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BANACH ALGEBRAS ASSOCIATED WITH LAPLACIANS ON SOLVABLE LIE GROUPS AND INJECTIVITY OF THE HARISH-CHANDRA TRANSFORM

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Dedicated to Andrzej Hulanicki from whom I learnt many things —as a mathematician and as a human being

Abstract. For any connected Lie group G and any Laplacian $\Lambda = X_1^2 + \cdots + X_n^2 \in \mathfrak{U}\mathfrak{g}$ ($X_1, \ldots, X_n$ being a basis of $\mathfrak{g}$) one can define the commutant $\mathfrak{B} = \mathfrak{B}(\Lambda)$ of Λ in the convolution algebra $\mathcal{L}^1(G)$ as well as the commutant $\mathfrak{C}(\Lambda)$ in the group C^*-algebra $C^*(G)$. Both are involutive Banach algebras. We study these algebras in the case of a "distinguished Laplacian" on the "Iwasawa part AN" of a semisimple Lie group. One obtains a fairly good description of these algebras by objects derived from the semisimple group. As a consequence one sees that both algebras are commutative (which is not immediate from the definition), $\mathfrak{B}$ is C^*-dense in $\mathfrak{C}$, and $\mathfrak{B}$ is a completely regular symmetric Wiener algebra. As a byproduct of our approach we give another proof of the injectivity of Harish-Chandra's spherical Fourier transform, which is based on a theorem on C^*-algebras of solvable Lie groups (due to N. V. Pedersen). The article closes with some open questions for more general solvable Lie groups. To some extent the article is written with a view to these questions, that is, we try to apply, as much as possible (at the moment), methods which work also outside the semisimple context.

1. Introduction. On any connected Lie group G one may study Laplace operators: For any basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{g}$ of G one may form the operator $f \mapsto Lf = (X_1^2 + \cdots + X_n^2)(f)$, which, of course, depends on the chosen basis. At least since the seminal work of E. Nelson and W. Stinespring [22, 23], in particular on analytic vectors in representation spaces, those operators were intensely studied, for instance as regards the asymptotic behaviour of the associated heat kernels p_t, "$p_t = e^{tL}$", $t > 0$, or functional calculus on L.

A. Hulanicki and his school made important contributions to this circle of questions. The present article is closely related to one of his ideas. In [16]

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A. Hulanicki introduced the commutative closed subalgebra of $\mathcal{L}^1(G)$ which is generated by the kernels p_t, $t > 0$. This algebra was studied in several subsequent articles. A couple of years ago we investigated another, in general larger subalgebra of $\mathcal{L}^1(G)$, the "commutant" of L (see [27]), which can be defined directly by Λ, that is, without knowing the heat kernels. For the convenience of the reader we briefly recall the relevant definitions.

Any $X \in \mathfrak{g}$, considered as an element of the tangent space of G at the origin, acts on smooth functions f on G via

$$(X * f)(y) = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tX)y),$$
$$(f * X)(y) = \frac{d}{dt} f(y \exp(-tX)) + \operatorname{tr}\operatorname{ad}(X) f(y)$$

for $y \in G$, thus defining right-invariant (in the first case) and left-invariant vector fields; $\operatorname{tr}\operatorname{ad}(X)$ denotes the trace of $\mathfrak{g} \ni Y \mapsto [X, Y] \in \mathfrak{g}$. If, as usual, an involution is defined by $f^*(y) = \delta(y)^{-1}\,\overline{f(y^{-1})}$, where δ denotes the modular function of G, one has

$$(f * X)^* = -X * f^*.$$

Observe that $\delta(\exp X) = e^{-\operatorname{tr}\operatorname{ad}(X)}$. Both actions extend to the universal enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$.

Definition 1.1. If $\Lambda := X_1^2 + \cdots + X_n^2 \in \mathfrak{U}\mathfrak{g}$ for a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ then the subalgebra $\mathfrak{B} = \mathfrak{B}(\Lambda)$ consists of all $f \in \mathcal{L}^1(G)$ such that $\Lambda * f = f * \Lambda$; this is short for the equations $(\varphi * \Lambda) * f * \psi = \varphi * f * (\Lambda * \psi)$ which have to hold for all $\varphi, \psi \in \mathcal{D}(G) = C_c^\infty(G)$. Evidently, $\mathfrak{B}$ is a closed involutive subalgebra of $\mathcal{L}^1(G)$ containing the heat kernels p_t.

Likewise one may form the commutant of Λ in the C^*-hull $C^*(G)$ of $\mathcal{L}^1(G)$.

Definition 1.2. If Λ is as above then the closed involutive subalgebra $\mathfrak{C} = \mathfrak{C}(\Lambda)$ of $C^*(G)$ consists of all $f \in C^*(G)$ with $\Lambda * f = f * \Lambda$. (See Prop. 3.1 in [27] for a more detailed discussion.)

For the $(ax + b)$-group and the Heisenberg group these algebras $\mathfrak{B}$ and $\mathfrak{C}$ were studied in detail in [27]. Parts of the results obtained will be used here, where we study the case of so-called distinguished Laplacians. They are certain Laplacians on the "AN-part" of a semisimple Lie group (see below for a precise definition), and they were already studied by A. Hulanicki and others (cf. e.g. [4, 5, 11, 13]).

Also in this case we get a fairly good picture of the algebras $\mathfrak{B}(\Lambda)$ and $\mathfrak{C}(\Lambda)$ expressed by some objects derived from the semisimple group which was taken as point of departure (cf. Proposition 4.7 and Theorem 4.8 below). As a byproduct of our approach we obtain another proof of the injectivity of the Harish-Chandra transform, actually on a space which is somewhat bigger than Harish-Chandra's Schwartz space. Interestingly enough, a basic tool is a theorem (due to Niels Vigand Pedersen [25]) on C^*-algebras of certain solvable Lie groups. This proof was already formulated in the (unpublished) manuscript [26] where we focused on this aspect. At several points the present article goes beyond that manuscript.

2. A criterion for $T : C^*(G) \to C^*(G/N)$ being injective on $\mathfrak{C}(\Lambda)$. In this section let the Lie algebra $\mathfrak{g}$ be the semidirect product of an abelian subalgebra $\mathfrak{a}$ and a nilpotent ideal $\mathfrak{n}$, such that the action of $\mathfrak{a}$ on $\mathfrak{n}$ is diagonalizable: There is a basis $X_1, \ldots, X_r$ of $\mathfrak{n}$ (fixed for the rest of this section), and there are $\alpha_1, \ldots, \alpha_r \in \mathfrak{a}' = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$ with $[H, X_j] = \alpha_j(H)X_j$ for $H \in \mathfrak{a}$ and $j = 1, \ldots, r$. For a chosen basis $A_1, \ldots, A_\ell$ of $\mathfrak{a}$ (also fixed for the rest of this section) we form, as above, $\Lambda = A_1^2 + \cdots + A_\ell^2 + X_1^2 + \cdots + X_r^2 \in \mathfrak{U}\mathfrak{g}$. By capital letters A, N, G we denote the simply connected Lie groups corresponding to $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{g}$. One has $G = A \ltimes N$, $G/N \cong A$. By integration over N, $(Tf)(a) = \int_N f(ax)\, dx$, where dx is a chosen Haar measure on N, we obtain a surjective map $T : \mathcal{L}^1(G) \to \mathcal{L}^1(A)$. This map extends to a map $C^*(G) \to C^*(A)$ denoted by the same letter. The algebra $C^*(A)$ is, via Fourier transform, isomorphic to $C_\infty(\widehat{A})$ where $\widehat{A} \cong \mathfrak{a}'$ denotes the Pontryagin dual of A.

Proposition 2.1. *In addition to the above assume that there is an element $A_0 \in \mathfrak{a}$ such that $\alpha_j(A_0) > 0$ for $j = 1, \ldots, r$. Let π be a continuous irreducible unitary representation of G in the Hilbert space $\mathfrak{H}$. By differentiation, $\pi(X) = \frac{d}{dt}\big|_{t=0}\pi(\exp tX)$, $X \in \mathfrak{g}$, π yields a representation of $\mathfrak{U}\mathfrak{g}$, and Λ leads to a self-adjoint operator $\pi(\Lambda)$, unbounded in general, on $\mathfrak{H}$* (cf. [23]). *If $\pi(\Lambda)$ happens to have an eigenvector then necessarily π factors through $G \to G/N = A$. The latter property is equivalent to π being one-dimensional (or finite-dimensional). In other words, $\pi(\Lambda)$ has eigenvectors for no infinite-dimensional irreducible representation π.*

Proof. Suppose that there is a non-zero ξ in the domain of $\pi(\Lambda)$ such that $\pi(\Lambda)\xi = \lambda\xi$ for some (real) scalar λ. For short, put $a_t = \exp(tA_0)$ and $\xi_t = \pi(a_t)\xi$. One has

$$\lambda\xi_t = \lambda\pi(a_t)\xi = \pi(a_t)\pi(\Lambda)\pi(a_t)^{-1}\pi(a_t)\xi = \pi(\Lambda_t)\xi_t$$

with

$$\Lambda_t = \mathrm{Ad}(a_t)(\Lambda) = A_1^2 + \cdots + A_l^2 + \sum_{j=1}^{r} e^{2t\alpha_j(A_0)}X_j^2 = \Delta + \sum_{j=1}^{r} e^{2t\alpha_j(A_0)}X_j^2.$$

For the following computations observe that ξ and ξ_t are in the domain of a Laplacian, which by [22, Lemma 6.1] dominates all operators derived from linear or quadratic expressions in $\mathfrak{U}\mathfrak{g}$. Plugging into $\langle \lambda\xi, \xi_t\rangle = \langle \xi, \lambda\xi_t\rangle$ the two eigenvalue equations one obtains

$$\Big\langle \pi(\Delta)\xi + \sum_{j=1}^{r} \pi(X_j^2)\xi, \xi_t \Big\rangle = \Big\langle \xi, \pi(\Delta)\xi_t + \sum_{j=1}^{r} e^{2t\alpha_j(A_0)}\pi(X_j)^2\xi_t \Big\rangle,$$

whence

$$\Big\langle \sum_{j=1}^{r} \pi(X_j^2)\xi, \xi_t \Big\rangle = \Big\langle \sum_{j=1}^{r} e^{2t\alpha_j(A_0)}\pi(X_j^2)\xi, \xi_t \Big\rangle,$$

or

$$\sum_{j=1}^{r} (e^{2t\alpha_j(A_0)} - 1)\langle \pi(X_j^2)\xi, \xi_t\rangle = 0.$$

The latter equation holds true for all real t. Dividing by $t \neq 0$ and taking the limit for $t \to 0$ yields

$$\sum_{j=1}^{r} 2\alpha_j(A_0)\langle \pi(X_j^2)\xi, \xi\rangle = 0.$$

As $\alpha_j(A_0) > 0$ and $\langle \pi(X_j^2)\xi, \xi\rangle \leq 0$ for all j we conclude that $\pi(X_j)\xi = 0$ for all j, which implies that ξ is in the space $\mathfrak{H}^N$ of N-fixed vectors. But N being normal in G, the latter space is G-invariant, hence $\mathfrak{H}^N = \mathfrak{H}$ as π is irreducible. Therefore, π factors through $G \to G/N = A$, as was claimed. ∎

The above result can also be expressed in terms of bounded operators. By a result of Nelson and Stinespring, [23], the closure $\Lambda^{(1)}$ of the operator $f \mapsto (\mathrm{Id} - \Lambda) * f$, $f \in \mathcal{D}(G)$, in $\mathcal{L}^1(G)$ has an inverse: there is an $\mathcal{L}^1$-function k_1 such that $\Lambda^{(1)}(k_1 * f) = f$ for all $f \in \mathcal{L}^1(G)$, and $k_1 * \Lambda^{(1)}(g) = g$ for all g in the domain of $\Lambda^{(1)}$. The above proposition can be rephrased by saying that $\pi(k_1)$ has eigenvectors for no infinite-dimensional irreducible unitary representation π of G.

Theorem 2.2. *The restriction of $T : C^*(G) \to C^*(A)$ to $\mathfrak{C}(\Lambda)$ is one-to-one. A fortiori, $T : \mathcal{L}^1(G) \to \mathcal{L}^1(A)$ is injective on $\mathfrak{B}(\Lambda)$.*

Remark 2.3. The theorem implies in particular that $\mathfrak{C}(\Lambda)$ and $\mathfrak{B}(\Lambda)$ are commutative, which is not obvious from their definition.

Proof. We first note that clearly $\mathfrak{C}$ may also be described as $\mathfrak{C} = \{f \in C^*(G) \mid f * k_1 = k_1 * f\}$, with k_1 as above. For the following argument it is crucial that the postliminal C^*-algebra $C^*(G)$ (for the definition of postliminal C^*-algebras see [8]) has in fact a *finite* composition series as was proved by N. V. Pedersen [25], that is, there is a sequence $0 = J_{n+1} \subsetneq J_n \subsetneq \cdots \subsetneq J_1 \subsetneq J_0 = C^*(G)$ of closed two-sided involutive ideals in $C^*(G)$ such that for each k, $0 \leq k \leq n$, each irreducible involutive representation π of

J_k/J_{k+1} in the Hilbert space $\mathfrak{H}_\pi$ maps J_k onto the algebra of compact operators of $\mathfrak{H}_\pi$. For further reading on the representation theory of exponential groups we recommend two articles of B. Currey [6, 7], as well as the book [21] by H. Leptin and J. Ludwig.

Clearly the above composition series may be chosen such that J_1 is just the kernel of T. Assume, contrary to our claim, that $J_1 \cap \mathfrak{C} = \ker T \cap \mathfrak{C} \neq 0$. Let $k \le n+1$ be the largest number such that $J_k \cap \mathfrak{C} \neq 0$ (hence $1 \le k \le n$). If a is any non-zero element in $J_k \cap \mathfrak{C}$ then $b := a^*a$ is also in $J_k \cap \mathfrak{C}$ and different from zero. Even the image $b^\bullet$ of b under the quotient map $J_k \to J_k/J_{k+1}$ is different from zero because otherwise b would be contained in $\mathfrak{C} \cap J_{k+1}$, which is zero by the maximality of k. Hence there exists an irreducible involutive representation π of J_k/J_{k+1} in the Hilbert space $\mathfrak{H}$ with $\pi(b^\bullet) \neq 0$. The representation π extends uniquely to a representation of $C^*(G)$, denoted by the same letter π, which vanishes on J_{k+1}. By the properties of composition series, $\pi(b^\bullet) = \pi(b)$ is a compact self-adjoint operator in $\mathfrak{H}$, hence there exists a non-zero λ such that the eigenspace $\mathfrak{H}^\lambda = \{\xi \in \mathfrak{H} \mid \pi(b)\xi = \lambda\xi\}$ is non-zero and finite-dimensional. As b commutes with k_1, the operators $\pi(b)$ and $\pi(k_1)$ commute as well. Therefore, $\mathfrak{H}^\lambda$ is invariant under $\pi(k_1)$, and there exist eigenvectors for $\pi(k_1)$ in $\mathfrak{H}^\lambda$, which implies by Proposition 2.1 (and the observation preceding the theorem) that π is trivial on $\ker T = J_1$. This contradicts $J_k \subset J_1$ and $\pi(J_k) \neq 0$, whence our assumption $J_1 \cap \mathfrak{C} \neq 0$ was false. ■

For later use we include a technical lemma on a transformation law for Λ under positive characters. Each positive character on A is given by a linear functional $\sigma \in \mathfrak{a}'$, namely $\widetilde{\sigma}(\exp X) = e^{\sigma(X)}$ defines a positive character on A, and hence on $G = A \ltimes N$. The chosen basis $A_1, \dots, A_\ell$ of $\mathfrak{a}$ implicitly yields an Euclidean structure $\langle\,,\rangle$ on $\mathfrak{a} : A_1, \dots, A_\ell$ is assumed to be orthonormal. Using $\langle\,,\rangle$ we will occasionally identify $\mathfrak{a}$ with $\mathfrak{a}'$ in the usual manner: for $\beta \in \mathfrak{a}'$ we have $H_\beta \in \mathfrak{a}$ such that $\beta(X) = \langle H_\beta, X\rangle$ for all $X \in \mathfrak{a}$. In a later application we shall also need the right action of $\mathfrak{g}$ (or $\mathfrak{U}\mathfrak{g}$) on smooth functions on G without the correction term $\operatorname{tr}\operatorname{ad}$.

For $X \in \mathfrak{g}$ and a smooth function f on G we define

$$(f *' X)(y) = \frac{d}{dt}\bigg|_{t=0} f(y\exp(-tX)).$$

Lemma 2.4. *For $\sigma \in \mathfrak{a}'$ with associated character $\widetilde{\sigma} \in \operatorname{Hom}(G, \mathbb{R}_+)$ the transformed operator $f \mapsto \widetilde{\sigma}(\Lambda * (\widetilde{\sigma}^{-1} f))$ can be written as*

$$\widetilde{\sigma}(\Lambda * (\widetilde{\sigma}^{-1} f)) = \Lambda * f + 2H_\sigma * f + \|H_\sigma\|^2 f.$$

Further, we have

$$\widetilde{\sigma}((\widetilde{\sigma}^{-1} f) * \Lambda) = f *' \Lambda + 2f *' H_{\sigma+\operatorname{tr}\operatorname{ad}} + \|H_{\sigma+\operatorname{tr}\operatorname{ad}}\|^2 f.$$

Proof. Evidently, only the $\mathfrak{a}$-part of Λ contributes to the transformation law. For $H \in \mathfrak{a}$ one has $\widetilde{\sigma}(H * (\widetilde{\sigma}^{-1}f)) = H * f + \sigma(H)f_0$, as is easily computed. Applying this formula twice one obtains

$$\widetilde{\sigma}(H^2 * (\widetilde{\sigma}^{-1}f)) = H^2 * f + 2\sigma(H)H * f + \sigma(H)^2 f,$$

from which we conclude that

$$\begin{aligned}\widetilde{\sigma}(\Lambda * (\widetilde{\sigma}^{-1}f)) &= \sum_{j=1}^{r} X_j^2 * f + \sum_{k=1}^{\ell} A_k^2 * f \\ &\quad + 2\sum_{k=1}^{r} \sigma(A_k)A_k * f + \sum_{k=1}^{r} \sigma(A_k)^2 f \\ &= \Lambda * f + 2H_\sigma * f + \|H_\sigma\|^2 f.\end{aligned}$$

Also for the action from the right we have

$$\widetilde{\sigma}((\widetilde{\sigma}^{-1}f) * H) = f * H + \sigma(H)f,$$

yielding

$$\widetilde{\sigma}((\widetilde{\sigma}^{-1}f) * H) = f *' H + (\sigma + \operatorname{tr}\operatorname{ad})(H)f,$$

which gives

$$\widetilde{\sigma}((\widetilde{\sigma}^{-1}f) * H^2) = f *' H^2 + 2(\sigma + \operatorname{tr}\operatorname{ad})(H)f *' H + (\sigma + \operatorname{tr}\operatorname{ad})(H)^2 f.$$

Summation, similar to the above, leads to the claimed formula. ∎

3. Distinguished Laplacians, notations. We recall the construction of a distinguished Laplacian on the Iwasawa part AN of a semisimple Lie group S with finite center, and we introduce several notations. Denote, as usual, by $\mathfrak{s}$ the Lie algebra of S, and by Θ a fixed Cartan involution on $\mathfrak{s}$. Decompose $\mathfrak{s}$ into the Θ-eigenspaces, $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$, where the $(+1)$-eigenspace $\mathfrak{k}$ corresponds to a maximal compact subgroup K of S. Further, choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Then $\mathfrak{s}$ decomposes into $\mathfrak{a}$-eigenspaces,

$$\mathfrak{s} = \mathfrak{s}^0 \oplus \sum_{\alpha \in R} \mathfrak{s}^\alpha,$$

where R is a subset of $\mathfrak{a}' \setminus \{0\}$, and $\mathfrak{s}^0 = \mathfrak{a} + \mathfrak{m}$ with $\mathfrak{m} = \mathfrak{s}^0 \cap \mathfrak{k}$. Moreover, choose a point A_0 in $\mathfrak{a}$ such that $\alpha(A_0) \neq 0$ for all $\alpha \in R$, and let

$$R_+ = \{\alpha \in R \mid \alpha(A_0) > 0\}, \qquad \mathfrak{n} = \sum_{\alpha \in R_+} \mathfrak{s}^\alpha, \qquad \mathfrak{g} = \mathfrak{a} + \mathfrak{n}.$$

If A, N, G denote the corresponding groups one has the Iwasawa decomposition $S = ANK = GK$. The group G is an exponential Lie group, its modular function δ is trivial on N, and for $H \in \mathfrak{a}$ one has

$$\delta(\exp H) = e^{-2\rho(H)}$$

where $\rho \in \mathfrak{a}'$ is defined by $\rho = \frac{1}{2}\sum_{\alpha \in R_+}(\dim \mathfrak{s}^\alpha)\alpha$.

By means of the Iwasawa decomposition the manifolds S/K and G can be identified, hence functions on these spaces can be identified. Explicitly, if h on S is K-right-invariant define $r(h)$ on G by $r(h)(x) = h(x)$; if f is a function on G define Ef on S by $(Ef)(xk) = f(x)$ for $x \in G$ and $k \in K$. In order to make these operations commute, at least partly, with the involutions on the unimodular group S and the non-unimodular group G we have to modify them by an appropriate power of the modular function δ: For a K-right-invariant function h on S define $r_0(h)$ on G by

$$r_0(h) = r(h)\delta^{-1/2}, \quad \text{that is,} \quad r_0(h)(x) = h(x)\delta(x)^{-1/2} \quad \text{for } x \in G;$$

and for a function f on G define $E_0 f$ on S by $(E_0 f)(xk) = f(x)\delta(x)^{1/2}$ for $x \in G$ and $k \in K$. Then $r_0(h^*) = r_0(h)^*$ if h is K-biinvariant.

Also, we wish to compare some $\mathcal{L}^p$-norms under these identifications. By da, dx, dk we denote chosen Haar measures on the unimodular groups A, N, K (dk being normalized). Then $dg = da\,dx$ gives a left Haar measure on $G = AN$ with the above mentioned modular function δ. The Iwasawa decomposition $S = GK$, $s = gk$, yields a Haar measure $ds = dg\,dk$ on S, while the decomposition $S = KG$ leads to $ds = \delta(g)^{-1} dk\,dg$. For any p, the extension operator E and the restriction operator r yield isometries between $\mathcal{L}^p(G)$ and $\mathcal{L}^p(S/K)$, where $\mathcal{L}^p(S/K)$ denotes the space of K-right-invariant functions in $\mathcal{L}^p(S, ds)$. The case of r_0 is a little more subtle; first we consider $p = 2$.

Lemma 3.1. *The map r_0 defines an isometry from $\mathcal{L}^2(S//K, ds)$, the space of K-biinvariant functions in $\mathcal{L}^2(S, ds)$, onto a (closed) subspace of $\mathcal{L}^2(G, dg)$.*

Proof. For $h \in \mathcal{L}^2(S//K, ds)$ put $f = r_0(h)$, that is, $f(x) = h(x)\delta(x)^{-1}$ for $x \in G$. Then

$$\begin{aligned}\|f\|_2^2 &= \int_G |f(x)|^2\,dx = \int_G |h(x)|^2\delta(x)^{-1}\,dx \\ &= \int_K\int_G |h(kx)|^2\delta(x)^{-1} dk\,dx = \int_S |h(s)|^2\,ds\end{aligned}$$

by the above relation $ds = \delta(g)^{-1} dk\,dg$. ∎

For a K-biinvariant function h on S we put, as above, $f = r_0(h)$, and we wish to express the $\mathcal{L}^1$-norm of f in terms of h. To this end we recall the 0^{th} elementary zonal spherical function Ξ given by

$$\Xi(s) = \int_K (E\delta)(ks)^{-1/2}\,dk, \qquad s \in S,$$

or by

$$\Xi(s) = \int_K (\widetilde{E}\delta)(sk)^{1/2}\,dk,$$

if $\widetilde{E}\delta$ denotes the K-left-invariant extension of δ, that is, $\widetilde{E}\delta(kx) = \delta(x)$ for $x \in G$ and $k \in K$.

This function is a matrix coefficient of a unitary representation, in particular it is less than or equal 1, and its asymptotic behaviour is very well understood (cf. for instance [10, Chap. 4.6]). Moreover, it is invariant under involution, $\Xi(s^{-1}) = \Xi(s)$ for all $s \in S$. Now we simply compute, using the K-biinvariance of h:

$$\begin{aligned}\int_G |f(x)|\,dx &= \int_G |h(x)|\delta(x)^{-1/2}\,dx = \int_S |h(s)|(E\delta)(s)^{-1/2}\,ds\\ &= \int_K\int_G |h(kg)|(E\delta)(kg)^{-1/2}\delta(g)^{-1}\,dk\,dg\\ &= \int_G |h(g)|\Xi(g)\delta(g)^{-1}\,dg = \int_S |h(s)|\Xi(s)\,ds.\end{aligned}$$

We conclude that r_0 defines an isometry from the weighted space $\mathcal{L}^1(S//K, \Xi(s)ds)$ onto a closed subspace of $\mathcal{L}^1(G)$ (with inverse E_0). Moreover, it is easy to see that r_0 is multiplicative with convolution on both sides. As observed earlier, r_0 commutes with the involution. Since later the resulting subalgebra of $\mathcal{L}^1(G)$ will play a decisive role, we introduce a name for it.

PROPOSITION and DEFINITION 3.2. *Denote by $\mathfrak{A}$ the image of $\mathcal{L}^1(S//K, \Xi(s)ds)$ under r_0. This is a closed involutive subalgebra of $\mathcal{L}^1(G)$, via r_0/E_0 isometrically $*$-isomorphic to $\mathcal{L}^1(S//K, \Xi(s)ds)$.*

REMARK 3.3. A common way to obtain more general convolution algebras (so-called Beurling algebras) than just the ordinary $\mathcal{L}^1$-algebras is to introduce weight functions which are submultiplicative and greater than or equal to 1 (cf. e.g. [28]). The submultiplicativity guarantees the crucial property $\|ab\| \le \|a\|\,\|b\|$ of Banach algebras. The "weight" Ξ is neither submultiplicative nor greater than 1. Here, the submultiplicativity of the norm in $\mathcal{L}^1(S//K, S/K, \Xi(s)ds)$ is deduced from the corresponding property in $\mathcal{L}^1(G)$. In addition, it may be worthwhile to notice that the underlying space $K\backslash S/K$ of the algebra $\mathcal{L}^1(S//K)$ is not a group at all. Moreover, the measure $\Xi(s)ds$ is defined on the whole group S, while what actually counts, is the induced measure on $K\backslash S/K$. The algebra $\mathcal{L}^1(S//K, \Xi(s)ds)$ may be viewed as a subalgebra, namely as those functions which are summable with respect to this measure, of the measure algebra associated with a hypergroup structure on $K\backslash S/K$.

For the construction of a distinguished Laplacian of G we also need the Killing form B on $\mathfrak{s}$. Using this form we choose a basis $X_1, \ldots, X_r$ of $\mathfrak{n}$ consisting of $\mathfrak{a}$-eigenvectors, say $[H, X_j] = \alpha_j(H)X_j$ for $H \in \mathfrak{a}$, $1 \le j \le r$, such that

$$B(X_j, \Theta X_k) = -2\delta_{jk}.$$

Clearly such a basis exists. Moreover, we choose an orthonormal basis $A_1, \ldots,$ A_ℓ of $\mathfrak{a}$ (with respect to the Killing form). Then we define the distinguished Laplacian Λ by

$$\Lambda = A_1^2 + \cdots + A_2^2 + X_1^2 + \cdots + X_r^2 \in \mathfrak{U}\mathfrak{g} \subset \mathfrak{U}\mathfrak{s}.$$

4. The algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$. We study the relations between the algebras mentioned in the title of this section in the case of a distinguished Laplacian. At the end we shall see that $\mathfrak{A} = \mathfrak{B}$, and that $\mathfrak{B}$ is C^*-dense in $\mathfrak{C}$. Also, we determine the Gelfand space of the commutative Banach algebra $\mathfrak{B} = \mathfrak{B}(\Lambda)$. In a first step we show that $\mathfrak{A} \subset \mathfrak{B}$.

THEOREM 4.1. *If $\mathfrak{A} \subset \mathcal{L}^1(G)$ is as in* 3.2 *and $\mathfrak{B} = \mathfrak{B}(\Lambda)$ is the commutant of a distinguished Laplacian then $\mathfrak{A}$ is contained in $\mathfrak{B}$.*

Proof. More or less, the proof consists of some well-known computations in the universal enveloping algebra $\mathfrak{U}\mathfrak{s}$. These computations are included because they are crucial, short and nice. In addition to $A_1, \ldots, A_\ell$, $X_1, \ldots, X_r$ choose a basis $W_1, \ldots, W_m$ of $\mathfrak{m}$ such that $B(W_j, W_k) = -\delta_{jk}$. Then $A_1, \ldots, A_\ell, W_1, \ldots, W_m, X_1, \ldots, X_r, \Theta X_1, \ldots, \Theta X_r$ is a basis of $\mathfrak{s}$ with dual basis (with respect to the Killing form B) $A_1, \ldots, A_\ell, -W_1, \ldots, -W_m$, $-\frac{1}{2}\Theta X_1, \ldots, -\frac{1}{2}\Theta X_r, -\frac{1}{2}X_1, \ldots, -\frac{1}{2}X_r$. Therefore (cf., e.g., [31, Ch. 3, Section 11]),

$$\Omega := A_1^2 + \cdots + A_\ell^2 - W_1^2 - \cdots - W_m^2 - \tfrac{1}{2}\sum_{j=1}^{r} X_j \Theta X_j - \tfrac{1}{2}\sum_{j=1}^{r} (\Theta X_j) X_j \in \mathfrak{U}\mathfrak{s}$$

is the Casimir element. This particular representation of the Casimir element played a decisive role, in particular in the rank 1 case, in Lepowsky's [20] treatment of the Harish-Chandra homomorphism.

For each j, the bracket $[X_j, \Theta X_j]$ sits in the eigenspace $\mathfrak{s}^0 = \mathfrak{a} + \mathfrak{m}$; as this bracket is a (-1)-eigenvalue of Θ it has to be in $\mathfrak{a}$. From $-B([X_j, H], \Theta X_j) = B(H, [X_j, \Theta X_j])$ for $H \in \mathfrak{a}$, we obtain $B(H, [X_j, \Theta X_j]) = \alpha_j(H) B(X_j, \Theta X_j)$ $= -2\alpha_j(H)$. The Killing form yields an isomorphism from $\mathfrak{a}'$ onto $\mathfrak{a}$; as in Section 2, for $\beta \in \mathfrak{a}'$ we denote by H_β the corresponding element in $\mathfrak{a}$, that is, $\beta(H) = B(H, H_\beta)$ for all $H \in \mathfrak{a}$. With this notation the above relation reads

$$[X_j, \Theta X_j] = -2H_j,$$

where we put $H_j = H_{\alpha_j}$ for brevity. Using this relation, or rather its equivalent form $\frac{1}{2}\{X_j(\Theta X_j) + (\Theta X_j)X_j\} = H_j + X_j(\Theta X_j)$, we find that $\Omega = \sum_{k=1}^{\ell} A_k^2 - \sum_{\mu=1}^{m} W_\mu^2 - 2H_\rho + \sum_{j=1}^{r} X_j^2 - \sum_{j=1}^{r} X_j(X_j + \Theta X_j)$; recall that $2\rho = \sum_{j=1}^{r} \alpha_j = \operatorname{tr}\operatorname{ad} \in \mathfrak{a}'$. This implies that we can write

(Cas) $\Omega = \Lambda - 2H_\rho + \Gamma$ with $\Gamma := -\sum_{\mu=1}^{m} W_\mu^2 - \sum_{j=1}^{r} X_j(X_j + \Theta X_j) \in (\mathfrak{U}\mathfrak{s})\mathfrak{k}$,

or, applying the involution in $\mathfrak{U}\mathfrak{s}$,

(Cas′) $$\Omega = \Lambda + 2H_\rho + \Gamma^* \quad \text{with } \Gamma^* \in \mathfrak{k}\,\mathfrak{U}\mathfrak{s}.$$

Let f be a smooth function on G. We wish to interpret the equation $\Lambda * f = f * \Lambda$ by means of $E_0 f$. Define $f_0 : G \to \mathbb{C}$ by $f_0(x) = f(x)\delta(x)^{1/2}$, that is, $E_0 f = E(f_0)$. In order to apply Lemma 2.4 we write

$$\delta^{1/2}(\Lambda * f - f * \Lambda) = \delta^{1/2}(\Lambda * (\delta^{-1/2} f_0)) - \delta^{1/2}((\delta^{-1/2} f_0) * \Lambda).$$

Choosing $\widetilde{\sigma} = \delta^{1/2}$, that is, $\sigma = -\frac{1}{2}\operatorname{tr}\operatorname{ad} = -\rho$, in Lemma 2.4 we obtain

$$\begin{aligned}\delta^{1/2}(\Lambda * f - f * \Lambda) &= \Lambda * f_0 - 2H_\rho * f_0 + \|H_\rho\|^2 f_0 - f_0 *' \Lambda \\ &= 2f_0 *' H_\rho - \|H_\rho\|^2 f_0 \\ &= (\Lambda - 2H_\rho) * f_0 - f_0 *' (\Lambda + 2H_\rho).\end{aligned}$$

We would like to apply the extension operator E to this equation. While $E : C^\infty(G) \to C^\infty(S/K) \subset C^\infty(S)$ commutes with the left action of $\mathfrak{U}\mathfrak{g} \subset \mathfrak{U}\sigma$ (and likewise $r : C^\infty(S/K) \to C^\infty(G)$), this is no longer true for the right $\mathfrak{U}\mathfrak{g}$-action. (In fact, $\mathfrak{U}\mathfrak{g}$ does not act at all from the right on $C^\infty(S/K)$.) But, of course, if h is any C^∞-function on S and if $u \in \mathfrak{U}\mathfrak{g}$ then $h * u|_G = h|_G *' u$ (this obvious formula is the reason why we had to use $*'$ temporarily—for transition from a non-unimodular situation to a unimodular one). If both h and $h * u$ happen to be in $C^\infty(S/K)$ then $r(h * u) = r(h) *' u$ and $h * u = E(r(h) *' u)$. We apply this simple observation to $h = E(f_0)$ and $u = \Lambda + 2H_\rho$. As Ω is the Casimir element we have

$$\Omega * h = h * \Omega \in C^\infty(S/K).$$

From $\Omega = \Lambda + 2H_\rho + \Gamma^*$ (see (Cas′)), we conclude that $h * \Omega = h * (\Lambda + 2H_\rho)$ as $\Gamma^* \in \mathfrak{k}\,\mathfrak{U}\mathfrak{s}$ and h is K-right-invariant. Thus we get $E(f_0 *' (\Lambda + 2H_\rho)) = E(f_0) * (\Lambda + 2H_\rho)$ whence $E(\delta^{1/2}(\Lambda * f - f * \Lambda)) = E_0(\Lambda * f - f * \Lambda) = (\Lambda - 2H_\rho) * E(f_0) - E(f_0) * \Omega = (\Lambda - 2H_\rho - \Omega) * E(f_0) = -\Gamma * E(f_0)$.

As $\Gamma \in (\mathfrak{U}\mathfrak{s})\mathfrak{k}$ we see that the K-biinvariance of $E_0(f) = E(f_0)$, in particular the invariance from the left, implies that f commutes with Λ. This shows that $\mathfrak{A} \cap \mathcal{D}(G)$ is contained in $\mathfrak{B}$, and as $\mathfrak{A} \cap \mathcal{D}(G)$ is dense in $\mathfrak{A}$ with respect to the $\mathcal{L}^1$-norm we are done. ∎

The inclusion $\mathfrak{A} \subset \mathfrak{B}$ allows us to prove the injectivity of the Harish-Chandra transform. The results of Section 2 apply in the present situation. Thus, we know in particular that the canonical map $T : \mathcal{L}^1(G) \to \mathcal{L}^1(G/N = A)$ is injective on $\mathfrak{A}$. Hence also the composition

$$\mathcal{H}' := T \circ r_0 : \mathcal{L}^1(S/\!/K, \Xi(s)ds) \to \mathcal{L}^1(A)$$

is injective.

Definition 4.2. If $\mathcal{F} : \mathcal{L}^1(A) \to C_\infty(\widehat{A})$ denotes the (Euclidean) Fourier transform then we define $\mathcal{H} : \mathcal{L}^1(S//K, \Xi(s)ds) \to C_\infty(A)$ by $\mathcal{H} = \mathcal{F} \circ \mathcal{H}' = \mathcal{F} \circ T \circ r_0$.

This is one of the possible descriptions of the Harish-Chandra transform. The relation to the other possible form (perhaps more traditional, using the elementary spherical functions φ_λ) is established in [10, p. 107].

Corollary 4.3. *The Harish-Chandra transform* $\mathcal{H} : \mathcal{L}^1(S//K, \Xi(s)ds) \to C_\infty(\mathfrak{a}')$ *is one to one.*

Remark 4.4. It is pretty obvious that $\mathcal{L}^1(S//K,\ \Xi(s)ds)$ contains Harish-Chandra's Schwartz space $\mathcal{C}(S//K)$ (we mean the "classical space" corresponding to $p = 2$, see [10, p. 253]). Originally, Harish-Chandra [12] proved the injectivity on $\mathcal{C}(S//K)$ using discrete series representations. Later, J. Rosenberg [30] gave another proof based on results of S. Helgason and R. Gangolli [14, 9]. For further results in this direction, also involving Harish-Chandra's Schwartz spaces for $p \neq 2$, see also J.-Ph. Anker [1]. Observe that our proof uses no particular information on the elementary spherical functions φ_λ. In fact, they do not appear at all (except for $\lambda = 0$). When I circulated the manuscript [26], some people seemed to be surprised that I considered $\mathcal{H}$ on an $\mathcal{L}^1$-space rather than on an $\mathcal{L}^2$-space. But as $|\varphi_\lambda| \leq \Xi$ for all (purely imaginary) λ (see [10, p. 168]), the space $\mathcal{L}^1(S//K, \Xi(s)ds)$ appears as a natural candidate for being the domain of $\mathcal{H}$—just as in the classical Fourier analysis the $\mathcal{L}^1$-space is a natural candidate for the domain of the Fourier transform.

In order to obtain some information on the image of $\mathfrak{C} \subset C^*(G)$ under the canonical map T we need some facts on root systems. Let $\mathfrak{v}$ be an $\mathfrak{a}$-invariant complement of the commutator algebra $[\mathfrak{n}, \mathfrak{n}]$, $\mathfrak{n} = \mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}]$. Assume now that our numbering of the eigenvectors X_k (which fixes the numbering of the α_k's) has the property that for $j \leq q \leq r$ the form α_j appears as an eigenfunctional in $\mathfrak{v}$, while for $q < j \leq r$ it does not. Observe that this does not mean that X_k $(k \leq q)$ occurs in $\mathfrak{v}$. Among the α_k's there may be many repetitions, $r = \dim \mathfrak{n}$ is not (in general) the number of different (positive) roots. Also observe that $\mathfrak{v}$ is isomorphic to $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ as an $\mathfrak{a}$-space. Therefore, the eigenfunctionals for the action of $\mathfrak{a}$ on $\mathfrak{v}$ are intrinsically defined. The space $\mathfrak{v}$ was chosen because we want to use that $\mathfrak{n}$ is generated (as a Lie algebra) by $\mathfrak{v}$. This is a general fact on nilpotent Lie algebras, true for any complement of the commutator algebra. For each k let s_k be the reflection corresponding to $\alpha_k \sim H_k$, that is, $s_k : \mathfrak{a} \to \mathfrak{a}$ is given by $s_k(X) = X - 2\frac{\langle H, H_k\rangle}{\langle H_k, H_k\rangle} H_k$. With this notation we have

Lemma 4.5. *The Weyl group W of the root system R is generated by s_k, $k \leq q$.*

Proof. This is a corollary to some results in Bourbaki [3], or other sources. As $\mathfrak{n}$ is generated by $\mathfrak{v}$ we conclude that each root a_k, $k > q$, which occurs only in $[\mathfrak{n}, \mathfrak{n}]$, is a proper linear combination of the α_j's, $j \le q$, with non-negative integral coefficients. According to our choice $R_+ = \{\alpha_1, \dots, \alpha_r\}$ of positive roots there is a well-defined "basis" $B \subset R_+$. The above property of the α_k's, $k > q$, shows (see [3, Ch. VI, 1.6, Corollaire 1, p. 159]) that those roots do not qualify for being members of B, hence $B \subset \{\alpha_1, \dots, \alpha_q\}$. But since the reflections corresponding to B already generate W (see [3, Ch. VI, 1.5, Théorème 2, Remarque 1, p. 153]), the lemma follows. ∎

Proposition 4.6. *If Λ is a distinguished Laplacian then the image of $\mathfrak{C} = \mathfrak{C}(\Lambda) \subset C^*(G)$ under T is contained in $C^*(A)^W \cong C_\infty(\mathfrak{a}')^W$, where, as usual, $(\ )^W$ denotes the elements fixed by the Weyl group W.*

Proof. We use the notation introduced before Lemma 4.5. Fix a $p \le q$ for a while. We may assume that the orthonormal basis $A_1, \dots, A_\ell$ of $\mathfrak{a}$ has the property that $\alpha_p(A_j) = 0$ for $j < \ell$. (The operator Λ remains unchanged after transition to another orthonormal basis.) In particular, A_ℓ is proportional to H_p. Further, we choose an $\mathfrak{a}$-invariant one-codimensional subspace $\mathfrak{n}_p$ of $\mathfrak{n}$, containing $[\mathfrak{n}, \mathfrak{n}]$, such that the action of $\mathfrak{a}$ on $\mathfrak{n}/\mathfrak{n}_p = \mathbb{R}Y$ is given by

$$[H, Y] = \alpha_p(H)Y.$$

Let $G_p = G/N_p = A \ltimes (N/N_p)$ with Lie algebra $\mathfrak{g}_p = \mathfrak{a} \ltimes (\mathfrak{n}/\mathfrak{n}_p)$. The image of Λ under the quotient map $\mathfrak{g} \to \mathfrak{g}_p$ is just

$$\Lambda_p := A_1^2 + \dots + A_\ell^2 + Y^2$$

if Y is properly normalized. Further, by integration over N/N_p, there is a morphism $T_p : \mathcal{L}^1(G_p) \to \mathcal{L}^1(A)$. The map T is the composition of the canonical map $\mathcal{L}^1(G) \to \mathcal{L}^1(G_p)$ with T_p (if Haar measures are properly adapted). These maps have C^*-analogues, and we conclude that $T(\mathfrak{C}(\Lambda))$ is contained in $T_p(\mathfrak{C}(\Lambda_p)) \subset C^*(A)$. Using the results of [27] on the $(ax+b)$-group we are going to show that $T_p(\mathfrak{C}(\Lambda_p))$ consists of those elements in $C^*(A)$ which are fixed by the reflection s_p. This is merely a matter of pure formalism. Write $\mathfrak{a} = \mathfrak{e} \oplus \mathfrak{d}$ with $\mathfrak{e} = \mathcal{L}_\mathbb{R}(A_1, \dots, A_{\ell-1})$ and $\mathfrak{d} = \mathbb{R}A_\ell$. Then $\mathfrak{g}_p = \mathfrak{e} \oplus (\mathfrak{d} \ltimes \mathbb{R}Y)$ and $G_p = E \times (D \ltimes \mathbb{R})$, whence $C^*(G)$ is the C^*-tensor product of $C^*(E)$ with $C^*(D \ltimes \mathbb{R})$, which can be identified with $C_\infty(\widehat{E}, C^*(D \ltimes \mathbb{R}))$ via Fourier transform with respect to E. Since $A_1, \dots, A_{\ell-1}$ play no role when considering the commutant, under the above identification $\mathfrak{C}(\Lambda_p)$ corresponds to $C_\infty(\widehat{E}, \mathfrak{C}(\Lambda'_p))$ with $\Lambda'_p = A_\ell^2 + Y^2 \in \mathfrak{U}(\mathfrak{d} \ltimes \mathbb{R}Y)$. Consistently, we denote the canonical map $C^*(D \ltimes \mathbb{R}) \to C^*(D)$ by T'_p. Further, we denote by $\mathcal{F}_E$, $\mathcal{F}_A$, $\mathcal{F}_D$ the Euclidean Fourier transforms with respect to the corresponding variables. Then we have the diagram

$$\begin{tikzcd}
C^*(G_p = E \times (D \ltimes \mathbb{R})) \arrow[r, "\mathcal{F}_E"] \arrow[d, "T_p"'] & C_\infty(\widehat{E}, C^*(D \ltimes \mathbb{R})) \supset C_\infty(\widehat{E}, \mathfrak{C}(\Lambda'_p)) \arrow[d, "\mathrm{Id}\otimes T'_p"] \\
C^*(A) \arrow[d, "\mathcal{F}_A"'] & C_\infty(\widehat{E}, C^*(D)) \arrow[d, "\mathrm{Id}\otimes \mathcal{F}_D"] \\
C_\infty(\widehat{A}) & C_\infty(\widehat{E}, C_\infty(\widehat{D})) = C_\infty(\widehat{E} \times \widehat{D})
\end{tikzcd}$$

which is commutative when completed by the obvious identification of the spaces in the last line. By [27, Theorem 5.7], $\mathcal{F}_D \circ T'_p$ maps $\mathfrak{C}(\Lambda'_p)$ isomorphically onto the even functions in $C_\infty(\widehat{D})$. By what we have seen earlier this implies that $\mathcal{F}_A \circ T_p$ maps $\mathfrak{C}(\Lambda_p)$ onto the functions fixed by s_p. But since $(\mathcal{F}_A \circ T)(\mathfrak{C}(\Lambda))$ is contained in $(\mathcal{F}_A \circ T_p)(\mathfrak{C}(\Lambda_p))$, we conclude that all members of $(\mathcal{F}_A \circ T)(\mathfrak{C}(\Lambda))$ are fixed by all s_p, $p \le q$. Using Lemma 4.5 we are done. ∎

In order to obtain further information on $\mathfrak{C}$ (and later on $\mathfrak{A}$ and $\mathfrak{B}$) we have to use more substantial results from the theory of spherical functions.

PROPOSITION 4.7. *If Λ is a distinguished Laplacian then $T : C^*(G) \to C^*(A)$ induces an isomorphism from $\mathfrak{C}(\Lambda)$ onto $C^*(A)^W$. The intersection $\mathfrak{A} \cap C_c^\infty(G)$ is C^*-dense in $\mathfrak{C}(\Lambda)$.*

Proof. By an older result of S. Helgason, [14], T maps $\mathfrak{A} \cap C_c^\infty(G)$ onto $C_c^\infty(A)^W$. (In Helgason's paper an additional assumption was imposed, which was removed later. In fact, the above mentioned result is well established, and much more is known, in particular on extensions of this result to Schwartz spaces, which will be used below. The point is that it is just this result which helps at the moment.) As $\mathcal{D}(A)^W$ is C^*-dense in $C^*(A)^W$, it follows that $C^*(A)^W$ is contained in $T(\mathfrak{C}(\Lambda))$. In view of our earlier results the statements of the proposition are now clear. ∎

THEOREM 4.8. *If Λ is a distinguished Laplacian then $\mathfrak{A} = \mathfrak{B}(\Lambda)$, and this algebra is C^*-dense in $\mathfrak{C}(\Lambda)$. In particular, $\mathfrak{B}(\Lambda)$ is isometrically isomorphic to $\mathcal{L}^1(S//K, \Xi(s)ds)$.*

Proof. Recall (Lemma 3.1) that r_0 induces an isometry from $\mathcal{L}^2(S//K, ds)$ onto a closed subspace $\mathfrak{H}$ of $\mathcal{L}^2(G)$. Since $\mathcal{L}^2(S//K, ds)$ is invariant under convolution with $C_c(S//K)$, and since r_0 is multiplicative on K-biinvariant functions, it follows that $\lambda(\mathfrak{A} \cap C_c(G))(\mathfrak{H})$ is contained in $\mathfrak{H}$, where λ denotes the left regular representation on $\mathcal{L}^2(G)$. Using Proposition 4.7 we see that $\mathfrak{H}$ is invariant under $\lambda(\mathfrak{C}(\Lambda))$ and, a fortiori, under $\lambda(\mathfrak{B}(\Lambda))$, that is, $\mathfrak{B} * \mathfrak{H} \subset \mathfrak{H}$. Let $b \in \mathfrak{B}$ be a fixed element, and φ be any element in $C_c(G) \cap \mathfrak{A} = C_c \cap \mathfrak{H}$. Then $b*\varphi$ is in $\mathfrak{H} \cap C(G) \cap \mathcal{L}^1(G)$. Hence $E_0(b*\varphi)$ is a well defined (continuous) function in $\mathcal{L}^2(S//K)$. As $b * \varphi \in \mathcal{L}^1(G)$ it follows that $E_0(b * \varphi)$ is in $\mathcal{L}^1(S//K, \Xi(s)ds)$, hence $b * \varphi \in \mathfrak{A}$. But there exists a sequence (φ_j) in $\mathfrak{A} \cap C_c(G)$ which is a bounded approximate identity for $\mathcal{L}^1(G)$. As $b * \varphi_j \in \mathfrak{A}$ and $\mathfrak{A}$ is closed it follows that b is in $\mathfrak{A}$. ■

Now we consider some properties of the commutative Banach $*$-algebra $\mathfrak{A} = \mathfrak{B}(\Lambda)$. The Gelfand space of $\mathfrak{A}$ was also determined in [11].

Theorem 4.9. *The Gelfand space of* $\mathfrak{B}(\Lambda)$ *is homeomorphic to the orbit space* $\widehat{A}/W$. *Moreover,* $\mathfrak{B}(\Lambda)$ *is symmetric and completely regular in the sense of* [29], *and it is a Wiener algebra in the sense of* [28], *that is, the ideal of all elements in* $\mathfrak{B}(\Lambda)$ *with compactly supported Gelfand transform is dense in* $\mathfrak{B}(\Lambda)$.

Proof. For the following see [10, in particular Theorem 6.4.1, p. 273]. If again $\mathcal{F} : \mathcal{L}^1(A) \to C_\infty(\widehat{A})$ denotes the Fourier transform then $(\mathcal{F} \circ T)(\mathfrak{A})$ contains $\mathcal{S}(\widehat{A})^W$, the Schwartz functions fixed by W. The preimage of $\mathcal{S}(\widehat{A})^W$ is just $r_0(\mathcal{C}(S//K))$, where, as in Remark 4.4, $\mathcal{C}(S//K)$ denotes Harish-Chandra's Schwartz space. The inverse of $\mathcal{F} \circ T$ provides a continuous multiplicative map $\mathcal{K} : \mathcal{S}(\widehat{A})^W \to \mathfrak{A}$ with dense image (where $\mathcal{S}(\widehat{A})$ is endowed with the pointwise operations and the usual Schwartz topology).

Now, if $\chi : \mathfrak{A} \to \mathbb{C}$ is a (continuous) multiplicative linear functional then composing with the above map $\mathcal{K} : \mathcal{S}(\widehat{A})^W \to \mathfrak{A}$ yields a continuous multiplicative functional $\widetilde{\chi} : \mathcal{S}(\widehat{A})^W \to \mathbb{C}$. The topological algebra $\mathcal{S}(\widehat{A})^W$ is a standard function algebra with the Wiener property which implies that $\widetilde{\chi}$ is point evaluation. (For all this see [28, pp. 18–22, in particular 2.4 on p. 22].) The density of $r_0(\mathcal{C}(S//K))$ in $\mathfrak{A}$ shows that χ is what it should be: there is a point η (only determined up to W-conjugation) in $\widehat{A}$ such that $\chi(f) = ((\mathcal{F} \circ T)(f))(\eta)$ for all $f \in \mathfrak{A}$. In particular, $\mathfrak{A}$ is symmetric. As already observed, the image of the Gelfand transform contains sufficiently many functions (namely $\mathcal{S}(\widehat{A})^W$), which guarantees complete regularity. As $C_c^\infty(\widehat{A})^W$ is dense in $\mathcal{S}(\widehat{A})^W$, the Wiener property is an immediate consequence. ■

Remark 4.10. We have seen that $\mathfrak{B}(\Lambda)$ is a symmetric subalgebra of $\mathcal{L}^1(G)$. In the case of the $(ax + b)$-group such a situation was studied by A. Hulanicki [18] already in 1976. It should be noticed that in many cases the ambient algebra $\mathcal{L}^1(G)$ is not symmetric. In fact, as was observed by T. Nomura [24], in the present situation $\mathcal{L}^1(G)$ is not symmetric if S is simple and $\dim A > 1$, while for $\dim A = 1$ it is symmetric.

Translating back into the semisimple situation, we conclude from Theorem 4.9 that the convolution algebra $\mathcal{L}^1(S//K, \Xi(s)ds)$ is symmetric while $\mathcal{L}^1(S//K, ds)$ is never symmetric as was noticed by J. Jenkins [19]. Observe en passant that $\mathcal{L}^1(S//K, ds)$ is a dense subalgebra of $\mathcal{L}^1(S//K, \Xi(s)ds)$ because $\Xi \leq 1$ being a matrix coefficient.

5. Concluding remarks. Let us return to the situation studied in Section 2, $G = A \ltimes N$ with a more general action of the vector group A on the simply connected nilpotent Lie group N. Also in this case the map $T : \mathcal{L}^1(G) \to \mathcal{L}^1(G/N = A)$ provides an injection of $\mathfrak{B}(\Lambda)$ into $\mathcal{L}^1(A)$. Each "root" α for the action of $\mathfrak{a}$ on $\mathfrak{n}$ yields a reflection s_α on $\mathfrak{a}$ (or $\mathfrak{a}'$, which is isomorphic to $\mathfrak{a}$ as we used a basis of $\mathfrak{a}$ to define Λ, giving a scalar product on $\mathfrak{a}$). Let F be the closure in the orthogonal group of $\mathfrak{a}$ of the group generated by the s_α, where α is a root of the action of $\mathfrak{a}$ on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. The arguments of Section 4, notably the proof of Proposition 4.6, tell us that $T(\mathfrak{B}(\Lambda))$ is contained in $\mathcal{L}^1(A)^F$. In our previous considerations we used results from the theory of spherical functions (and from root systems) in order to show that the image $T(\mathfrak{B}(\Lambda))$ is sufficiently rich. There is an obvious question: Are there suitable assumptions which guarantee, for instance, that $T(\mathfrak{B}(\Lambda))$ contains $C_c^\infty(A)^F$ or $\mathcal{S}(A)^F$, or that even $T(\mathfrak{B}(\Lambda) \cap C_c^\infty(G))$ is equal to $C_c^\infty(A)^F$? It seems to me that already the case of $\mathfrak{n}$ abelian might lead to some interesting questions. Even if there are no satisfactory results in this direction one may still ask if $\mathfrak{B}(\Lambda)$ is a symmetric (commutative) Banach algebra.

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