequence of random variables*

$$\frac{X_1 + \cdots + X_n}{n^\alpha}$$

converges in distribution to an α-stable symmetric random variable in standard form.

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