COLLOQUIUM MATHEMATICUM

VOL. 94 2002 NO. 1

COMPACTNESS CRITERIA IN FUNCTION SPACES

BY

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Abstract. The classical criterion for compactness in Banach spaces of functions can be reformulated into a simple tightness condition in the time-frequency domain. This description preserves more explicitly the symmetry between time and frequency than the classical conditions. The result is first stated and proved for $\mathbf{L}^2(\mathbb{R}^d)$, and then generalized to coorbit spaces. As special cases, we obtain new characterizations of compactness in Besov–Triebel–Lizorkin, modulation and Bargmann–Fock spaces.

1. Introduction. Compactness in function spaces is usually characterized by conditions of the Arzelà–Ascoli type. Typically, what is necessary is an equicontinuity condition with respect to the norm of the space under consideration. If the underlying topological space is not compact, then in addition a tightness condition is required, i.e., all functions have the same "essential" support. The prototype of such a result is the characterization of compactness in $\mathbf{L}^p$-spaces, which in its general form on locally compact abelian groups is due to A. Weil. In what follows, χ_U will denote the indicator function of a compact set U.

THEOREM 1 ([15]). *A closed and bounded subset S of $\mathbf{L}^p(\mathbb{R}^d)$ for $1 \le p < \infty$ is compact if and only if the following conditions are satisfied*:

(i) *Equicontinuity*: *for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{f\in S} \sup_{|h|\le\delta} \|f(\cdot - h) - f\|_p < \varepsilon. \tag{1}$$

(ii) *Tightness*: *for all $\varepsilon > 0$ there exists a compact set U in $\mathbb{R}^d$ such that*

$$\sup_{f\in S} \|f\chi_U - f\|_p < \varepsilon. \tag{2}$$

Far-reaching generalizations of Theorem 1 to general translation invariant Banach spaces of distributions with a so-called double module structure were proved in [5].

2000 *Mathematics Subject Classification*: 46B50, 42B35.

Key words and phrases: compactness, Besov spaces, modulation spaces, Bargmann–Fock spaces, coorbit spaces, wavelet transform, short-time Fourier transform.

Specializing to $\mathbf{L}^2(\mathbb{R}^d)$, it is well known and not difficult to see that the equicontinuity condition (1) is equivalent to the tightness of the Fourier transforms $\widehat{S} = \{\widehat{f} : f \in S\}$ in $\mathbf{L}^2(\mathbb{R}^d)$. In particular, *a closed and bounded set* $S \subseteq \mathbf{L}^2(\mathbb{R}^d)$ *is compact if and only if* S *and* $\widehat{S}$ *are both tight in* $\mathbf{L}^2(\mathbb{R}^d)$ (see [5] and [12]).

The symmetry of this characterization under the Fourier transform motivated us to look at analytic tools which are designed expressedly to deal with situations that treat a function and its Fourier transform *simultaneously* and to search for a characterization of compactness by means of these tools. In this regard the short-time Fourier transform is the tool that is used most frequently to describe both time and frequency simultaneously, i.e., a function and its Fourier transform.

Definition 1 (Short-time Fourier transform). Let M_ω and T_x denote the frequency shift by ω and time shift by x, respectively, i.e., $M_\omega T_x g(t) = e^{2\pi i t\omega} g(t-x)$ for $(x,\omega) \in \mathbb{R}^{2d}$. The *short-time Fourier transform* (STFT) of a function $f \in \mathbf{L}^2(\mathbb{R}^d)$ with respect to a window function $g \in \mathbf{L}^2(\mathbb{R}^d)$ is defined as

$$\mathcal{S}_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g}(t-x)e^{-2\pi i\omega t}\,dt = \langle f, M_\omega T_x g\rangle. \tag{3}$$

With slightly different normalization, the short-time Fourier transform also occurs under the names "(radar) ambiguity function" or "(cross-) Wigner distribution" (see [10]). For suitable windows g, e.g. g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, the value $\mathcal{S}_g f(x,\omega)$ can be interpreted as a measure for the energy of f at $z = (x,\omega) \in \mathbb{R}^{2d}$. An important property in the study of compactness is the isometry property of the short-time Fourier transform, which states that for any $f, g \in \mathbf{L}^2(\mathbb{R}^d)$,

$$\|\mathcal{S}_g f\|_2 = \|g\|_2 \|f\|_2. \tag{4}$$

It becomes intuitively obvious that a condition comprising the support conditions given in Theorem 1 for time and frequency separately can be formulated as a simultaneous tightness condition in time and frequency via the short-time Fourier transform.

Theorem 2 (Compactness in $\mathbf{L}^2(\mathbb{R}^d)$). *For a closed and bounded set* $S \subseteq \mathbf{L}^2(\mathbb{R}^d)$ *the following statements are equivalent*:

(i) S *is compact in* $\mathbf{L}^2(\mathbb{R}^d)$.

(ii) *The set* $\{\mathcal{S}_g f : f \in S\}$ *is tight in* $\mathbf{L}^2(\mathbb{R}^{2d})$; *this means that for all* $\varepsilon > 0$ *there exists a compact set* $U \subseteq \mathbb{R}^{2d}$ *such that*

$$\sup_{f\in S}\Big(\int_{U^c} |\mathcal{S}_g f(x,\omega)|^2\,dx\,d\omega\Big)^{1/2} < \varepsilon. \tag{5}$$

Proof. To get an idea about possible generalizations we give the pretty proof of this theorem right here. Without loss of generality we assume that $\|g\|_2 = 1$, so that $\mathcal{S}_g$ is an isometry on $\mathbf{L}^2(\mathbb{R}^d)$.

(i)⇒(ii). By compactness of S we can find $f_1, \ldots, f_n$ such that

$$\min_{j=1,\ldots,n} \|f - f_j\|_2 < \varepsilon/2 \quad \text{for all } f \in S.$$

Since $\mathcal{S}_g f_j \in \mathbf{L}^2(\mathbb{R}^{2d})$ by (4), we may choose a compact set $U \subseteq \mathbb{R}^{2d}$ such that $\int_{U^c} |\mathcal{S}_g f_j(x,\omega)|^2 \, dx \, d\omega < \varepsilon^2/4$ for $j = 1, \ldots, n$. From (4) we deduce for arbitrary $f \in S$ that

$$\begin{aligned}\Big(\int_{U^c} |\mathcal{S}_g f(x,\omega)|^2 \, dx \, d\omega\Big)^{1/2} &\le \min_{j=1,\ldots,n} \Big\{\Big(\int_{U^c} |\mathcal{S}_g (f - f_j)(x,\omega)|^2 \, dx \, d\omega\Big)^{1/2} \\ &\qquad + \Big(\int_{U^c} |\mathcal{S}_g f_j(x,\omega)|^2 \, dx \, d\omega\Big)^{1/2}\Big\} \\ &\le \min_j \|f - f_j\|_2 + \varepsilon/2 < \varepsilon.\end{aligned}$$

(ii)⇒(i). It suffices to show that every sequence (f_n) in S contains a convergent subsequence. By (ii) we can choose a compact set $U \subseteq \mathbb{R}^{2d}$ such that

$$\int_{U^c} |\mathcal{S}_g f(x,\omega)|^2 \, dx \, d\omega < \varepsilon^2 \tag{6}$$

for all $f \in S$, in particular for the sequence (f_n). Since by assumption S is bounded, it is weakly compact in $\mathbf{L}^2(\mathbb{R}^d)$ and thus (f_n) has a weakly convergent subsequence $f_j = f_{n_j}$ with limit f, i.e., $\langle f_j, h\rangle \to \langle f, h\rangle$ for all $h \in \mathbf{L}^2(\mathbb{R}^d)$. If we choose $h = M_\omega T_x g$ for $(x,\omega) \in \mathbb{R}^{2d}$, this implies the pointwise convergence of the short-time Fourier transforms

$$\mathcal{S}_g f_j(x,\omega) \to \mathcal{S}_g f(x,\omega) \qquad \text{for } x, \omega \in \mathbb{R}^d. \tag{7}$$

Since by (3) and the Cauchy–Schwarz inequality we have, for all (x,ω),

$$|\mathcal{S}_g (f - f_j)(x,\omega)| \le \|f - f_j\|_2 \le \sup_j \|f_j\|_2 + \|f\|_2 < C,$$

the restriction of $|\mathcal{S}_g(f - f_j)|^2$ to U is dominated by the constant function $C^2 \chi_U \in \mathbf{L}^1(\mathbb{R}^d)$. In view of (7) we may now apply the dominated convergence theorem to obtain

$$\int_U |\mathcal{S}_g (f - f_j)(x,\omega)|^2 \, dx \, d\omega \to 0. \tag{8}$$

The combination of (8) and (6) now yields

$$\overline{\lim_{j\to\infty}} \|f - f_j\|_2 = \overline{\lim_{j\to\infty}} \|\mathcal{S}_g(f - f_j)\|_2$$

$$\leq \overline{\lim_{j\to\infty}} \Big(\int_U |\mathcal{S}_g(f-f_j)(x,\omega)|^2 \, dx \, d\omega \Big)^{1/2} + \overline{\lim_{j\to\infty}} \Big(\int_{U^c} |\mathcal{S}_g(f-f_j)(x,\omega)|^2 \, dx \, d\omega \Big)^{1/2}$$

$$\leq 0 + 2\varepsilon.$$

Therefore $\lim_{j\to\infty} \|f - f_j\|_2 = 0$ and thus S is compact. ∎

Theorem 2 and its proof suggest several extensions. On the one hand, we may replace the $\mathbf{L}^2$-norm of the short-time Fourier transform by other norms and ask for which function spaces we can still characterize compactness as in Theorem 2. Pursuing this idea leads to the characterization of compactness in the so-called modulation spaces (Section 3.1).

On the other hand, if we are willing to give up the time-frequency interpretation of Theorems 1 and 2, we may replace the short-time Fourier transform by other transforms. As a further example occurring in modern analysis we consider the wavelet transform, which shares the important isometry property with the STFT [4].

Definition 2 (Continuous wavelet transform). Let

$$T_x D_s g(t) = s^{-d/2} g(s^{-1}(t-x))$$

for $(x,s) \in \mathbb{R}^d \times \mathbb{R}^+$. The *continuous wavelet transform* of a function $f \in \mathbf{L}^2(\mathbb{R}^d)$ with respect to a wavelet $g \in \mathbf{L}^2(\mathbb{R}^d)$ is defined to be

$$\mathcal{W}_g f(x,s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) g\Big(\frac{t-x}{s}\Big) \, dt = \langle f, T_x D_s g\rangle. \tag{9}$$

If g is radial and satisfies the admissibility condition

$$\int_{\mathbb{R}^+} |\widehat{g}(t\omega)|^2 \, \frac{dt}{t} = 1 \quad \text{for all } \omega \in \mathbb{R}^d \setminus \{0\}, \tag{10}$$

then $\int |\mathcal{W}_g f(x,s)|^2 \, dx \, ds/|s|^{d+1} = \|f\|^2$ and thus $\mathcal{W}_g$ is an isometry for $\mathbf{L}^2(\mathbb{R}^d)$ (see [4]).

The same proof as for Theorem 2 with the wavelet transform in place of the STFT now yields the following criterion for compactness in $\mathbf{L}^2(\mathbb{R}^d)$.

Theorem 3 (Compactness in $\mathbf{L}^2(\mathbb{R}^d)$ via wavelet transform). *Let $g \in \mathbf{L}^2(\mathbb{R}^d)$ satisfy condition* (10). *A closed and bounded set $S \subseteq \mathbf{L}^2(\mathbb{R}^d)$ is compact in $\mathbf{L}^2(\mathbb{R}^d)$ if and only if for all $\varepsilon > 0$ there exists a compact set $U \subseteq \mathbb{R}^d \times \mathbb{R}^+$ such that*

$$\sup_{f\in S} \Big(\int_{U^c} |\mathcal{W}_g f(x,s)|^2 \, \frac{dx\,ds}{s^{d+1}} \Big)^{1/2} < \varepsilon. \tag{11}$$

The STFT and the wavelet transform do have other properties in common. Both are defined as the inner product of f with the action of a group of unitary operators on a fixed function g. More precisely, both the STFT and the wavelet transform are *representation coefficients* of a certain unitary continuous representation π of a group $\mathcal{G}$ on a Hilbert space $\mathcal{H}$. This observation has been very fruitful for the evolution of the general wavelet theory [6, 7, 9]. In our context we shall take the proof of Theorem 2 as an outline to obtain compactness criteria for a general class of function spaces defined by means of other group representations.

With each irreducible unitary continuous representation π of a locally compact group on a Hilbert space $\mathcal{H}$ satisfying some additional integrability condition, we associate a family of abstract function spaces, the so-called *coorbit spaces*. For these spaces we will prove compactness criteria analogous to those of Theorems 2 and 3. Upon choosing a particular group and a natural representation, we will recover the above statements. In addition we will obtain compactness criteria for modulation spaces by means of the short-time Fourier transform, similar to Theorem 2, and using the wavelet transform we will characterize compactness in Besov–Triebel–Lizorkin spaces. Another modification leads to a new compactness criterion for Bargmann–Fock spaces.

The paper is organized as follows. Section 2 introduces the concept of coorbit spaces and deals with technical difficulties arising in the generalization of Theorem 2 to coorbit spaces. These concern the validity of dominated convergence and the theorem of Alaoglu–Bourbaki. In Section 2.2 we state the main theorem for general coorbit spaces. In Section 3 we treat the application of this theorem to several classes of well known function spaces.

In a subsequent project we will apply the new compactness criteria to study operators on coorbit spaces.

2. Coorbit spaces

2.1. *Preliminaries and definition.* We first recall the theory of coorbit spaces. For simplicity we omit some technical details and refer the reader to [6, 7, 9] where the theory has been thoroughly investigated.

The theory of coorbit spaces requires the following basic structures:

- a locally compact group $\mathcal{G}$ with Haar measure dz,
- an irreducible unitary representation π of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$,
- a continuous submultiplicative weight function ν on $\mathcal{G}$, i.e., ν satisfies $\nu(z_1 + z_2) \le \nu(z_1)\nu(z_2)$ and $\nu(z_1) \ge 1$ for all $z_1, z_2 \in \mathcal{G}$,
- a Banach space $(Y, \|\cdot\|_Y)$ of functions on $\mathcal{G}$.

Functions of the form $z \in \mathcal{G} \mapsto \langle f, \pi(z)g\rangle$ are called *representation coefficients* of π. Upon inspection we see that the short-time Fourier transform defined in (3) is (up to a trivial factor) a representation coefficient of the Schrödinger representation of the Heisenberg group $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ on $\mathbf{L}^2(\mathbb{R}^d)$, given by $\pi(x, y, \tau)f(t) = \tau e^{2\pi i y(t-x)} f(t - x)$. Likewise the wavelet transform is a representation coefficient of the $ax + b$-group $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}^+$ of the representation $\varrho(x, s)f(t) = s^{-d/2} f(s^{-1}(t - x))$.

For reasons of compatibility and well-definedness we impose the following conditions on $\mathcal{G}, (\pi, \mathcal{H}), \nu$. We refer to [6] for a detailed justification of these assumptions.

(A) π is irreducible, unitary, continuous and ν-integrable, i.e., there exists $g \in \mathcal{H}$, $g \neq 0$, such that

$$\int_{\mathcal{G}} |\langle \pi(z)g, g\rangle| \nu(z)\, dz < \infty. \tag{12}$$

(B) Y is a *solid* Banach function space on $\mathcal{G}$, i.e., if $F \in Y$ and G is measurable with $|G(z)| \leq |F(z)|$ for almost all $z \in \mathcal{G}$, then $G \in Y$ and $\|G\|_Y \leq \|F\|_Y$.

(C) Y is invariant under right and left translations and satisfies the convolution relation $Y * \mathbf{L}^1_\nu(\mathcal{G}) \subseteq Y$, with $\|F * G\|_Y \leq \|F\|_Y \|G\|_{\mathbf{L}^1_\nu(\mathcal{G})}$ for $F \in Y$, $G \in \mathbf{L}^1_\nu(\mathcal{G})$.

It follows that $\mathbf{L}^\infty_0(\mathcal{G})$, the space of bounded functions with compact support on $\mathcal{G}$, is contained in Y. This property will be crucial in the proof of our main statement.

We introduce the following notation for the representation coefficient of π:

$$\mathcal{V}_g f(z) = \langle f, \pi(z)g\rangle \quad \text{ for } z \in \mathcal{G}.$$

Definition 3 (Abstract test functions and distributions). We fix $g_0 \in \mathcal{H} \setminus \{0\}$ satisfying (12). Then the space $\mathcal{A}_\nu$ of test functions is defined as

$$\mathcal{A}_\nu = \{g \in \mathcal{H} : \|g\|_{\mathcal{A}_\nu} = \|\mathcal{V}_{g_0} g\|_{\mathbf{L}^1_\nu(\mathcal{G})} < \infty\}$$

The space $\mathcal{A}_\nu$ is dense in $\mathcal{H}$, its dual $\mathcal{A}'_\nu$, the space of all (conjugate-) linear continuous functionals on $\mathcal{A}_\nu$, contains $\mathcal{H}$ and plays the role of a space of distributions. It will serve us as a reservoir of selection.

Definition 4 (Coorbit spaces). Under the hypotheses (A), (B), (C) imposed on $\mathcal{G}, \pi, \mathcal{H}, \nu$, fix any $g \in \mathcal{A}_\nu \setminus \{0\}$. Then the *coorbit space* of Y under the representation π is defined as

$$\mathcal{C}o_\pi Y \ = \{f \in \mathcal{A}'_\nu : \mathcal{V}_g f \in Y\}$$

with norm $\|f\|_{\mathcal{C}o_\pi Y} = \|\mathcal{V}_g f\|_Y$.

Then $\mathcal{C}o_\pi Y$ has the following properties (see [6] for details):

(i) $\mathcal{C}o_\pi Y$ is a Banach space invariant under the action of π. Specifically,

$$\|\pi(z)f\|_{\mathcal{C}o_\pi Y} \le C\nu(z)\|f\|_{\mathcal{C}o_\pi Y} \quad \text{for } f \in \mathcal{C}o_\pi Y. \tag{13}$$

(ii) The definition of $\mathcal{C}o_\pi Y$ is independent of the choice of $g \in \mathcal{A}_\nu$.

(iii) Different functions $g \in \mathcal{A}_\nu \setminus \{0\}$ define equivalent norms on $\mathcal{C}o_\pi Y$.

(iv) By definition $\mathcal{C}o_\pi Y$ is a subspace of $\mathcal{A}'_\nu$ and we also have

$$\|f\|_{\mathcal{A}'_\nu} \le C\|f\|_{\mathcal{C}o_\pi Y} \tag{14}$$

(v) Special cases: $\mathcal{C}o_\pi \mathbf{L}^2(\mathcal{G}) = \mathcal{H}$, $\mathcal{C}o_\pi \mathbf{L}^1_\nu(\mathcal{G}) = \mathcal{A}_\nu$, and $\mathcal{C}o_\pi \mathbf{L}^\infty_{1/\nu}(\mathcal{G}) = \mathcal{A}'_\nu$.

In order to obtain compactness criteria analogous to Theorem 2 for general coorbit spaces, we have to impose further assumptions on Y. As pointed out at the end of Section 1, we need to use a norm $\|\cdot\|_Y$ for which dominated convergence holds. In our treatment of dominated convergence we follow [2, Ch. 1.3].

Definition 5. A Banach function space Y on $\mathcal{G}$ is said to have *absolutely continuous norm* if $\|f\chi_{E_n}\|_Y \to 0$ for all f and for every sequence $\{E_n\}_{n=1}^\infty$ of measurable subsets of $\mathcal{G}$ satisfying $E_n \to \emptyset$ almost everywhere with respect to Haar measure.

Proposition 1 ([2]). *For a Banach function space Y the following are equivalent*:

(i) *Y has absolutely continuous norm.*

(ii) *Dominated convergence holds for all $f \in Y$: If $f_n \in Y$, $n = 1, 2, \ldots$, and $g \in Y$ satisfy $|f_n| \le |g|$ for all n and $f_n(z) \to f(z)$ a.e., then $\|f_n - f\|_Y \to 0$.*

(iii) *The dual space Y' of Y coincides with its associate space Y^* defined as*

$$Y^* = \Big\{ g \text{ measurable} : \sup_{f\in Y,\, \|f\|_Y \le 1} \int_{\mathcal{G}} |f(z)g(z)|\, dz < \infty \Big\}. \tag{15}$$

Example (Mixed-norm spaces). Let m be a weight function on $\mathbb{R}^{2d}$ and let $1 \le p, q \le \infty$. Then the weighted mixed norm space $\mathbf{L}^{p,q}_m(\mathbb{R}^{2d})$ consists of all measurable functions on $\mathbb{R}^{2d}$ such that the norm

$$\|F\|_{\mathbf{L}^{p,q}_m} = \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |F(x,\omega)|^p m(x,\omega)^p \, dx \Big)^{q/p} d\omega \Big)^{1/q}$$

is finite, with the usual modifications when $p = \infty$ or $q = \infty$.

If m is a "moderate" weight with respect to the submultiplicative weight ν, i.e., $m(z_1 + z_2) \le C\nu(z_1)m(z_2)$, then hypotheses (B) and (C) are always satisfied (see [10, Prop. 11.1.3]).

If $p, q < \infty$, then $\mathbf{L}^{p,q}_m$ has absolutely continuous norm. For $p = q = 1$ this is just Lebesgue's theorem on dominated convergence. If $p, q < \infty$, then the dual space is $\mathbf{L}^{p',q'}_{1/m}$, where $1/p + 1/p' = 1$ (cf. [1]). As a consequence of

Hölder's inequality the dual space coincides with the associate space defined in (15). By Proposition 1, $\mathbf{L}_m^{p,q}$ has absolutely continuous norm.

2.2. *Compactness in coorbit spaces.* We are now ready to state and prove our main theorem, a criterion for compactness in coorbit spaces.

THEOREM 4 (Compactness in $\mathcal{C}o_\pi Y$). *In addition to the general assumptions* (A), (B), *and* (C), *assume that* Y *has absolutely continuous norm. For a closed and bounded set* $S \subseteq \mathcal{C}o_\pi Y$ *the following statements are equivalent*:

(i) S *is compact in* $\mathcal{C}o_\pi Y$.

(ii) *For all* $\varepsilon > 0$ *there exists a compact set* $U \subseteq \mathcal{G}$ *such that*

$$\sup_{f\in S} \|\chi_{U^c}\mathcal{V}_g f\|_Y < \varepsilon. \tag{16}$$

Proof. The argument follows the simpler proof of Theorem 2.

(i)$\Rightarrow$(ii). Assume that S is compact in $\mathcal{C}o_\pi Y$ and let $\varepsilon > 0$. Then there exist $f_1, \ldots, f_n \in S$ such that

$$\min_{j=1,\ldots,n} \|f - f_j\|_{\mathcal{C}o_\pi Y} < \varepsilon/2 \qquad \text{for all } f \in S.$$

Since $\mathbf{L}_0^\infty(\mathcal{G})$ is contained in Y as a consequence of (B) and (C) and since Y has absolutely continuous norm, $\mathbf{L}_0^\infty(\mathcal{G})$ is even dense in Y; see [5, Prop. 1.4]. Hence, there exist $H_j \in \mathbf{L}_0^\infty \subseteq Y$ with $\|H_j - \mathcal{V}_g f_j\|_Y < \varepsilon/2$. By solidity of Y, H_j can be chosen as the restriction $\chi_U \mathcal{V}_g f_j$ for some compact set $U \subseteq \mathcal{G}$, and thus we see that

$$\|\chi_{U^c}\mathcal{V}_g f_j\|_Y < \varepsilon/2 \qquad \text{for } j = 1, \ldots, n.$$

Then for general $f \in S$ we find that

$$\begin{aligned}\|\chi_{U^c}\mathcal{V}_g f\|_Y &\le \min_{j=1,\ldots,n} (\|\chi_{U^c}\mathcal{V}_g(f - f_j)\|_Y + \|\chi_{U^c}\mathcal{V}_g f_j\|_Y)\\ &\le \min_{j=1,\ldots,n} \|\mathcal{V}_g(f - f_j)\|_Y + \varepsilon/2\\ &= \min_{j=1,\ldots,n} \|f - f_j\|_{\mathcal{C}o_\pi Y} + \varepsilon/2 < \varepsilon\end{aligned}$$

In the second inequality we have applied condition (B) to the pointwise estimate

$$|\chi_{U^c}\mathcal{V}_g(f - f_j)(z)| \le |\mathcal{V}_g(f - f_j)(z)|.$$

(ii)$\Rightarrow$(i). Assume that (16) holds. Again it suffices to show that every sequence $(f_n) \subseteq S$ contains a convergent subsequence.

To extract a weak-star convergent subsequence of $(f_n) \subseteq S$, we modify the argument for $\mathbf{L}^2$ as follows: Since S is bounded and closed in $\mathcal{C}o_\pi Y$, it is also bounded and closed in $\mathcal{A}'_\nu$ by (14), and therefore S is weak-star compact in $\mathcal{A}'_\nu$ by Alaoglu's theorem. Consequently, we can find a weak-star convergent subsequence f_{n_j} of (f_n), which we again denote by f_j, with limit

f_∞ in S, i.e. $\langle f_j, h\rangle \to \langle f_\infty, h\rangle$ for all $h \in \mathcal{A}_\nu$. In particular, for $h = \pi(z)g$, we obtain *pointwise* convergence of the representation coefficients on $\mathcal{G}$:

$$\mathcal{V}_g f_j(z) \to \mathcal{V}_g f_\infty(z) \quad \text{for all } z \in \mathcal{G}.$$

Next we show that the sequence $\{\mathcal{V}_g(f_\infty - f_j) : j \in \mathbb{N}\}$ is uniformly bounded on any compact set $U \subseteq \mathcal{G}$. We have

$$|\langle f_\infty - f_j, \pi(z)g\rangle| \le \|f_\infty - f_j\|_{\mathcal{A}'_\nu} \|\pi(z)g\|_{\mathcal{A}_\nu},$$

by duality, and $\|\pi(z)g\|_{\mathcal{A}_\nu} \le \nu(z)\|g\|_{\mathcal{A}_\nu}$ by (13). Therefore

$$\sup_{z\in U} |\mathcal{V}_g(f_\infty - f_j)(z)| \le \|g\|_{\mathcal{A}_\nu} \sup_{z\in U} \nu(z) \sup_{j\in\mathbb{N}} \|f_\infty - f_j\|_{\mathcal{A}'_\nu} \le C\chi_U(z).$$

Since $\chi_U \in Y$ and Y has absolutely continuous norm, we can apply dominated convergence (Proposition 1(ii)) to obtain

$$\lim_{j\to\infty} \|\chi_U \mathcal{V}_g(f_\infty - f_j)\|_Y = 0. \tag{17}$$

To deal with the behavior of $\mathcal{V}_g(f_\infty - f_j)$ on the complement U^{c}, we use the assumption (16). Given $\varepsilon > 0$, we choose $U \subseteq \mathcal{G}$ so that $\|\chi_{U^{\mathrm{c}}} \mathcal{V}_g f\|_Y < \varepsilon/2$ for all $f \in S \cup \{f_\infty\}$. The combination of these steps now yields

$$\begin{aligned}
\lim_{j\to\infty} \|f_\infty - f_j\|_{\mathcal{C}o_\pi Y} &= \lim_{j\to\infty} \|\mathcal{V}_g(f_\infty - f_j)\|_Y \\
&\le \lim_{j\to\infty} \|\chi_U \mathcal{V}_g(f_\infty - f_j)\|_Y + \overline{\lim_{j\to\infty}} \|\chi_{U^{\mathrm{c}}} \mathcal{V}_g(f_\infty - f_j)\|_Y \\
&\le 0 + 2 \sup_{f\in S\cup\{f_\infty\}} \|\chi_{U^{\mathrm{c}}} \mathcal{V}_g f\|_Y < 2\varepsilon.
\end{aligned}$$

Therefore any sequence in S has a subsequence that converges in $\mathcal{C}o_\pi Y$ and so S is compact. ∎

REMARKS. 1. Loosely speaking, Theorem 4 states that a set in $\mathcal{C}o_\pi Y$ is compact if and only if the set of representation coefficients is tight in Y.

2. Note that in the first part of the proof we have only used the fact that $\mathbf{L}_0^\infty(\mathcal{G})$ is dense in Y. On the other hand, the absolutely continuous norm of Y is only needed for the proof of sufficiency of (ii) for S to be compact. If $\mathbf{L}_0^\infty(\mathcal{G})$ is not dense in Y, then condition (ii) characterizes the compactness in the closed subspace $\mathcal{C}o_\pi Y_0$ of $\mathcal{C}o_\pi Y$, where Y_0 is the closure of $\mathbf{L}_0^\infty(\mathcal{G})$ in Y.

3. If Y does not have absolutely continuous norm, then one may alternatively apply the compactness criterion to the coorbit corresponding to the closed subspace $Y_a \subseteq Y$ of all "functions of absolutely continuous norm" (see [2, Ch. 1.3]).

3. Examples. Theorem 4 yields a handy compactness criterion for most function spaces commonly used in analysis. We now give several concrete manifestations of Theorem 4. In order to apply it we have to verify that all the conditions on $\mathcal{G}, \pi, \nu$ and Y are satisfied. Once the general setting is described, this is an easy task.

3.1. *Modulation spaces.* Modulation spaces are those function spaces which are associated with the short-time Fourier transform (3).

Their standard definition is as follows. Fix a non-zero "window function" $g \in \mathcal{S}(\mathbb{R}^d)$ and consider moderate functions m satisfying $m(z_1 + z_2) \leq C(1 + |z_1|)^s m(z_2)$, $z_1, z_2 \in \mathbb{R}^{2d}$, for some constants $C, s \geq 0$; for instance $m(z) = (1 + |z|)^a$ for $a \in \mathbb{R}$ is moderate with respect to $\nu(z) = (1 + |z|)^{|a|}$. Then the *modulation space* $\mathbf{M}_m^{p,q}(\mathbb{R}^d)$ is defined as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\mathcal{V}_g f \in \mathbf{L}_m^{p,q}(\mathbb{R}^{2d})$, with norm

$$\|f\|_{\mathbf{M}_m^{p,q}(\mathbb{R}^d)} = \|\mathcal{S}_g f\|_{\mathbf{L}_m^{p,q}(\mathbb{R}^{2d})}.$$

For a detailed theory of the modulation spaces we refer to [10, Chs. 11–13] where they are treated for even more general classes of weight functions. As particularly important modulation space we mention $\mathbf{M}_m^{1,1}$ with constant weight $m \equiv 1$. In the abstract notation this is just $\mathcal{A}_\nu$; it is a Segal algebra and is denoted by S_0 in harmonic analysis.

To interpret modulation spaces as coorbit spaces, we extend the time-frequency shifts $(x, \omega) \mapsto T_x M_\omega$ to a unitary representation of the Heisenberg group. Let $\mathbb{H} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ be the d-dimensional reduced Heisenberg group with multiplication

$$(x_1, \omega_1, e^{2\pi i \tau_1})(x_2, \omega_2, e^{2\pi i \tau_2}) = (x_1 + x_2, \omega_1 + \omega_2, e^{2\pi i(\tau_1 + \tau_2)} e^{\pi i x_2 \cdot \omega_1})$$

and let π be the Schrödinger representation of $\mathbb{H}$ acting on $\mathbf{L}^2(\mathbb{R}^d)$ by time-frequency shifts

$$\pi(x, \omega, \tau) = e^{2\pi i \tau} T_x M_\omega. \tag{18}$$

Then π is an irreducible unitary representation of $\mathbb{H}$ on $\mathbf{L}^2(\mathbb{R}^d)$. The representation coefficient for the Gaussian $\phi(t) = e^{-\pi t \cdot t}$ is $\langle \phi, \pi(x, \omega, \tau)\phi \rangle = 2^{-d/2}\, \overline{\tau} e^{\pi i x \cdot \omega}\, e^{-\pi(x \cdot x + \omega \cdot \omega)}$, therefore π is integrable with respect to any weight $\nu(x) = \mathcal{O}(e^{\alpha |z|}), \alpha \geq 0$ (see [10]). Furthermore observe that

$$|\mathcal{V}_g f(x, \omega)| = |\langle f, \pi(x, \omega, \tau) g \rangle|.$$

Now consider the auxiliary space $\widetilde{\mathbf{L}_m^{p,q}}$ consisting of all measurable functions f on $\mathbb{H}$ such that

$$\Big(\int_{\tau \in \mathbb{T}} |f(x, \omega, \tau)|^2 \, d\tau \Big)^{1/2} \in \mathbf{L}_m^{p,q};$$

then the modulation spaces can be interpreted as coorbit spaces by

$$\mathbf{M}_m^{p,q}(\mathbb{R}^d) = \mathcal{C}o_\pi \widetilde{\mathbf{L}_m^{p,q}}.$$

If $1 \leq p, q < \infty$, then $\mathbf{L}_0^\infty(\mathbb{H})$ is dense in $\widetilde{\mathbf{L}_m^{p,q}}$. We have already verified after Proposition 1 that the spaces $\widetilde{\mathbf{L}_m^{p,q}}$ have absolutely continuous norm.

Therefore all conditions of Theorem 4 are satisfied, and we obtain the following more explicit characterization of compactness in modulation spaces.

THEOREM 5 (Compactness in $\mathbf{M}^{p,q}_m(\mathbb{R}^d)$). *Let* $0 \neq g \in \mathbf{M}^1_\nu(\mathbb{R}^d)$, $1 \leq p, q < \infty$ *and* S *be a closed and bounded subset of* $\mathbf{M}^{p,q}_m(\mathbb{R}^d)$. *Then* S *is compact in* $\mathbf{M}^{p,q}_m(\mathbb{R}^d)$ *if and only if for all* $\varepsilon > 0$ *there exists a compact set* $U \subseteq \mathbb{R}^{2d}$ *such that*

$$\sup_{f \in S} \|\chi_{U^c} \mathcal{S}_g f\|_{\mathbf{L}^{p,q}_m} < \varepsilon.$$

REMARK. Clearly such a characterization cannot hold when $p = \infty$ or $q = \infty$. In this case $\mathbf{M}^{p,q}_m(\mathbb{R}^d)$ is the dual of a non-reflexive Banach space, and compactness in norm takes on a different shape.

3.2. *Besov–Triebel–Lizorkin spaces.* The Besov–Triebel–Lizorkin spaces are those function spaces that can be associated with the wavelet transform (9). Let $g \in \mathcal{S}(\mathbb{R}^d)$ be a fixed non-zero radial function with all moments vanishing. Then the homogeneous Besov space $\dot{\mathbf{B}}^\alpha_{p,q}(\mathbb{R}^d)$ contains all tempered distributions (modulo polynomials) such that

$$\|f\|^q_{\mathbf{B}^\alpha_{p,q}} = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |\langle f, \varrho(x,s)g\rangle|^p \, dx \Big)^{q/p} s^{-q(\alpha + d/2 - d/q)} \, \frac{ds}{s^{d+1}} < \infty. \tag{19}$$

This definition is equivalent to the standard definition given in [13] (see also [14]).

To interpret the Besov spaces as coorbit spaces, we consider the $(ax+b)$-group $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}^+$ with multiplication $(b,a) \cdot (x,y) = (ax + b, ay)$ for $b, x \in \mathbb{R}^d$, $a, y \in \mathbb{R}^+$, and the representation of $\mathcal{G}$ on $\mathbf{L}^2(\mathbb{R}^d)$ by translations and dilations

$$\varrho(x,s) f(t) = s^{-d/2} f(s^{-1}(t - x)). \tag{20}$$

Again it is easy to see that this representation is integrable with respect to all weights of the form $\nu(x,s) = \max(1, s^\alpha)$ for some $\alpha \geq 0$ by choosing $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp} \widehat{g} \subseteq \{t \in \mathbb{R}^d : 0 < c \leq |t| \leq d < \infty\}$. However, this representation is reducible, and thus the general theory of Section 2 is not immediately applicable. To save the situation, we take the extended group $\mathbb{R}^d \times (\mathbb{R}^+ \times \mathrm{SO}(d))$ with the representation $\pi(x,s,\mathcal{O}) f(t) = s^{-d/2} f(s^{-1}(\mathcal{O}^{-1}(t - x)))$, $\mathcal{O} \in \mathrm{SO}(d)$, acting on $\mathbf{L}^2(\mathbb{R}^d)$. Then π is again irreducible. Now take a wavelet g that is rotation invariant; then

$$\langle f, \pi(x,s,\mathcal{O}) g\rangle = \langle f, \varrho(x,s) g\rangle.$$

Comparing with (19), we see that

$$\dot{\mathbf{B}}^\alpha_{p,q}(\mathbb{R}^d) = \mathcal{C}o_\pi \mathbf{L}^{p,q}_{\alpha + d/2 - d/q}$$

where the subscript refers to the weight $\nu(x,t,\mathcal{O}) = t^{-(\alpha + d/2 - d/q)}$ on the extended $(ax+b)$-group. As before all assumptions of Theorem 4 are satisfied, and we obtain the following new characterization of compactness in Besov spaces.

THEOREM 6 (Compactness in $\dot{\mathbf{B}}^\alpha_{p,q}(\mathbb{R}^d)$). *A closed and bounded set $S \subseteq \dot{\mathbf{B}}^\alpha_{p,q}(\mathbb{R}^d)$, $1 \le p, q < \infty$, is compact in $\dot{\mathbf{B}}^\alpha_{p,q}(\mathbb{R}^d)$ if and only if for all $\varepsilon > 0$ there exists a compact set $U \subseteq \mathbb{R}^d \times \mathbb{R}^+$ such that*

$$\sup_{f \in S} \|\chi_{U^c} \mathcal{W}_g f\|_{\mathbf{L}^{p,q}_\nu} < \varepsilon.$$

Similarly, all Triebel–Lizorkin spaces $\mathbf{F}^\alpha_{p,q}(\mathbb{R}^d)$, among them $\mathbf{L}^p$ and the Hardy spaces, can be defined as the coorbits of so-called tent spaces $\mathbf{T}^{p,q}_\nu$ on $\mathcal{G}$ (cf. [3]). The compactness in $\mathbf{F}^\alpha_{p,q}(\mathbb{R}^d)$ can be characterized as in Theorem 6 with $\mathbf{L}^{p,q}_\nu$ replaced by $\mathbf{T}^{p,q}_\nu$. Since the classical criterion of Theorem 1 is much simpler to use, we omit the explicit formulation of Theorem 6 for $\mathbf{F}^\alpha_{p,q}(\mathbb{R}^d)$-spaces.

3.3. *Bargmann–Fock spaces.* Finally we study a class of function spaces occurring in complex analysis (see [11, 8]).

DEFINITION 6. The *Bargmann–Fock spaces* $\mathcal{F}^p = \mathcal{F}^p(\mathbb{C}^d)$, $p < \infty$, are the Banach spaces of entire functions F on $\mathbb{C}^d$ for which the norm

$$\|F\|_{\mathcal{F}^p(\mathbb{C}^d)} = \Big(\int_{\mathbb{C}^d} |F(z)|^p e^{-\pi p |z|^2/2} \, dz \Big)^{1/p}$$

is finite.

REMARK. $\mathcal{F}^2$ is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{F}^2} = \int_{\mathbb{C}^d} F(z) \overline{G(z)} e^{-\pi |z|^2} \, dz,$$

which is isometrically isomorphic to $\mathbf{L}^2(\mathbb{R}^d)$ via the Bargmann transform (see [10, Ch. 3]).

By identifying $\mathbb{H} = \mathbb{R}^{2d} \times \mathbb{T}$ with $\mathbb{C}^d \times \mathbb{T}$ and using the notation of [10, p. 183], the Heisenberg group acts on $\mathcal{F}^2$ via the Bargmann–Fock representation β as follows:

$$\beta(z,\tau) F(w) = e^{2\pi i \tau} e^{\pi z \cdot w} F(w - \overline{z}) e^{-\pi |z|^2/2}, \qquad z, w \in \mathbb{C}^d,\ |\tau| = 1.$$

Then β is irreducible on $\mathcal{F}^2$ and is in fact equivalent to the Schrödinger representation π of (18). Therefore β enjoys all properties required to apply Theorem 4. The following compactness criterion for Bargmann–Fock spaces seems to be new.

THEOREM 7 (Compactness in $\mathcal{F}^p(\mathbb{C}^d)$). *Let $1 \le p < \infty$. A closed and bounded set $S \subseteq \mathcal{F}^p(\mathbb{C}^d)$ is compact in $\mathcal{F}^p(\mathbb{C}^d)$ if and only if for all $\varepsilon > 0$*

there exists a compact set $U \subseteq \mathbf{C}^d$ such that

$$\sup_{F\in S}\Big(\int_{U^c}|F(z)|^p e^{-\pi p|z|^2/2}\,dz\Big)^{1/p} < \varepsilon.$$

Proof. We show the identification $\mathcal{F}^p(\mathbb{C}^d) = \mathcal{C}o_\beta\mathbf{L}^p(\mathbb{R}^d)$ by using the properties of $\mathcal{F}^2$ as a reproducing kernel Hilbert space [10, Thm. 3.4.2].

We know that $F(\xi) = \langle F, e^{\pi z\overline{\xi}}\rangle$. In particular, for the constant $\mathbf{1}$ we obtain

$$|\langle \mathbf{1}, \beta(z,\tau)\mathbf{1}\rangle| = |\langle \mathbf{1}, e^{\pi\xi\overline{z}}\rangle e^{-\pi|z|^2/2}| = e^{-\pi|z|^2/2},$$

which implies that β is integrable with respect to arbitrary weights $\nu(x) = \mathcal{O}(e^{\alpha|z|})$ and that $\mathbf{1}\in\mathcal{A}_\nu$. Furthermore, the identity

$$|\langle F, \beta(z,\tau)\mathbf{1}\rangle| = |\langle F, e^{\pi\xi\overline{z}}\, e^{-\pi|z|^2/2}\rangle| = |F(z)|e^{-\pi|z|^2/2}$$

implies that

$$\langle F, \beta(z,\tau)\mathbf{1}\rangle \in \mathbf{L}^p(\mathbb{R}^d) \;\Leftrightarrow\; F\in\mathcal{F}^p(\mathbb{C}^d).$$

This means that $\mathcal{F}^p(\mathbb{C}^d) = \mathcal{C}o_\beta\mathbf{L}^p(\mathbb{R}^d)$. Thus $\mathcal{F}^p(\mathbb{C}^d)$ is a coorbit space and so the statement follows from Theorem 4. ■

REMARK. We leave it to the reader to generalize the result to *weighted* Bargmann–Fock spaces $\mathcal{F}^p_\nu(\mathbb{C}^d)$ or to "mixed-norm" Bargmann–Fock spaces.

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Received 4 December 2001 (4141)