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SPHERICAL HARMONICS ON GRASSMANNIANS

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Abstract. We propose a generalization of the theory of spherical harmonics to the context of symmetric subgroups of reductive groups acting on flag manifolds. We give some sample results for the case of the orthogonal group acting on Grassmann manifolds, especially the case of 2-planes.

1. Introduction. The theory of spherical harmonics is a classical piece of mathematics, with manifold applications to physics, in any situation where the dependence of a quantity on direction in space needs to be described.

A representation-theoretic understanding of the theory is that it describes the behavior of polynomials under the action of the orthogonal group. If $\mathcal{P}(\mathbb{R}^n) = \mathcal{P}$ is the space of complex-valued polynomial functions in n (real) variables, then the group $\mathrm{GL}_n(\mathbb{R}) = \mathrm{GL}_n$ of linear transformations on $\mathbb{R}^n$ also acts on the polynomial functions by the standard recipe:

$$g(p)(\vec{x}) = p(g^{-1}(\vec{x})). \tag{1.1}$$

Here $g \in \mathrm{GL}_n$, $p \in \mathcal{P}$, and $\vec{x}$ is a point in $\mathbb{R}^n$. (We do not distinguish notationally here between g as a linear transformation on $\mathbb{R}^n$ and on $\mathcal{P}$; which is meant should be clear from context.)

Let $\mathcal{P}^d(\mathbb{R}^n) = \mathcal{P}^d$ be the space of polynomials homogeneous of degree d. Then it is well known that

$$\mathcal{P} \simeq \bigoplus_{d=0}^{\infty} \mathcal{P}^d, \tag{1.2}$$

that the action (1.1) of GL_n preserves each space $\mathcal{P}^d$, and that the $\mathcal{P}^d$ are irreducible representations for GL_n. However, if $\mathrm{O}_n \subset \mathrm{GL}_n$ is the group of orthogonal transformations, defined to be the linear transformations which preserve the usual Euclidean (squared) length $r^2 = \sum_{i=1}^n x_i^2$, then O_n does not act irreducibly on the homogeneous components $\mathcal{P}^d$. Indeed, it is easy

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to find operators on $\mathcal{P}$ that commute with the action of elements of O_n, and these operators allow one to display subspaces of $\mathcal{P}^d$ that are invariant under O_n.

The most obvious such operator is multiplication by $r^2 : p \mapsto r^2 p$. Since r^2 is homogeneous of degree two, multiplying by it raises the degree of a polynomial by two, so we get mappings

$$r^2 : \mathcal{P}^d \to \mathcal{P}^{d+2} \tag{1.3}$$

for each d. Since operators g defined in (1.1) are algebra automorphisms of $\mathcal{P}$, it is easy to check that $g(r^2 p) = r^2 g(p)$ for $g \in \mathrm{O}_n$. It follows that the space $r^2(\mathcal{P}^d) \subset \mathcal{P}^{d+2}$ is invariant under O_n.

Another operator that commutes with O_n, less obvious than r^2, but which presented itself in physics, is the Laplacian $\Delta = \sum_{i=0}^{n} \partial^2/\partial x_i^2$. The Laplacian reduces the degree of polynomials by two; it defines mappings

$$\Delta : \mathcal{P}^d \to \mathcal{P}^{d-2}. \tag{1.4}$$

Since these maps commute with the action of O_n, it follows that the kernel

$$\mathcal{H}^d(\mathbb{R}^n) = \mathcal{H}^d = \{p \in \mathcal{P}^d : \Delta p = 0\} \tag{1.5}$$

of Δ, commonly known as the *harmonic polynomials*, is an O_n-invariant subspace of $\mathcal{P}^d$.

The main assertions of the theory of spherical harmonics are

PROPOSITION 1.1 (Theory of Spherical Harmonics).

(a) *The space $\mathcal{H}^d$ of harmonic polynomials of degree d is an irreducible representation of O_n.*
(b) $\mathcal{P}^d \simeq \mathcal{H}^d \oplus r^2 \mathcal{P}^{d-2}$.
(c) $\mathcal{P}^d \simeq \bigoplus_{k=0}^{[d/2]} r^{2k} \mathcal{H}^{d-2k}$ *is a decomposition of $\mathcal{P}^d$ into irreducible representations for O_n.*

Evidently, the main statement (c) follows from (a) and (b).

One way to prove these results is to study the interaction between the operators r^2 and Δ. They do not commute with each other, but it turns out that there is an elegant formula for their commutator:

$$[\Delta, r^2] = \Delta r^2 - r^2 \Delta = 4E + 2n, \tag{1.6}$$

where $E = \sum_{i=1}^{n} x_i \partial/\partial x_i$ is the Euler degree operator, which acts on $\mathcal{P}^d$ by the scalar d. From this, one can show by induction that, if $h \in \mathcal{H}^\ell$, then

$$\Delta(r^{2k} h) = 2k(2(\ell + k - 1) + n) r^{2(k-1)} h. \tag{1.7}$$

From (1.7) it is evident that, if the decomposition (c) is true for $\mathcal{P}^{d-2}$, then $\ker \Delta \cap r^2 \mathcal{P}^{d-2} = \{0\}$. Hence $\dim \ker \Delta \leq \dim \mathcal{P}^d - \dim \mathcal{P}^{d-2}$. On the other hand, since Δ maps $\mathcal{P}^d$ to $\mathcal{P}^{d-2}$, it is likewise clear that $\dim \ker \Delta \geq$

$\dim \mathcal{P}^d - \dim \mathcal{P}^{d-2}$. Hence $\dim \ker \Delta = \dim \mathcal{P}^d - \dim \mathcal{P}^{d-2}$, and equation (b) holds for $\mathcal{P}^d$, whence equation (c) does also.

In this situation, a remarkable fact is that the three operators r^2, Δ and

$$[\Delta, r^2] = 4E + 2n \tag{1.8}$$

span a three-dimensional Lie algebra isomorphic to the three-dimensional simple Lie algebra $\mathfrak{sl}_2$. Furthermore, the associative algebra generated by this Lie algebra is the full algebra of polynomial coefficient differential operators which commute with the action of O_n on $\mathcal{P}$ ([GW], [Ho]). This in turn implies statement (a) of Proposition 1.1. (There are also ways of proving (a) that do not involve observing the existence of the commuting $\mathfrak{sl}_2$.)

We are interested in a generalization of the theory of spherical harmonics. This theory is usually thought of as describing functions on the sphere, but statement (c) of Proposition 1.1 in fact describes the relationship between the space $\mathcal{H}^d$, which is the O_n components of the functions on the sphere, and the space $\mathcal{P}$ of all polynomials. Therefore, it is reasonable in thinking about generalizing the theory of spherical harmonics to consider the nature of $\mathcal{P}$, in particular as a module for GL_n.

The polynomial ring $\mathcal{P}$ can be thought of as the "homogeneous coordinate ring" ([F]) of the projective space $\mathbb{P}^{n-1}$ of lines through the origin in $\mathbb{R}^n$. Projective space is the simplest of the flag manifolds associated to GL_n. Other flag manifolds for GL_n include the Grassmann varieties of k-dimensional subspaces of $\mathbb{R}^n$, or higher flag manifolds associated to nested sequences of subspace of specified dimensions. Each flag manifold has an associated homogeneous coordinate ring, which is a module for GL_n, with a well understood decomposition into irreducible representations. One could think of the theory of "flag harmonics" to be concerned with the decomposition of these GL_n representations into irreducible subspaces for O_n.

But there is more to the theory of spherical harmonics than just the irreducible decomposition. There is the action of the O_n invariant r^2, and even more directly implicated in the name, there is the understanding of the operator Δ, and especially its kernel, the harmonic functions.

One can ask if there is some analog of this structure, in particular, analogs of r^2 and Δ, for other flag manifolds. This is easy to answer for r^2. The invariants of O_n in representations of GL_n are well-understood; this is a special case of the Cartan–Helgason Theorem (see Theorem 4.1 on page 535 of [He]). In particular, for each Grassmann variety, the algebra of O_n invariants is generated by a single well-understood element that bears a clear relation to r^2.

It is rather less obvious, but it turns out that there also exist differential operators analogous to the Laplacian. Together, they generate an algebra of operators that commute with the action of O_n. One can think of the theory of spherical harmonics as a description of the action of this algebra, and most particularly, of the action of the Laplacian analog.

As we have seen above, in the classical theory of spherical harmonics, one has the remarkable circumstance that the Laplacian and r^2 generate a Lie algebra. The commutation relations in this Lie algebra let one give a precise description, not just of the kernel of Δ, but of the action of Δ on all of $\mathcal{P}$. Although one cannot expect that the analogs of r^2 and Δ will live inside a finite-dimensional Lie algebra for more general flag manifolds, still one might hope to give a reasonably explicit description of the action of the Δ analog. In this paper, we present an example that provides some encouragement for such a hope.

2. Notation. The action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ can be complexified to give a representation of $\mathrm{GL}_n(\mathbb{R})$ on the algebra $\mathcal{P}(\mathbb{C}^n)$ of polynomial functions on $\mathbb{C}^n$. This action can be extended to an action by the complexification of $\mathrm{GL}_n(\mathbb{R})$, which is the complex general linear group $\mathrm{GL}_n(\mathbb{C})$ consisting of all $n \times n$ invertible complex matrices. The complexification of O_n is the complex orthogonal group $\mathrm{O}_n(\mathbb{C})$. By a suitable change of coordinates on $\mathbb{C}^n$, we can assume that $\mathrm{O}_n(\mathbb{C})$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ which preserves the symmetric bilinear form

$$\left\langle \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} u_j v_{n+1-j} \tag{2.1}$$

on $\mathbb{C}^n$. In the rest of the paper, we shall work with $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{O}_n(\mathbb{C})$. So from now on, GL_n and O_n shall stand for $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{O}_n(\mathbb{C})$ respectively.

We now introduce notation for the irreducible representations of GL_n. Let $B_n = A_n U_n$ be the standard Borel subgroup of upper triangular matrices in GL_n, where A_n is the diagonal torus in GL_n and U_n is the maximal unipotent subgroup consisting of all the upper triangular matrices with 1's on the diagonal. Recall that a *Young diagram* λ is an array of square boxes arranged in left-justified horizontal rows, with each row no longer than the one above it ([F]). If λ has at most m rows, then we shall write it as

$$\lambda = (\lambda_1, \ldots, \lambda_m)$$

where for each i, λ_i is the number of boxes in the ith row of λ. We shall denote the number of rows in λ by $d(\lambda)$, and call it the *depth* of λ. For later use, we let $\mathbf{1}_k$ be the Young diagram with only one column of k boxes, i.e.

$$\mathbf{1}_k = (\overbrace{1, \ldots, 1}^{k}). \tag{2.2}$$

For a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n rows, let $\psi_n^\lambda : A_n \to \mathbb{C}^\times$ be the character given by

$$\psi_n^\lambda[\mathrm{diag}(a_1, \ldots, a_n)] = a_1^{\lambda_1} \cdots a_n^{\lambda_n}.$$

Here $\mathrm{diag}(a_1, \ldots, a_n)$ is the $n \times n$ diagonal matrix such that its diagonal entries are $a_1, \ldots, a_n$. Then ψ_n^λ is a dominant weight for GL_n with respect to the Borel subgroup B_n ([GW]), and we shall denote the irreducible representation of GL_n with highest weight ψ_n^λ by ρ_n^λ. We shall abuse notation and say that the highest weight of ρ_n^λ is λ. If $\lambda = m\mathbf{1}_n$, then we also say that the highest weight of ρ_n^λ is $\det_n^m$.

The irreducible finite-dimensional representations of O_n are parameterized by Young diagrams λ such that the sum of the lengths of the first two columns of λ does not exceed n ([Wy],[GW],[Ho]). For such a Young diagram λ, we shall denote the O_n representation associated with λ by σ_n^λ. Specifically, σ_n^λ is the irreducible representation of O_n generated by the GL_n highest weight vector in ρ_n^λ. See Section 3.6 of [Ho] for more details.

Let SO_n denote the subgroup of O_n consisting of all elements of O_n with determinant 1, and let

$$A_{\mathrm{SO}_n} = A_n \cap \mathrm{SO}_n, \qquad N_n = U_n \cap \mathrm{SO}_n. \tag{2.3}$$

Explicitly,

$$A_{\mathrm{SO}_n} = \begin{cases} \{\mathrm{diag}(a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}) : a_1, \ldots, a_m \in \mathbb{C}^\times\}, & n = 2m, \\ \{\mathrm{diag}(a_1, \ldots, a_m, 1, a_m^{-1}, \ldots, a_1^{-1}) : a_1, \ldots, a_m \in \mathbb{C}^\times\}, & n = 2m+1. \end{cases}$$

Let λ be a Young diagram such that the sum of the lengths of its first two columns does not exceed n. If $2d(\lambda) \neq n$, then the restriction of σ_n^λ to SO_n is irreducible. If in addition $2d(\lambda) < n$ and $\phi_n^\lambda : A_{\mathrm{SO}_n} \to \mathbb{C}^\times$ is the restriction of the character ψ_n^λ to A_{SO_n}, then as an SO_n module, σ_n^λ has highest weight ϕ_n^λ. In this case, we shall abuse notation and say that λ is the highest weight of σ_n^λ.

3. The O_n highest weight vectors on Grassmannians. Let $2k < n$ and let $\mathbb{G}_k^n$ be the set of all k-dimensional subspaces of $\mathbb{C}^n$. It has a structure of a projective variety. Let $\mathcal{R}(\mathbb{G}_k^n)$ be the homogeneous coordinate ring ([1]) of $\mathbb{G}_k^n$. It carries an action by $\mathrm{GL}_n \times \mathrm{GL}_k$, and O_n acts by re-

([1]) If G is a reductive algebraic group, then a *flag manifold* for G is a coset space G/P, where P is a parabolic subgroup of G. The *homogeneous coordinate ring* for G/P is the ring $\mathcal{R}(G/P^{(2)})$ of regular functions on the variety $G/P^{(2)}$, where $P^{(2)}$ is the commutator subgroup of P. The variety G/P is a projective variety, and therefore has no nonconstant regular functions. However, the variety $G/P^{(2)}$ is a torus bundle over G/P, and is quasi-affine, so it has a large collection of regular functions, which can be thought of as (sums of) sections of appropriate line bundles over G/P.

striction from GL_n. In this section, we shall describe the $\mathrm{O}_n \times \mathrm{GL}_k$ module structure of $\mathcal{R}(\mathbb{G}_k^n)$.

Let $\mathrm{M}_{nk} = \mathrm{M}_{nk}(\mathbb{C})$ be the space of $n \times k$ complex matrices, and let $\mathcal{P}(\mathrm{M}_{nk})$ be the algebra of polynomial functions on M_{nk}, that is, each $p \in \mathcal{P}(\mathrm{M}_{nk})$ is of the form

$$p(x) = \sum_{\alpha} a_\alpha x^\alpha,$$

where $x = (x_{ij}) \in \mathrm{M}_{nk}$, each $\alpha = (\alpha_{ij})$ appearing in the sum is an $n \times k$ matrix of nonnegative integers, $a_\alpha \in \mathbb{C}$ and

$$x^\alpha = \prod_{i,j} x_{ij}^{\alpha_{ij}}.$$

We now define an action of $\mathrm{GL}_n \times \mathrm{GL}_k$ on M_{nk}. For each $n \times n$ complex matrix A, there exists a unique $n \times n$ complex matrix A^τ such that

$$\langle Au, v\rangle = \langle u, A^\tau v\rangle,$$

where $\langle \cdot, \cdot\rangle$ is given in (2.1). For $g \in \mathrm{GL}_n, h \in \mathrm{GL}_k$, and $T \in \mathrm{M}_{nk}$, let

$$(g,h)(T) = (g^{-1})^\tau T h^{-1}, \qquad g \in \mathrm{GL}_n,\ h \in \mathrm{GL}_k,\ T \in \mathrm{M}_{nk}. \tag{3.1}$$

This action induces an action of $\mathrm{GL}_n \times \mathrm{GL}_k$ on $\mathcal{P}(\mathrm{M}_{nk})$ in the usual way. We have used $(g^{-1})^\tau$ in the action by GL_n so that this gives rise to a more symmetrical decomposition of $\mathcal{P}(\mathrm{M}_{nk})$ into irreducible $\mathrm{GL}_n \times \mathrm{GL}_k$ representations (see (3.2) below). Moreover, $g \in \mathrm{O}_n$ if and only if $g = (g^{-1})^\tau$.

Let SL_k be the subgroup of GL_k consisting of all elements of GL_k with determinant 1. Then it is well known ([F]) that $\mathcal{R}(\mathbb{G}_k^n)$ can be identified with the algebra

$$\mathfrak{A} = \mathcal{P}(\mathrm{M}_{nk})^{\mathrm{SL}_k}$$

of SL_k invariants in $\mathcal{P}(\mathrm{M}_{nk})$. Thus in the remaining part of the paper, we shall replace $\mathcal{R}(\mathbb{G}_k^n)$ by $\mathfrak{A}$.

By the $(\mathrm{GL}_n, \mathrm{GL}_k)$-duality ([Ho]), under the action by $\mathrm{GL}_n \times \mathrm{GL}_k$, we have the decomposition

$$\mathcal{P}(\mathrm{M}_{nk}) = \bigoplus_{d(\lambda)\le k} \rho_n^\lambda \otimes \rho_k^\lambda. \tag{3.2}$$

By extracting the SL_k invariants, we obtain

$$\mathfrak{A} = \mathcal{P}(\mathrm{M}_{nk})^{\mathrm{SL}_k} = \bigoplus_{d(\lambda)\le k} \rho_n^\lambda \otimes (\rho_k^\lambda)^{\mathrm{SL}_k},$$

where $(\rho_k^\lambda)^{\mathrm{SL}_k}$ denotes the space of SL_k invariant vectors in ρ_n^λ. Now $(\rho_k^\lambda)^{\mathrm{SL}_k} \neq 0$ if and only if $\lambda = m\mathbf{1}_k$ for some nonnegative integer m, i.e. ρ_n^λ is the one-dimensional space on which GL_k acts by the character $h \mapsto (\det h)^m$.

We shall write $\rho_k^{m\mathbf{1}_k}$ as $\det_k^m$. Then under the action by $\mathrm{GL}_n \times \mathrm{GL}_k$,

$$\mathfrak{A} = \bigoplus_{m=0}^{\infty} \mathfrak{A}_m \tag{3.3}$$

where for each m,

$$\mathfrak{A}_m \cong \rho_n^{m\mathbf{1}_k} \otimes \det_k^m .$$

Next, recall that N_n is the standard maximal unipotent subgroup in SO_n given in (2.3). Let

$$\mathfrak{A}^{N_n} = \mathcal{P}(\mathrm{M}_{nk})^{N_n \times \mathrm{SL}_k}$$

be the algebra of N_n invariants in $\mathfrak{A}$. We now describe the generators of $\mathfrak{A}^{N_n}$. For $1 \le i, j \le k$ and $x \in \mathrm{M}_{nk}$, let

$$r_{ij}^2(x) = \langle x_i, x_j \rangle$$

where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form given in (2.1), and x_i and x_j are the ith and jth columns of x respectively. Then the algebra $\mathcal{P}(\mathrm{M}_{nk})^{\mathrm{O}_n}$ of O_n invariants in $\mathcal{P}(\mathrm{M}_{nk})$ is a polynomial algebra on the generators $\{r_{ij}^2 : 1 \le i \le j \le k\}$ ([Ho], [GW]). Define

$$\gamma_j = \begin{vmatrix} & & & x_{11} & \cdots & x_{1k} \\ & 0 & & \vdots & & \vdots \\ & & & x_{j1} & \cdots & x_{jk} \\ x_{11} & \cdots & x_{j1} & r_{11}^2 & \cdots & r_{1k}^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{1k} & \cdots & x_{jk} & r_{k1}^2 & \cdots & r_{kk}^2 \end{vmatrix} \qquad (0 \le j \le k-1)$$

and

$$\gamma_k = \begin{vmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{vmatrix} .$$

Here, the vertical lines in γ_j indicate determinant. These polynomials are joint $\mathrm{SO}_n \times \mathrm{GL}_k$ highest weight vectors with the following weights:

	SO_n weight	GL_k weight
γ_0	0	$\det_k^2$
$\gamma_j \quad (1 \le j \le k-1)$	$2\mathbf{1}_j$	$\det_k^2$
γ_k	$\mathbf{1}_k$	$\det_k$

Proposition 3.1 ([ATZ]). *The algebra $\mathfrak{A}^{N_n}$ is a polynomial algebra on the generators $\gamma_0, \gamma_1, \ldots, \gamma_k$.*

Let $\mathbf{a} = (a_0, a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^{k+1}$, and

$$\gamma^{\mathbf{a}} = \gamma_0^{a_0}\gamma_1^{a_1}\cdots\gamma_k^{a_k}.$$

Then $\gamma^{\mathbf{a}}$ is an $\mathrm{SO}_n \times \mathrm{GL}_k$ highest weight vector, and it has SO_n weight $\lambda = (\lambda_1, \ldots, \lambda_k)$ and GL_k weight $\det_k^{m(\mathbf{a})}$ where

$$\lambda_j = 2\sum_{i=j}^{k-1} a_i + a_k \quad (1 \leq j \leq k-1), \quad \lambda_k = a_k$$

and

$$m(\mathbf{a}) = 2\sum_{i=0}^{k-1} a_i + a_k. \tag{3.4}$$

Let $\mathfrak{A}_{m,\mathbf{a}}$ be the irreducible O_n module generated by $\gamma^{\mathbf{a}}$. Then as a representation of O_n, $\mathfrak{A}_m$ admits the decomposition

$$\mathfrak{A}_m = \bigoplus_{\mathbf{a}} \mathfrak{A}_{m,\mathbf{a}}$$

where the sum is taken over all $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{k+1}$ such that $m(\mathbf{a}) = m$. For each $\mathfrak{A}_{m,\mathbf{a}}$ which appears in the sum, its SO_n highest weight λ is such that $\lambda_1 \leq m$ and $\lambda_j \equiv m \pmod 2$ for all j. Conversely, every such λ is the SO_n highest weight of a unique $\mathfrak{A}_{m,\mathbf{a}}$ in the sum. It follows that

$$\mathfrak{A}_m \cong \bigoplus_{\substack{\lambda_1 \leq m \\ \lambda_j \equiv m \,(\mathrm{mod}\, 2)}} \sigma_n^{\lambda}, \tag{3.5}$$

which is O_n multiplicity free: any two irreducible O_n submodules are non-isomorphic.

4. The map $\partial : \mathcal{P}(\mathrm{M}_{nk}) \to \mathcal{D}(\mathrm{M}_{nk})$. Let $\mathcal{D}(\mathrm{M}_{nk})$ be the space of constant-coefficient differential operators on $\mathcal{P}(\mathrm{M}_{nk})$, that is, $\mathcal{D}(\mathrm{M}_{nk})$ consists of the operators of the form

$$\sum_{\alpha} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where each $\alpha = (\alpha_{ij})$ appearing in the sum is an $n \times k$ matrix of nonnegative integers, $a_\alpha \in \mathbb{C}$,

$$|\alpha| = \sum_{i,j} \alpha_{ij}$$

and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_{11}^{\alpha_{11}} \partial x_{12}^{\alpha_{12}} \cdots \partial x_{nk}^{\alpha_{nk}}}.$$

The algebra $\mathcal{D}(\mathrm{M}_{nk})$ is naturally isomorphic to the symmetric algebra $S(\mathrm{M}_{nk})$ on M_{nk}. In fact, for each $u \in \mathrm{M}_{nk}$ and $f \in \mathcal{P}(\mathrm{M}_{nk})$, we let

$$D_u(f)(v) = \lim_{t\to 0} \frac{f(v+tu)-f(v)}{t}, \qquad v \in \mathrm{M}_{nk}.$$

Then the map $D : \mathrm{M}_{nk} \to \mathcal{D}(\mathrm{M}_{nk})$ is linear, and extends uniquely to an algebra isomorphism $D : S(\mathrm{M}_{nk}) \to \mathcal{D}(\mathrm{M}_{nk})$. Let GL_n act on M_{nk} as in (3.1), that is,

$$g.v = (g^{-1})^{\tau} v, \qquad g \in \mathrm{GL}_n,\ v \in \mathrm{M}_{nk}.$$

We extend this action to an action by GL_n on $S(\mathrm{M}_{nk})$ by algebra automorphisms, and this in turn induces an action on $\mathcal{D}(\mathrm{M}_{nk})$ via the map D.

For $1 \le i \le n$ and $1 \le j \le k$, let

$$\partial(x_{ij}) = \frac{\partial}{\partial x_{n+1-i,j}}.$$

Then ∂ extends uniquely to an algebra isomorphism

$$\partial : \mathcal{P}(\mathrm{M}_{nk}) \to \mathcal{D}(\mathrm{M}_{nk}).$$

Specifically, if $p = \sum_\alpha a_\alpha x^\alpha \in \mathcal{P}(\mathrm{M}_{nk})$, then

$$\partial(p) = \sum_\alpha a_\alpha \frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}} \tag{4.1}$$

where $\alpha' = (\alpha'_{ij})$ and $\alpha'_{ij} = \alpha_{n+1-i,j}$ for all $1 \le i \le n$ and $1 \le j \le k$.

LEMMA 4.1. *The map $\partial : \mathcal{P}(\mathrm{M}_{nk}) \to \mathcal{D}(\mathrm{M}_{nk})$ is an O_n map, that is, it commutes with the action of O_n on $\mathcal{P}(\mathrm{M}_{nk})$ and $\mathcal{D}(\mathrm{M}_{nk})$ defined by the restriction from the actions of GL_n.*

Proof. Define for $u, v \in \mathrm{M}_{nk}$,

$$(u, v) = [\partial^{-1}(D_v)](u).$$

For $1 \le j \le k$, let u_j (respectively v_j) be the jth column of u (respectively v). Then

$$(u, v) = \sum_{j=1}^{k} \langle u_j, v_j \rangle$$

where $\langle \cdot, \cdot \rangle$ is the symmetric form on $\mathbb{C}^n$ given in (2.1). Hence for each $n \times n$ complex matrix A, we have

$$(Au, v) = (u, A^{\tau} v).$$

Now for $g \in \mathrm{GL}_n$,

$$\begin{aligned}[\partial^{-1}(g.D_v)](u) &= [\partial^{-1}(D_{g.v})](u) = (u, g.v) = (g^{\tau}.u, v) \\ &= [\partial^{-1}(D_v)](g^{\tau}.u) = [(g^{-1})^{\tau}.(\partial^{-1}(D_v))](u).\end{aligned}$$

This shows that $\partial^{-1}(g.D_v) = (g^{-1})^\tau.(\partial^{-1}(D_v))$ for every $v \in \mathrm{M}_{nk}$. Since GL_n acts on $\mathcal{P}(\mathrm{M}_{nk})$ and $\mathcal{D}(\mathrm{M}_{nk})$ by algebra automorphisms and ∂^{-1} is an algebra isomorphism, we have $\partial^{-1}g = (g^{-1})^\tau\partial^{-1}$ on $\mathcal{D}(\mathrm{M}_{nk})$, or equivalently, $g\partial = \partial(g^{-1})^\tau$ on $\mathcal{P}(\mathrm{M}_{nk})$. In particular, $g\partial = \partial g$ for all $g \in \mathrm{O}_n$. This proves the lemma. ∎

For $1 \le i, j \le k$, let

$$\Delta_{ij} = \partial(r_{ij}^2) = \sum_{a=1}^{n} \frac{\partial^2}{\partial x_{a,i}\partial x_{n+1-a,j}}.$$

We also let

$$L = \partial(\gamma_0) = \begin{vmatrix} \Delta_{11} & \cdots & \Delta_{1k} \\ \vdots & & \vdots \\ \Delta_{n1} & \cdots & \Delta_{nk} \end{vmatrix}.$$

COROLLARY 4.2. *The map $L : \mathcal{P}(\mathrm{M}_{nk}) \to \mathcal{P}(\mathrm{M}_{nk})$ is an O_n map.*

Proof. This is because L is an O_n invariant in $\mathcal{D}(\mathrm{M}_{nk})$. ∎

Next, we shall define an inner product on $\mathcal{P}(\mathrm{M}_{nk})$ such that multiplication by γ_0 and the operator L are adjoints of each other. If $p = \sum_\alpha a_\alpha x^\alpha \in \mathcal{P}(\mathrm{M}_{nk})$, we let

$$\widetilde{p}(x) = \sum_\alpha \overline{a_\alpha} x^\alpha.$$

Here for each α, $\overline{a_\alpha}$ is the complex conjugate of a_α. Then for $p, q \in \mathcal{P}(\mathrm{M}_{nk})$, define

$$\langle p, q\rangle = \{[\partial(p)](\widetilde{q})\}(0). \tag{4.2}$$

Explicitly, if $p(x) = \sum_\alpha a_\alpha x^\alpha \in \mathcal{P}(\mathrm{M}_{nk})$ and $q(x) = \sum_\beta b_\beta x^\beta \in \mathcal{P}(\mathrm{M}_{nk})$, then by (4.1),

$$\langle p, q\rangle = \sum_{\alpha,\beta} a_\alpha \overline{b_\beta}\left[\frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}(x^\beta)\right](0) = \sum_\alpha \alpha'! a_\alpha \overline{b_{\alpha'}},$$

where for each $\alpha = (\alpha_{ij})$,

$$\alpha'! = \prod_{i,j} \alpha'_{ij}!.$$

From this it is easy to see that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{P}(\mathrm{M}_{nk})$, and homogeneous polynomials with distinct total degrees are orthogonal. In particular, the sum (3.3) is an orthogonal sum.

Moreover, for $p, q, r \in \mathcal{P}(\mathrm{M}_{nk})$,

$$\langle pq, r\rangle = \{[\partial(qp)](\widetilde{r})\}(0) = \{[\partial(q)\partial(p)](\widetilde{r})\}(0) = \langle q, \widetilde{[\partial(p)](\widetilde{r})}\rangle = \langle q, [\partial(\widetilde{p})](r)\rangle.$$

Hence the operator $\partial(\widetilde{p})$ is the adjoint to multiplication by p. In particular, since $\widetilde{\gamma_0} = \gamma_0$, $L = \partial(\widetilde{\gamma_0})$ is the adjoint to multiplication by γ_0.

5. Generalized Laplacian on Grassmannians. From the discussion in Section 3, we see that γ_0 generates the subalgebra of O_n invariants in $\mathfrak{A}$, that is,

$$\mathfrak{A}^{\mathrm{O}_n} = \mathbb{C}[\gamma_0].$$

Multiplication by γ_0 defines an injective O_n map on $\mathfrak{A}$. We shall abuse notation and denote this operator also by γ_0. Since γ_0 is also a GL_k eigenvector corresponding to $\det_k^2$, γ_0 defines an O_n map

$$\gamma_0 : \mathfrak{A}_m \to \mathfrak{A}_{m+2},$$

so its image $\gamma_0\mathfrak{A}_m$ in $\mathfrak{A}_{m+2}$ is an O_n submodule isomorphic to $\mathfrak{A}_m$.

We recall that the "dual" operator $L = \partial(\gamma_0)$ is the adjoint of γ_0 with respect to the inner product defined in (4.2). Let

$$\mathcal{H} = \{p \in \mathfrak{A} : L(p) = 0\}$$

be the space of all "harmonic polynomials" in $\mathfrak{A}$. For each $m \geq 0$, the restriction of L to $\mathfrak{A}_m$ also defines an O_n map

$$L : \mathfrak{A}_m \to \mathfrak{A}_{m-2}.$$

In fact, if $p \in \mathfrak{A}_\ell$, $q \in \mathfrak{A}_m$ and $\ell + 2 \neq m$, then

$$0 = \langle \gamma_0 p, q\rangle = \langle p, L(q)\rangle.$$

This shows that $L(q)$ is orthogonal to $\mathfrak{A}_\ell$ for $\ell \neq m-2$, so that $L(q) \in \mathfrak{A}_{m-2}$.

Next, we let

$$\mathcal{H}_m = \{f \in \mathfrak{A}_m : L(f) = 0\}.$$

Then we have

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

PROPOSITION 5.1. *For each* $m \geq 2$,

$$\mathcal{H}_m = \bigoplus_{\substack{\lambda_1 = m \\ \lambda_j \equiv m \,(\mathrm{mod}\, 2)}} \sigma_n^\lambda \quad \textit{and} \quad \mathfrak{A}_m = \mathcal{H}_m \oplus \gamma_0\mathfrak{A}_{m-2}.$$

Proof. Let $p \in \mathfrak{A}_{m-2}$ and $q \in \mathfrak{A}_m$. Then $\langle \gamma_0 p, q\rangle = \langle p, L(q)\rangle$. Thus if $q \in \mathcal{H}_m$, then

$$\langle \gamma_0 p, q\rangle = \langle p, 0\rangle = 0,$$

so that

$$q \in (\gamma_0\mathfrak{A}_m)^\perp,$$

the orthogonal complement of $\gamma_0\mathfrak{A}_m$ in $\mathfrak{A}_m$. It follows that $\mathcal{H}_m \subseteq (\gamma_0\mathfrak{A}_m)^\perp$.

Let σ_n^λ be an O_n representation which occurs in $\mathfrak{A}_m$ with $\lambda_1 = m$. Since $L : \mathfrak{A}_m \to \mathfrak{A}_{m-2}$ and σ_n^λ does not occur in $\mathfrak{A}_{m-2}$,

$$L(\sigma_n^\lambda) = 0.$$

So $\sigma_n^\lambda \subseteq \mathcal{H}_m$. But these O_n representations σ_n^λ together with those in $\gamma_0\mathfrak{A}_{m-2}$ have exhausted all the O_n representations in $\mathfrak{A}_m$. So the lemma follows. ∎

COROLLARY 5.2. *The space $\mathcal{H}^{N_n}$ of N_n invariants in $\mathcal{H}$ is the subalgebra of $\mathfrak{A}$ generated by $\{\gamma_1, \ldots, \gamma_k\}$.*

THEOREM 5.3. *For $m \geq 2$,*

$$\mathfrak{A}_m = \bigoplus_{j=0}^{[m/2]} \gamma_0^j \mathcal{H}_{m-2j}.$$

Consequently,

$$\mathfrak{A} \cong \mathcal{H} \otimes \mathfrak{A}^{\mathrm{O}_n}.$$

Proof. The first assertion follows from (3.5) by induction on m together with the observation that $\mathfrak{A}_0 = \mathcal{H}_0$ and $\mathfrak{A}_1 = \mathcal{H}_1$. The second assertion follows from the first. ∎

6. Eigenvalues of $L\gamma_0$ in the case $k = 2$. For each $m \geq 0$, we have the O_n maps

$$\gamma_0 : \mathfrak{A}_m \to \mathfrak{A}_{m+2} \quad \text{and} \quad L : \mathfrak{A}_{m+2} \to \mathfrak{A}_m.$$

By composing these two maps, we obtain the O_n map

$$L\gamma_0 : \mathfrak{A}_m \to \mathfrak{A}_m.$$

Since $\mathfrak{A}_m$ is multiplicity free as a representation of O_n, by Schur's Lemma, it acts by a scalar on each irreducible O_n submodule of $\mathfrak{A}_m$. Thus if $\mathfrak{A}_{m,\mathbf{a}}$ is an irreducible O_n submodule of $\mathfrak{A}_m$, then there exists a complex number $c(\mathbf{a})$ such that

$$L\gamma_0(v) = c(\mathbf{a})v, \quad \forall v \in \mathfrak{A}_{m,\mathbf{a}}.$$

In particular, if we take $v = \gamma^{\mathbf{a}}$, then

$$L\gamma_0(\gamma^{\mathbf{a}}) = c(\mathbf{a})\gamma^{\mathbf{a}}. \tag{6.1}$$

It is easy to compute $c(\mathbf{a})$ when $k = 1$, which has been discussed in the Introduction. We shall compute the scalar $c(\mathbf{a})$ in the case $k = 2$ in this section. In this case,

$$L = \begin{vmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12} & \Delta_{22} \end{vmatrix},$$

$$\gamma_0 = \begin{vmatrix} r_{11}^2 & r_{12}^2 \\ r_{12}^2 & r_{22}^2 \end{vmatrix}, \qquad \gamma_1 = \begin{vmatrix} 0 & x_{11} & x_{12} \\ x_{11} & r_{11}^2 & r_{12}^2 \\ x_{12} & r_{12}^2 & r_{22}^2 \end{vmatrix}, \qquad \gamma_2 = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

THEOREM 6.1. *If* $k = 2$ *and* $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{Z}^3_{\geq 0}$, *then*

$$c(\mathbf{a}) = 2(a_0+1)(2a_0+2a_1+3)(2a_0+2a_1+2a_2+n-1)(2a_0+4a_1+2a_2+n).$$

The theorem follows by induction on a_0 from Lemma 6.2 below. If T_1 and T_2 are two linear operators on $\mathfrak{A}$, then the commutator $[T_1, T_2]$ is the linear operator

$$[T_1, T_2] = T_1T_2 - T_2T_1.$$

LEMMA 6.2. *If* $k = 2$ *and* $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{Z}^3_{\geq 0}$, *then*

$$[L, \gamma_0](\gamma^{\mathbf{a}}) = 2(4a_0 + 4a_1 + 2a_2 + n)$$
$$\times (8a_0^2 + 4a_1^2 + 16a_0a_1 + 8a_0a_2 + 4a_1a_2 + 4a_0n + 2a_1n + 4a_1 + 6a_2 + 3n - 3)\gamma^{\mathbf{a}}.$$

Before we prove Lemma 6.2, we need to introduce some notations. Let $\mathcal{PD}(\mathrm{M}_{nk})$ be the algebra of polynomial-coefficient differential operators on $\mathcal{P}(\mathrm{M}_{nk})$. For $1 \leq i, j \leq k$, let

$$E_{ij} = \sum_{p=1}^{n} x_{pi} \frac{\partial}{\partial x_{pj}} + \delta_{ij} \frac{n}{2}.$$

Let

$$\mathfrak{p}^+ = \mathrm{Span}\{r_{ij}^2 : 1 \leq i \leq j \leq k\},$$
$$\mathfrak{p}^- = \mathrm{Span}\{\Delta_{ij} : 1 \leq i \leq j \leq k\},$$
$$\mathfrak{k} = \mathrm{Span}\{E_{ij} : 1 \leq i, j \leq k\}.$$

Then $\mathfrak{k}$ is a Lie algebra isomorphic to $\mathfrak{gl}_k$, and

$$\mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$$

forms a Lie algebra isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2k}$. We shall denote $\mathfrak{k}$ and $\mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ by $\mathfrak{gl}_k$ and $\mathfrak{sp}_{2k}$ respectively. Then $\mathfrak{sp}_{2k}$ generates the algebra $\mathcal{PD}(\mathrm{M}_{nk})^{\mathrm{O}_n}$ of operators in $\mathcal{PD}(\mathrm{M}_{nk})$ commuting with O_n ([Ho], [GW]). Consequently, $\mathcal{PD}(\mathrm{M}_{nk})^{\mathrm{O}_n}$ is a homomorphic image of the universal enveloping algebra $\mathcal{U}(\mathfrak{sp}_{2k})$ of $\mathfrak{sp}_{2k}$.

Proof of Lemma 6.2. Since both L and γ_0 are elements of $\mathcal{PD}(\mathrm{M}_{n2})^{\mathrm{O}_n}$, so is $[L, \gamma_0]$. We shall express $[L, \gamma_0]$ in the form

$$[L, \gamma_0] = X + YE_{12} + ZE_{21}$$

where X, Y, Z are elements $\mathcal{PD}(\mathrm{M}_{n2})^{\mathrm{O}_n}$ such that the expression for X does not involve the elements E_{12} and E_{21}. Since GL_2 acts on $\gamma^{\mathbf{a}}$ by a determinant character, the Lie algebra $\mathfrak{sl}_2$ of SL_2 will annihilate it. In particular,

$$E_{12}(\gamma^{\mathbf{a}}) = E_{21}(\gamma^{\mathbf{a}}) = 0.$$

Consequently,

$$[L, \gamma_0](\gamma^{\mathbf{a}}) = X(\gamma^{\mathbf{a}}) + YE_{12}(\gamma^{\mathbf{a}}) + ZE_{21}(\gamma^{\mathbf{a}}) = X(\gamma^{\mathbf{a}}). \tag{6.2}$$

Thus it suffices to compute $X(\gamma^{\mathbf{a}})$.

We now determine the elements X, Y and Z. We have

$$\begin{aligned}[L, \gamma_0] &= \left[\begin{vmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12} & \Delta_{22} \end{vmatrix}, \gamma_0\right] = \begin{vmatrix} [\Delta_{11}, \gamma_0] & \Delta_{12} \\ [\Delta_{12}, \gamma_0] & \Delta_{22} \end{vmatrix} + \begin{vmatrix} \Delta_{11} & [\Delta_{12}, \gamma_0] \\ \Delta_{12} & [\Delta_{22}, \gamma_0] \end{vmatrix} \\ &= [\Delta_{11}, \gamma_0]\Delta_{22} - [\Delta_{12}, \gamma_0]\Delta_{12} + \Delta_{11}[\Delta_{22}, \gamma_0] - \Delta_{12}[\Delta_{12}, \gamma_0]\end{aligned} \tag{6.3}$$

We will need the following commutation relations in $\mathfrak{sp}_4$ in our computations:

$$\begin{aligned}[\Delta_{ab}, r^2_{cd}] &= \delta_{bc}E_{da} + \delta_{ac}E_{db} + \delta_{bd}E_{ca} + \delta_{ad}E_{cb}, \\ [E_{ab}, r^2_{cd}] &= \delta_{bc}r^2_{ad} + \delta_{bd}r^2_{ac}, \\ [E_{ab}, \Delta_{cd}] &= -\delta_{ac}\Delta_{bd} - \delta_{ad}\Delta_{cb}.\end{aligned}$$

Using these formulas, we obtain

$$\begin{aligned}[\Delta_{11}, \gamma_0] &= -2r^2_{22} + 4r^2_{22}E_{11} - 4r^2_{12}E_{21}, \\ [\Delta_{22}, \gamma_0] &= -2r^2_{11} + 4r^2_{11}E_{22} - 4r^2_{12}E_{12}, \\ [\Delta_{12}, \gamma_0] &= 2r^2_{12} - 2r^2_{12}(E_{11} + E_{22}) + 2r^2_{11}E_{21} + 2r^2_{22}E_{12}.\end{aligned}$$

Substituting these expressions into (6.3) and simplifying, we obtain

$$\begin{aligned}X &= 4(r^2_{22}\Delta_{22} + r^2_{12}\Delta_{12})E_{11} + 4(r^2_{11}\Delta_{11} + r^2_{12}\Delta_{12})E_{22} \\ &\quad - 14E_{11} + 2E_{22} + 16E_{11}E_{22} + 2(E_{11} + E_{22})^2, \\ Y &= -4r^2_{22}\Delta_{12} - 4r^2_{12}\Delta_{11} - 16E_{21}, \\ Z &= -4r^2_{12}\Delta_{22} - 4r^2_{11}\Delta_{12}.\end{aligned}$$

It follows from (6.2) that

$$\begin{aligned}[L, \gamma_0](\gamma^{\mathbf{a}}) &= X(\gamma^{\mathbf{a}}) \\ &= 4(r^2_{22}\Delta_{22} + r^2_{12}\Delta_{12})E_{11}(\gamma^{\mathbf{a}}) + 4(r^2_{11}\Delta_{11} + r^2_{12}\Delta_{12})E_{22}(\gamma^{\mathbf{a}}) \\ &\quad + \{-14E_{11} + 2E_{22} + 16E_{11}E_{22} + 2(E_{11} + E_{22})^2\}(\gamma^{\mathbf{a}}).\end{aligned}$$

Now

$$E_{11}(\gamma^{\mathbf{a}}) = E_{22}(\gamma^{\mathbf{a}}) = \lambda\gamma^{\mathbf{a}} \quad \text{where} \quad \lambda = 2a_0 + 2a_1 + a_2 + n/2.$$

So

$$[L,\gamma_0](\gamma^{\mathbf{a}}) = 4\lambda T(\gamma^{\mathbf{a}}) + \{-14\lambda + 2\lambda + 16\lambda^2 + 2(2\lambda)^2\}\gamma^{\mathbf{a}} = 4\lambda T(\gamma^{\mathbf{a}}) + 12\lambda(2\lambda-1)\gamma^{\mathbf{a}} \tag{6.4}$$

where T is the operator

$$T = r_{11}^2 \varDelta_{11} + r_{22}^2 \varDelta_{22} + 2r_{12}^2 \varDelta_{12}.$$

We note that the operator T corresponds to the trivial representation of GL_2 in the tensor product $S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^{2*})$. So it will act as a scalar on $\gamma^{\mathbf{a}}$. We now compute this scalar.

Routine calculations show

$$\begin{aligned}
\varDelta_{11}(\gamma^{\mathbf{a}}) &= 2a_0(2a_0+4a_1+2a_2+n-3) r_{22}^2 \gamma_0^{a_0-1}\gamma_1^{a_1}\gamma_2^{a_2} \\
&\quad - 2a_1(2a_1+2a_2+n-4) x_{12}^2 \gamma_0^{a_0}\gamma_1^{a_1-1}\gamma_2^{a_2}, \\
\varDelta_{22}(\gamma^{\mathbf{a}}) &= 2a_0(2a_0+4a_1+2a_2+n-3) r_{11}^2 \gamma_0^{a_0-1}\gamma_1^{a_1}\gamma^{a_2} \\
&\quad - 2a_1(2a_1+2a_2+n-4) x_{11}^2 \gamma_0^{a_0}\gamma_1^{a_1-1}\gamma_2^{a_2}, \\
\varDelta_{12}(\gamma^{\mathbf{a}}) &= -2a_0(2a_0+4a_1+2a_2+n-3) r_{12}^2 \gamma_0^{a_0-1}\gamma_1^{a_1}\gamma_2^{a_2} \\
&\quad + 2a_1(2a_1+2a_2+n-4) x_{11}x_{12} \gamma_0^{a_0}\gamma_1^{a_1-1}\gamma_2^{a_2}.
\end{aligned}$$

Using these formulas, we obtain

$$\begin{aligned}
T(\gamma^{\mathbf{a}}) &= 2a_0(2a_0+4a_1+2a_2+n-3)\{r_{11}^2 r_{22}^2 + r_{22}^2 r_{11}^2 - 2(r_{12}^2)^2\}\gamma_0^{a_0-1}\gamma_1^{a_1}\gamma_2^{a_2} \\
&\quad + 2k(2a_1+2a_2+n-4)\{-x_{12}^2 r_{11}^2 - x_{11}^2 r_{22}^2 + 2x_{11}x_{12} r_{12}^2\}\gamma_0^{a_0}\gamma_1^{k-1}\gamma_2^{a_2} \\
&= 2a_0(2a_0+4a_1+2a_2+n-3)\{2\gamma_0\}\gamma_0^{a_0-1}\gamma_1^{k}\gamma_2^{a_2} \\
&\quad + 2a_1(2a_1+2a_2+n-4)\{\gamma_1\}\gamma_0^{a_0}\gamma_1^{k-1}\gamma_2^{a_2} \\
&= \{4a_0(2a_0+4a_1+2a_2+n-3) + 2a_1(2a_1+2a_2+n-4)\}\gamma^{\mathbf{a}}.
\end{aligned}$$

This together with (6.4) gives

$$\begin{aligned}
[L,\gamma_0](\gamma^{\mathbf{a}}) &= 4\lambda T(\gamma^{\mathbf{a}}) + 12\lambda(2\lambda-1)\gamma^{\mathbf{a}} \\
&= 4\lambda\{4a_0(2a_0+4a_1+2a_2+n-3) + 2a_1(2a_1+2a_2+n-4)\}\gamma^{\mathbf{a}} \\
&\qquad + 12\lambda(2\lambda-1)\gamma^{\mathbf{a}} \\
&= 2(4a_0+4a_1+2a_2+n)(8a_0^2 + 4a_1^2 + 16a_0a_1 + 8a_0a_2 \\
&\qquad + 4a_1a_2 + 4a_0n + 2a_1n + 4a_1 + 6a_2 + 3n - 3)\gamma^{\mathbf{a}}.
\end{aligned}$$

This proves the lemma. ■

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