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A WEAK TYPE $(1,1)$ ESTIMATE FOR A MAXIMAL OPERATOR ON A GROUP OF ISOMETRIES OF A HOMOGENEOUS TREE

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Dedicated to the memory of Andrzej Hulanicki

Abstract. We give a simple proof of a result of R. Rochberg and M. H. Taibleson that various maximal operators on a homogeneous tree, including the Hardy–Littlewood and spherical maximal operators, are of weak type $(1,1)$. This result extends to corresponding maximal operators on a transitive group of isometries of the tree, and in particular for (nonabelian finitely generated) free groups.

1. Introduction. A *homogeneous tree* $\mathfrak{X}$ *of degree* $q+1$ is defined to be a connected graph with no loops, in which every vertex is adjacent to $q+1$ other vertices. It carries a natural distance function d, namely, $d(x,y)$ is the number of edges between vertices x and y, and a natural measure, the counting measure. The usual Lebesgue space $L^p(\mathfrak{X})$ is thus the set of all complex-valued functions f on $\mathfrak{X}$ such that $\|f\|_p<\infty$, where

$$\|f\|_p=\begin{cases}[\sum_{x\in\mathfrak{X}}|f(x)|^p]^{1/p} & \text{if } 1\le p<\infty,\\ \sup_{x\in\mathfrak{X}}|f(x)| & \text{if } p=\infty.\end{cases}$$

Any weak-star continuous linear operator $\mathcal{K}$ from $L^1(\mathfrak{X})$ to $L^\infty(\mathfrak{X})$ has an associated kernel $K:\mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$ such that

$$\mathcal{K}f(x)=\sum_{y\in\mathfrak{X}}K(x,y)\,f(y)\qquad\forall x\in\mathfrak{X}\ \forall f\in L^1(\mathfrak{X}),$$

which determines and is determined by $\mathcal{K}$. We shall be particularly interested in the invariant operators, i.e., those which commute with the action of the isometry group G of $\mathfrak{X}$. It is easy to see that the condition $\mathcal{K}(f\circ g)=(\mathcal{K}f)\circ g$ for all g in G is equivalent to the condition that $K(g\cdot x,g\cdot y)=K(x,y)$ for all x and y in $\mathfrak{X}$ and g in G, or the condition that $K(x,y)$ depends only on $d(x,y)$.

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Fix an arbitrary reference point o in $\mathfrak{X}$, and denote by G_o the stabiliser of o in the isometry group G. The map $g \mapsto g \cdot o$ identifies $\mathfrak{X}$ with the coset space G/G_o; thus a function f on $\mathfrak{X}$ gives rise to a G_o-invariant function f' on G by the formula $f'(g) = f(g \cdot o)$, and every G_o-invariant function arises in this way. A function f on $\mathfrak{X}$ is called *radial* if $f(x)$ depends only on $d(x, o)$, or equivalently if f is G_o-invariant, or if f' is G_o-bi-invariant. We endow the totally disconnected group G with the Haar measure such that the mass of the open subgroup G_o is 1. Thus

$$\int_G f'(g \cdot o)\, dg = \sum_{x \in \mathfrak{X}} f(x)$$

for all finitely supported functions on $\mathfrak{X}$. The reader can find much more on the group G in the book of A. Figà-Talamanca and C. Nebbia [5].

Suppose that $\mathcal{K}$ is an invariant continuous linear operator from $L^1(\mathfrak{X})$ to $L^\infty(\mathfrak{X})$. We write k' for the function on G such that

$$k'(g) = K(g \cdot o, o) \qquad \forall g \in G.$$

Then $k'(g_1 g g_2) = k'(g)$ for all g in G and g_1, g_2 in G_o, and so there exists a radial function k on $\mathfrak{X}$ such that $k'(g) = k(g \cdot o)$. Further, for f in $L^1(\mathfrak{X})$,

$$\begin{aligned} (1.1) \qquad (\mathcal{K}f)'(g) &= \mathcal{K}f(g \cdot o) = \sum_{y \in \mathfrak{X}} K(g \cdot o, y) f(y) \\ &= \int_G K(g \cdot o, h \cdot o) f(h \cdot o)\, dh = \int_G K(h^{-1} g \cdot o, o) f(h \cdot o)\, dh \\ &= \int_G f'(h) k'(h^{-1} g)\, dh = f' * k'(g) \qquad \forall g \in G. \end{aligned}$$

The study of invariant operators on $\mathfrak{X}$ is thus essentially a part of the harmonic analysis of G, namely the study of operators from $L^p(G/G_o)$ to $L^r(G/G_o)$ given by convolution on the right by G_o-bi-invariant functions.

If Γ is any subgroup of G which acts transitively on $\mathfrak{X}$, then the argument of (1.1) shows also that

$$(\mathcal{K}f)'|_\Gamma = (f'|_\Gamma) *_\Gamma (k'|_\Gamma),$$

where $*_\Gamma$ denotes convolution in the group Γ. Thus our work also includes, for example, the study of convolution of G_o-invariant functions and G_o-bi-invariant functions on $\mathrm{PGL}(2, \mathbb{F})$, where $\mathbb{F}$ is a local field, or the study of convolution of arbitrary functions and "radial" functions on a free group. Since the identifications of G_o-right-invariant and G_o-bi-invariant functions on G with functions and with radial functions on $\mathfrak{X}$ are standard, we shall henceforth usually not distinguish between these, and omit primes.

Our main result is a weak type $(1, 1)$ estimate for radial convolutors k satisfying an estimate of the form $k(x) = O(q^{-|x|})$ for all x in $\mathfrak{X}$. A corresponding result in the case of symmetric spaces of the noncompact type was obtained by J.-O. Strömberg [8]. Our proof is based on the discussion of Strömberg's work by J.-Ph. Anker, E. Damek and Ch. Yacoub [2]. For another version of Strömberg's proof, and more information about free groups, see [1].

This result also follows from a theorem of Rochberg and Taibleson [7], who proved the weak $(1,1)$ boundedness of the Green's operator (the inverse of the Laplacian) for a strongly reversible, not necessarily isotropic, random walk on a tree of bounded degree. Indeed, it is easy to verify that the convolution kernel of the Green's operator on a homogeneous tree of degree $q+1$ is given by

$$k(x) = \frac{q}{q-1}\, q^{-|x|}.$$

However, the proof of Rochberg and Taibleson uses probabilistic methods, and is based on decomposition formulae for the Laplace and Green's operators, and is more difficult than our proof, which is geometric, direct and elementary, yields explicit constants, and emphasises the analogy with real hyperbolic space. Further, by making clear the connections between operators on a tree and operators on a group of isometries of a tree above, we show that the result also holds for groups that act transitively on the tree, such as $\mathrm{PGL}(2,\mathbb{F})$ and the free group.

The main result is proved in Section 3, after we describe the geometry of a homogeneous tree in Section 2.

2. The geometry and boundary of a homogeneous tree. We review here some general facts about the geometry of the tree and two pictures of its boundary.

By $\mathfrak{X}$ we denote a homogeneous tree of degree $q+1$, where $q \geq 2$, with a chosen reference point o. We write $|x|$ for $d(x,o)$, where d is the natural distance on $\mathfrak{X}$. A *geodesic* γ in $\mathfrak{X}$ is a doubly infinite sequence $\{x_n : n \in \mathbb{Z}\}$ of points of $\mathfrak{X}$ such that $d(x_i, x_j) = |i-j|$ for all integers i and j. We say that x lies on γ, and write $x \in \gamma$, if $x = x_n$ for some n in $\mathbb{Z}$. The boundary Ω of $\mathfrak{X}$ is the set of equivalence classes of geodesics, where $\{x_n : n \in \mathbb{Z}\}$ and $\{y_n : n \in \mathbb{Z}\}$ are identified if there exist integers i and j such that $x_n = y_{n+i}$ for all n greater than j. If $\omega \in \Omega$, we write $\omega = \lim_{n\to\infty} x_n$ if $\{x_n : n \in \mathbb{Z}\}$ lies in the equivalence class of ω, and $\omega = \lim_{n\to-\infty} x_n$ if $\{x_{-n} : n \in \mathbb{Z}\} \in \omega$. We use interval notation: $[x,y]$ denotes the geodesic interval between x and y in $\mathfrak{X}$, including the endpoints, $[x,\omega)$ denotes the set of points $\{x_n : n \geq 0\}$, where $\{x_n : n \in \mathbb{Z}\}$ is a geodesic such that $x_0 = x$ and $\lim_{n\to\infty} x_n = \omega$, and so on. The boundary gives rise to a natural compactification of the discrete topological space $\mathfrak{X}$. A basis for the topology of $\mathfrak{X} \cup \Omega$ is given by the point

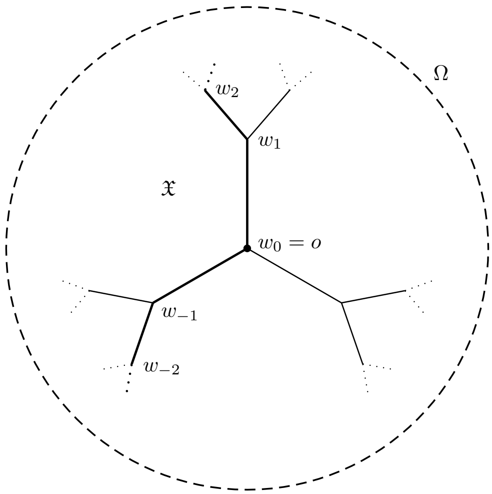

Fig. 1. The tree and a geodesic

sets $\{x\}$, where $x \in \mathfrak{X}$, together with the sets $\mathfrak{T}_x$ given by the rule

$$\mathfrak{T}_x = \bigcup_{\substack{\omega\in\Omega\\ x\in[o,\omega]}} [x,\omega].$$

It is possible to endow Ω with a natural probability measure σ such that $\sigma(\Omega \cap \mathfrak{T}_x) = (q-1)^{1-|x|}q^{-1}$ for all x in $\mathfrak{X} \setminus \{o\}$.

We fix an arbitrary infinite geodesic $\{\ldots, w_{-2}, w_{-1}, w_0, w_1, w_2, \ldots\}$ such that $w_0 = o$, so that $|w_j| = |j|$. We denote this geodesic by γ_0, and by ω_0 the boundary point $\lim_{n\to\infty} w_n$. A radial function f on $\mathfrak{X}$ is determined by its restriction to γ_0.

The height function h on $\mathfrak{X}$ is defined by the formula

$$h(x) = \lim_{m\to\infty} (d(o,w_m) - d(x,w_m)) \qquad \forall x \in \mathfrak{X}.$$

Define the *horocycle* $\mathfrak{H}_n$ to be the set $\{x \in \mathfrak{X} : h(x) = n\}$; then $\mathfrak{X}$ is the disjoint union of the horocycles $\mathfrak{H}_n$, for n in $\mathbb{Z}$.

For ω in $\Omega \setminus \{\omega_0\}$, denote by γ_ω the unique geodesic $\{x_n : n \in \mathbb{Z}\}$ such that

$$\lim_{n\to\infty} d(x_n, w_n) = 0, \qquad \lim_{n\to-\infty} x_n = \omega.$$

The horocycle decomposition is the analogue of the upper half space realisation of hyperbolic space. Pushing this analogy one step further, we define a second measure ρ on Ω, adapted to the horocyclical decomposition of Ω. Given x in $\mathfrak{X}$, we define Ω_x to be the compact open subset of Ω of points ω

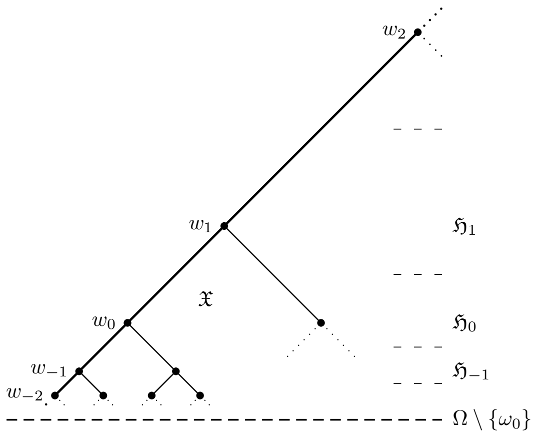

Fig. 2. The rooted tree and some horocycles

such that γ_ω passes through x, i.e.,

$$\Omega_x = \{\omega \in \Omega : x \in (\omega_0, \omega)\}.$$

Note that, for every integer n,

$$\bigcup_{x \in \mathfrak{H}_n} \Omega_x = \Omega \setminus \{\omega_0\}.$$

Define $\rho(\{\omega_0\}) = 0$ and $\rho(\Omega_x) = q^{h(x)}$; then it is simple to check that ρ extends uniquely to a Borel measure, still denoted by ρ, on Ω with the property that $\rho(\{\omega\}) = 0$ for all ω in Ω.

3. Weak $(1,1)$ estimates and multipliers. We denote by $\#\mathfrak{E}$ the cardinality of a subset $\mathfrak{E}$ of $\mathfrak{X}$.

THEOREM 3.1. *Suppose that the function $k\colon \mathfrak{X} \to \mathbb{C}$ is radial and also that $|k(x)| \le A\, q^{-|x|}$ for all x in $\mathfrak{X}$. Then the operator $\mathcal{K}$ of right convolution with k is of strong type (p,p) for all p in $(1,\infty)$, and of weak type $(1,1)$. More precisely,*

$$\#\{x \in \mathfrak{X} : |\mathcal{K}f(x)| > \lambda\} \le 4A\, \frac{q+1}{q-1}\, \frac{\|f\|_1}{\lambda} \qquad \forall f \in L^1(\mathfrak{X})\ \forall \lambda \in \mathbb{R}^+.$$

Proof. Since k is in $L^r(\mathfrak{X})$ when $r > 1$, the radial form of the Kunze–Stein phenomenon, proved by C. Nebbia [6] for isometry groups of trees, implies that $\mathcal{K}$ is of strong type (p,p) for all p in $(1,\infty)$ (a sharper version of this, involving Lorentz spaces, is proved in another paper of the authors [3]).

The hard part of the proof is to show that $\mathcal{K}$ is of weak type $(1,1)$. Since $\mathcal{K}f \le A\,\mathcal{K}^\sharp|f|$, where $\mathcal{K}^\sharp$ is the operator corresponding to the kernel $k^\sharp$, given by $q^{-|\cdot|}$, we may assume that $k = k^\sharp$ and $\mathcal{K} = \mathcal{K}^\sharp$.

We use the horocyclic decomposition of $\mathfrak{X}$ and the measure ρ on the boundary Ω, described in Section 2. Since $\omega \in \Omega_x$ if and only if x lies on γ_ω, given f in $L^1(\mathfrak{X})$ we may use Fubini's theorem to write

$$\begin{aligned}\sum_{x\in\mathfrak{X}} f(x) &= \sum_{x\in\mathfrak{X}} \int_{\Omega_x} \rho(\Omega_x)^{-1} f(x)\, d\rho(\omega) \\ &= \int_{\Omega} \sum_{x\in\gamma_\omega} q^{-h(x)} f(x)\, d\rho(\omega).\end{aligned} \tag{3.1}$$

In particular, $\sum_{x\in\gamma_\omega} q^{-h(x)}|f(x)|$ is finite for ρ-almost every ω if f is in $L^1(\mathfrak{X})$.

We now define operators $\mathcal{G}$ and $\mathcal{B}$ on $L^1(\mathfrak{X})$ by the rule

$$\mathcal{G}f(x) = \sum_{\substack{y\in\mathfrak{X}\\ h(y)\le h(x)}} q^{-d(x,y)} f(y) \quad \text{and} \quad \mathcal{B}f(x) = \sum_{\substack{y\in\mathfrak{X}\\ h(y)> h(x)}} q^{-d(x,y)} f(y)$$

for all x in $\mathfrak{X}$. Then

$$\mathcal{K}f(x) = \sum_{y\in\mathfrak{X}} q^{-d(x,y)} f(y) = \mathcal{G}f(x) + \mathcal{B}f(x) \qquad \forall x \in \mathfrak{X}.$$

We claim that $\mathcal{G}$ is bounded on $L^1(\mathfrak{X})$ with norm at most $\frac{q+1}{q-1}$. Indeed,

$$\|\mathcal{G}f\|_1 \le \sum_{x\in\mathfrak{X}} \sum_{\substack{y\in\mathfrak{X}\\ h(y)\le h(x)}} q^{-d(x,y)}|f(y)| = \sum_{y\in\mathfrak{X}} \sum_{\substack{x\in\mathfrak{X}\\ h(x)\ge h(y)}} q^{-d(x,y)}|f(y)|.$$

To compute the inner sum, we write

$$\sum_{\substack{x\in\mathfrak{X}\\ h(x)\ge h(y)}} q^{-d(x,y)} = \sum_{H\in\mathbb{N}} \sum_{\substack{x\in\mathfrak{X}\\ h(x)= h(y)+H}} q^{-d(x,y)}.$$

It is easy to verify, for every nonnegative integer H and y in $\mathfrak{X}$, that if $h(x) = h(y) + H$, then $d(x, y) - H \in \{0, 2, 4, \ldots\}$, and

$$\#\{x \in \mathfrak{X} : h(x) = h(y) + H,\ d(x, y) = 2j + H\} = 1$$

if $j = 0$, while if $j \in \mathbb{Z}^+$, then

$$\#\{x \in \mathfrak{X} : h(x) = h(y) + H,\ d(x, y) = 2j + H\} = (q - 1)q^{j-1}.$$

Therefore

$$\sum_{H\in\mathbb{N}} \sum_{\substack{x\in\mathfrak{X}\\ h(x)= h(y)+H}} q^{-d(x,y)} = \sum_{H\in\mathbb{N}} q^{-H}\Big(1 + \sum_{j\in\mathbb{Z}^+} (q-1)\, q^{-j-1}\Big) = \frac{q+1}{q-1},$$

and

$$\|\mathcal{G}f\|_1 \le \frac{q+1}{q-1}\, \|f\|_1,$$

thus proving our claim. We remark for future use that the calculation above also shows that the operator $\mathcal{G}_0$, defined by

$$\mathcal{G}_0 f(x) = \sum_{\substack{y\in\mathfrak{X}\\ h(y)=h(x)}} q^{-d(x,y)} f(y),$$

is bounded on $L^1(\mathfrak{X})$ with norm at most $(q+1)/q$.

The heart of the argument consists in proving that $\mathcal{B}$ satisfies a weak $(1,1)$ estimate. Fix λ in $\mathbb{R}^+$, and denote by $\mathfrak{E}$ the set $\{x\in\mathfrak{X} : |\mathcal{B}f(x)|>\lambda\}$. It follows from (3.1) that

$$\#\mathfrak{E} = \sum_{x\in\mathfrak{X}} \chi_{\mathfrak{E}}(x) = \sum_{x\in\mathfrak{X}} \int_{\Omega_x} q^{-h(x)} \chi_{\mathfrak{E}}(x)\, d\rho(\omega) = \int_{\Omega} \sum_{x\in\gamma_\omega} q^{-h(x)} \chi_{\mathfrak{E}}(x)\, d\rho(\omega).$$

We claim that for all ω in $\Omega\setminus\{\omega_0\}$,

$$\sum_{x\in\gamma_\omega} q^{-h(x)} \chi_{\mathfrak{E}}(x) \le \frac{q}{q-1}\,\frac{1}{\lambda} \sum_{x\in\gamma_\omega} q^{-h(x)} |\mathcal{G}_0 f(x)|.$$

This claim implies that

$$\begin{aligned}\#\mathfrak{E} &\le \frac{q}{q-1}\,\frac{1}{\lambda} \int_{\Omega} \sum_{x\in\gamma_\omega} q^{-h(x)} |\mathcal{G}_0 f(x)|\, d\rho(\omega)\\ &= \frac{q}{q-1}\,\frac{\|\mathcal{G}_0 f\|_1}{\lambda} \le \frac{q+1}{q-1}\,\frac{\|f\|_1}{\lambda},\end{aligned}$$

by (3.1) and our estimate for the operator norm of $\mathcal{G}_0$, which establishes the weak $(1,1)$ boundedness of $\mathcal{B}$.

To prove the claim, we fix ω in $\Omega\setminus\{\omega_0\}$, and recall that the points y_n of γ_ω are indexed so that $h(y_n)=n$. We also note that $d(y_n,y)=d(y_{n+H},y)+H$ whenever $y_n\in\gamma_\omega$, $h(y)=n+H$, and $H\in\mathbb{N}$. Therefore

$$\begin{aligned}\mathcal{B}f(y_n) &= \sum_{H\in\mathbb{Z}^+} \sum_{\substack{y\in\mathfrak{X}\\ h(y)=n+H}} q^{-d(x,y)} f(y)\\ &= \sum_{H\in\mathbb{Z}^+} q^{-H} \mathcal{G}_0 f(y_{n+H}) = \sum_{H\in\mathbb{Z}^+} q^{-H} F_\omega(n+H),\end{aligned}$$

where $F_\omega(n)=\mathcal{G}_0 f(y_n)$.

Thus, denoting by $W(\mathbb{Z})$ the L^1-space constructed relative to the measure m on $\mathbb{Z}$ given by

$$m(E) = \sum_{n\in E} q^{-n} \qquad \forall E\subset\mathbb{Z},$$

and by $\widetilde{\mathcal{B}}$ the operator on $W(\mathbb{Z})$ defined by

$$\widetilde{\mathcal{B}}F(n) = \sum_{H\in\mathbb{Z}^+} q^{-H} F(n+H) \qquad \forall F\in W(\mathbb{Z}),$$

our claim is equivalent to showing that, for all F in $W(\mathbb{Z})$,

$$m(\{n \in \mathbb{Z} : |\widetilde{\mathcal{B}}F(n)| > \lambda\}) \le \frac{q}{q-1} \frac{\|F\|_W}{\lambda}.$$

Now

$$|\widetilde{\mathcal{B}}F(n)| = \Big|\sum_{H\in\mathbb{Z}^+} q^{-H}\, F(n+H)\Big| = q^n \Big|\sum_{H\in\mathbb{Z}^+} q^{-(n+H)} F(n+H)\Big| \le q^n \|F\|_W,$$

which yields

$$\{n \in \mathbb{Z} : |\widetilde{\mathcal{B}}F(n)| > \lambda\} \subseteq \{n \in \mathbb{Z} : n \ge \log_q(\lambda/\|F\|_W)\},$$

and therefore

$$\begin{aligned} m(\{n \in \mathbb{Z} : |\widetilde{\mathcal{B}}F(n)| > \lambda\}) &\le \sum_{n\ge \log_q(\lambda/\|F\|_W)} q^{-n} \\ &= q^{-n_0} \frac{q}{q-1} \le \frac{q}{q-1} \frac{\|F\|_W}{\lambda}, \end{aligned}$$

where $n_0 = \min\{n \in \mathbb{Z} : n \ge \log_q(\lambda/\|F\|_W)\}$, and the proof of the claim is complete.

Summing up, we have

$$\begin{aligned} &\#\{x \in \mathfrak{X} : |\mathcal{K}f(x)| > \lambda\} \\ &\qquad \le \#\{x \in \mathfrak{X} : |\mathcal{G}f(x)| > \lambda/2\} + \#\{x \in \mathfrak{X} : |\mathcal{B}f(x)| > \lambda/2\} \\ &\qquad \le 4\, \frac{q+1}{q-1} \frac{\|f\|_1}{\lambda}, \end{aligned}$$

as required. ∎

This result depends crucially on the fact that the tree has exponential volume growth, and therefore that the degree is greater than or equal to three, or equivalently, $q \ge 2$. The constant in our proof becomes infinite when $q = 1$, and it is easy to verify that the analogous operator on $\mathbb{Z}$, defined by convolution with $(1+|\cdot|)^{-1}$, is not of strong type (p,p) for any p in $(1,\infty)$, nor of weak type $(1,1)$.

To conclude, we need more notation. We write $\mathfrak{S}_n$ and $\mathfrak{B}_n$ for the subsets $\{x \in \mathfrak{X} : |x| = n\}$ and $\{x \in \mathfrak{X} : |x| \le n\}$ of $\mathfrak{X}$. Clearly $\#\mathfrak{S}_0 = 1$, while $\#\mathfrak{S}_n = (q+1)q^{n-1}$ when $n \in \mathbb{Z}^+$, so $\#\mathfrak{S}_n \ge q^n$ for all n in $\mathbb{N}$. We also denote by $\mathfrak{M}$ the class of all radial probability measures on $\mathfrak{X}$.

Corollary 3.2. *The operator $\mathcal{M}_G$, defined by*

$$\mathcal{M}_G f(x) = \sup_{\nu\in\mathfrak{M}} |f| * \nu(x) \qquad \forall x \in \mathfrak{X},$$

is of strong type (p,p) for every p in $(1,\infty)$, and of weak type $(1,1)$. In *particular, the same holds for the operators* $\mathcal{M}_0$, $\mathcal{M}_{HL}$, *and* $\mathcal{M}_S$, *defined by*

$$\mathcal{M}_0 f(x) = \sup_{n\in\mathbb{N}} |f| * \nu_1^{(*n)},$$

$$\mathcal{M}_{HL} f(x) = \sup_{n\in\mathbb{N}} \frac{1}{\#\mathfrak{B}_n} \sum_{\substack{y\in\mathfrak{X}\\ d(x,y)\le n}} |f(y)|,$$

$$\mathcal{M}_S f(x) = \sup_{n\in\mathbb{N}} \frac{1}{\#\mathfrak{S}_n} \sum_{\substack{y\in\mathfrak{X}\\ d(x,y)= n}} |f(y)|,$$

where ν_1 *is the radial probability measure concentrated on* $\mathfrak{S}_1$, *and* $\nu_1^{(*n)}$ *denotes the nth convolution power of* ν_1.

Proof. Take ν in $\mathfrak{M}$ and f in $L^1(\mathfrak{X})$. Write ν as $\sum_{d\in\mathbb{N}} a_d\chi_d$, where $a_d \ge 0$. Then $\sum_{d\in\mathbb{N}} a_d \#\mathfrak{S}_d = 1$, whence $a_d\#\mathfrak{S}_d \le 1$ and so $a_d \le q^{-d}$ for all $d\in\mathbb{N}$. It follows that

$$\begin{aligned} |f| * \nu(x) &= \sum_{d\in\mathbb{N}} a_d\,(|f| * \chi_d)(x) \le \sum_{d\in\mathbb{N}} q^{-d} \sum_{\substack{y\in\mathfrak{X}\\ d(x,y)=d}} |f|(y) \\ &= \mathcal{K}|f|(x). \end{aligned}$$

Taking the supremum over ν in $\mathfrak{M}$, we see that

$$\mathcal{M}_G f(x) \le \sup_{\nu\in\mathfrak{M}} |f| * \nu(x) \le \mathcal{K}|f|(x),$$

and the corollary follows. ∎

An application of Theorem 3.1 above to multipliers for the spherical Fourier transform may be found in [4, Theorem 2.2(iii)].

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