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ELEMENTARY PROOFS OF SOME BASIC SUBTEMPERATURE THEOREMS

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Abstract. We present simple elementary proofs of several theorems about temperatures and subtemperatures. Most of these are concerned with mean values over heat spheres, heat balls, and modified heat balls, with applications to proving Harnack theorems and the monotone approximation of subtemperatures by smooth subtemperatures.

The basic theorems about classical subharmonic functions have been proved using only simple elementary techniques. However, the same cannot be said for the basic theorems about subtemperatures (subharmonic functions related to the heat equation, subparabolic functions). The harmonic space approach of Bauer [1] requires at the outset the knowledge that the regular sets for the heat equation form a basis for the Euclidean topology, and avoids results which can naturally be derived from Green's formula. The heat ball approach, begun by Watson in [15], also requires considerable knowledge of regular points at an early stage. The subparabolic function approach of Doob [4] is based upon the Poisson integral formula for solutions of the heat equation on a rectangle, and quickly becomes entangled in excessively complicated details.

The present paper is motivated by a desire to make the subject as accessible as possible. It is therefore partly an expository article and partly a research paper. Most of the results have been published before, but most of the proofs have not. The attempt to provide easy elementary proofs has led to some new theorems. Some very simple proofs which have appeared before are included for completeness.

As in [15], the approach is based on heat balls because they are much too important to ignore. Let

$$W(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-\|x\|^2/(4t)) & \text{if } t>0,\\ 0 & \text{if } t\le 0 \end{cases}$$

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be the fundamental solution of the heat equation

$$\Theta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = 0$$

on $\mathbb{R}^{n+1} = \{(x,t) : x \in \mathbb{R}^n,\ t \in \mathbb{R}\}$. Then, for any point $(x_0, t_0) \in \mathbb{R}^{n+1}$ and any number $c > 0$, the set

$$\Omega(x_0, t_0; c) = \{(y,s) : W(x_0 - y, t_0 - s) > (4\pi c)^{-n/2}\}$$

is called the *heat ball* with *centre* (x_0, t_0) and *radius* c. It is a convex domain contained in the circular cylinder

$$\{(y,s) : \|x_0 - y\|^2 < 2nce^{-1},\ t_0 - c < s < t_0\}.$$

The boundary of the heat ball, namely

$$\{(x_0,t_0)\} \cup \left\{ (y,s) : \|x_0 - y\| = \left(2n(t_0 - s) \log \frac{c}{t_0 - s} \right)^{1/2},\ t_0 - c \le s < t_0 \right\},$$

is the corresponding *heat sphere*.

Throughout this paper, E denotes an arbitrary open subset of $\mathbb{R}^{n+1}$, and $C^{2,1}(E)$ denotes the class of all functions u on E such that the partial derivatives $\partial^2 u/\partial x_i \partial x_j$ $(i,j \in \{1,\dots,n\})$ and $\partial u/\partial t$ all exist and are continuous on E.

Our starting point is Green's formula for the heat equation

$$(1) \qquad \iint_D (v\Theta w - w\Theta^* v)\, dy\, ds = \int_{\partial D} (\langle v\nabla_x w - w\nabla_x v, \nu_x\rangle - vw\nu_t)\, d\sigma,$$

in which D is a bounded open set whose boundary is piecewise smooth,

$$\Theta^* v = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial y_i^2} + \frac{\partial v}{\partial s},$$

$\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^n$,

$$\nabla_x w = \left(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n} \right),$$

$\nu = (\nu_x, \nu_t)$ is the outward unit normal to ∂D, and σ denotes surface area measure on ∂D. (See, for example, [16, p. 43].) The special case where $v = 1$, that is,

$$(2) \qquad \iint_D \Theta w\, dy\, ds = \int_{\partial D} (\langle \nabla_x w, \nu_x\rangle - w\nu_t)\, d\sigma,$$

is also important. The first five theorems are all derived from Green's formula. They are concerned with the mean values of functions in $C^{2,1}(E)$ over heat spheres. These mean values, described in detail below, are the ones that characterize temperatures (solutions of the heat equation) among the continuous functions, and which were used to define subtemperatures in [15].

Theorems 2 and 5, as well as part of Theorem 3, specialize to the case of functions u such that $\Theta u \geq 0$, which are the smooth subtemperatures. In Theorem 2 these functions are characterized in terms of their mean values over heat spheres; in Theorem 3 their mean values are shown to be increasing functions of the radius; and in Theorem 5 a convexity property of their means is established. The section ends with an elementary evaluation of the mean values of translates of W.

The next section contains analogues of the above-mentioned properties for the mean values of functions $u \in C^{2,1}(E)$ such that $\Theta u \geq 0$ over the heat balls themselves. These volume means, however, have serious deficiencies when applied to general subtemperatures, due to the unboundedness of the kernel. So, in the following section, volume mean values over modified heat balls are developed. This is done by considering a fuction $u \in C^{2,1}(E)$ as a function on $\mathbb{R}^m \times E$ that does not depend on the first m variables, taking the volume mean in $\mathbb{R}^m \times E$, and integrating out the extra variables. The modified heat ball $\Omega_m(x_0, t_0; c)$ has the form

$$\{(x_0, t_0)\} \cup \left\{(y,s) : \|x_0 - y\| = \left(2(m+n)(t_0 - s)\log\frac{c}{t_0 - s}\right)^{1/2},\ t_0 - c \leq s < t_0\right\}.$$

The kernel is bounded if $m \geq 3$, and has a smooth extension by zero to the whole of $\mathbb{R}^{n+1}$ if $m \geq 5$. The properties of the original volume means carry over to the modified volume means. But if $m \geq 5$, the modified means are much better for dealing with functions that are merely locally integrable. The key to this is Theorem 10, a new result which leads to a mean value characterization of temperatures in the class of locally integrable functions, and later to a new proof of the Harnack monotone convergence theorem.

In the subsequent section we define "m-subtemperatures" for every non-negative integer m, using the volume means over modified heat balls $\Omega_m(x_0, t_0; c)$. So the subtemperatures of [15] are 0-subtemperatures. We prove that the limit of a decreasing sequence of m-subtemperatures is itself an m-subtemperature if a mild finiteness condition is satisfied, and that m-subtemperatures are locally integrable functions, for all $m \geq 0$. The Harnack monotone convergence theorem for temperatures follows from these results and the characterization of temperatures among the locally integrable functions. Bauer's form of the Harnack inequality [1] is deduced as a consequence.

The final section contains further evidence that m-subtemperatures with $m \geq 5$ are preferable to those with $m = 0$. A theorem on the monotone approximation of m-subtemperatures by smooth subtemperatures is proved for $m \geq 5$. This enables us to deduce that the mean values of m-subtemperatures over heat spheres, heat balls, and modified heat balls, have similar properties to those of smooth subtemperatures. We also show that the class of all m-subtemperatures is independent of m for $m \geq 5$. Everything is proved by elementary methods.

For many results whose proofs require a mean value inequality, such as the maximum principle [15], the precise form of the kernel is immaterial. It therefore appears that the most elementary approach to the subject is obtained by taking $m \geq 5$ from the outset.

We now introduce the important mean values over heat spheres. For each $x \in \mathbb{R}^n$ and $t > 0$, we put

$$Q(x,t) = \|x\|^2(4\|x\|^2t^2 + (\|x\|^2 - 2nt)^2)^{-1/2};$$

we also put $Q(0,0) = 1$. The restriction to the heat sphere $\partial\Omega(x_0,t_0;c)$ of the function $(x,t) \mapsto Q(x_0 - x, t_0 - t)$ is continuous (see Lemma 1 below), and is positive except for a zero at $(0,c)$. We write $\tau(c) = (4\pi c)^{-n/2}$ for all $c > 0$, and put

$$\mathcal{M}(u;x_0,t_0;c) = \tau(c) \int_{\partial\Omega(x_0,t_0;c)} Q(x_0 - x, t_0 - t)u(x,t)\,d\sigma$$

for any function u such that the integral exists. We require the following details about Q.

Lemma 1. *Let $(y,s) \in \partial\Omega(x_0,t_0;c)$. Then*

(i) *$Q(x_0 - y, t_0 - s) \to 1$ as $(y,s) \to (x_0,t_0)$, and*
(ii) *$\tau(c)Q(x_0 - y, t_0 - s) = -\langle \nabla_x W(x_0 - y, t_0 - s), \nu_x(y,s)\rangle$.*

Proof. (i) Putting $x = x_0 - y$ and $t = t_0 - s$, we have

$$\begin{aligned} Q(x,t) &= \left(\frac{4t^2}{\|x\|^2} + \left(1 - \frac{2nt}{\|x\|^2}\right)^2\right)^{-1/2} \\ &= \left(\frac{2t}{n\log(c/t)} + \left(1 - \frac{1}{\log(c/t)}\right)^2\right)^{-1/2} \\ &\to 1 \quad \text{as } t \to 0. \end{aligned}$$

(ii) A routine calculation. ∎

Mean values of smooth functions over heat spheres

Theorem 1. *If $\overline{\Omega}(x_0,t_0;c) \subseteq E$, $u \in C^{2,1}(E)$, and $W_0(x,t) = W(x_0 - x, t_0 - t)$, then*

$$\mathcal{M}(u;x_0,t_0;c) - u(x_0,t_0) = \iint_{\Omega(x_0,t_0;c)} (W_0(y,s) - \tau(c))\Theta u(y,s)\,dy\,ds.$$

Proof. We apply Green's formula (1) with D the following truncation of $\Omega(c)$: For any t such that $t_0 - c < t < t_0$, we put

$$\Psi(t) = \{(y,s) \in \Omega(c) : s < t\}.$$

We divide $\partial\Psi(t)$ into two parts:

$$A(t) = \partial\Psi(t) \cap (\mathbb{R}^n \times \{t\}), \qquad B(t) = \partial\Psi(t) \setminus A(t).$$

Taking $D = \Psi(t)$, $v = W_0$, and $w = u$ in (1), we obtain

$$(3) \qquad \iint\limits_{\Psi(t)} W_0 \Theta u \, dy \, ds = \int\limits_{\partial\Psi(t)} (\langle W_0 \nabla_x u - u \nabla_x W_0, \nu_x \rangle - u W_0 \nu_t) \, d\sigma,$$

because $\Theta^* W_0 = 0$ on $\Psi(t)$. The right-hand side of (3) can be written as

$$(4) \qquad - \int\limits_{A(t)} u W_0 d\sigma + \tau(c) \int\limits_{B(t)} (\langle \nabla_x u, \nu_x \rangle - u\nu_t) \, d\sigma - \int\limits_{B(t)} u \langle \nabla_x W_0, \nu_x \rangle \, d\sigma,$$

because $\nu_x = 0$, $\nu_t = 1$ on $A(t)$, and $W_0 = \tau(c)$ on $B(t)$.

As $t \to t_{0-}$ we have

$$- \int\limits_{B(t)} u \langle \nabla_x W_0, \nu_x \rangle \, d\sigma \to \mathcal{M}(u; x_0, t_0; c)$$

in view of Lemma 1; and

$$\int\limits_{B(t)} (\langle \nabla_x u, \nu_x \rangle - u\nu_t) d\sigma \to \int\limits_{\partial\Omega(c)} (\langle \nabla_x u, \nu_x \rangle - u\nu_t) \, d\sigma = \iint\limits_{\Omega(c)} \Theta u \, dy \, ds$$

by Green's formula (2) with $w = u$ and $D = \Omega(c)$. We now consider the integral over $A(t)$ in (4). Writing $r = t_0 - t$ and $\varrho(r) = 2nr \log(c/r)$ for $r > 0$, we have

$$\int\limits_{A(t)} W_0 \, d\sigma = \int\limits_{\|x_0 - y\| < \sqrt{\varrho(r)}} (4\pi r)^{-n/2} \exp\left(- \frac{\|x_0 - y\|^2}{4r} \right) dy$$

$$= \frac{1}{\Gamma(n/2)} \int\limits_0^{\varrho(r)/(4r)} s^{n/2-1} e^{-s} \, ds \to 1^-$$

as $r \to 0^+$. Therefore, as $t \to t_{0-}$,

$$\Big| \int\limits_{A(t)} u(y,t) W_0(y,t) \, dy - u(x_0, t_0) \Big|$$

$$\leq \Big| \int\limits_{A(t)} (u(y,t) - u(x_0,t_0)) W_0(y,t) \, dy \Big| + \Big| u(x_0,t_0) \Big(\int\limits_{A(t)} W_0(y,t) \, dy - 1 \Big) \Big|$$

$$\leq \sup\{ |u(y,t) - u(x_0,t_0)| : \|x_0 - y\| < \varrho(r) \} + o(1) = o(1).$$

Taking the limit as $t \to t_0-$ in (3), we therefore obtain

$$\iint_{\Omega(c)} W_0 \Theta u \, dy \, ds = -u(x_0, t_0) + \tau(c) \iint_{\Omega(c)} \Theta u \, dy \, ds + \mathcal{M}(u; x_0, t_0; c),$$

and the result follows. ∎

Theorem 1 was proved by Smyrnélis [14, formula (7)]. It has the following consequences.

THEOREM 2. *Let $u \in C^{2,1}(E)$. If $\Theta u \geq 0$ on E, then the inequality*

$$u(x_0, t_0) \leq \mathcal{M}(u; x_0, t_0; c) \tag{5}$$

holds whenever $\overline{\Omega}(x_0, t_0; c) \subseteq E$. Conversely if, given any point $(x_0, t_0) \in E$ and $\varepsilon > 0$, we can find $c < \varepsilon$ such that (5) holds, then $\Theta u \geq 0$ on E.

Proof. If $\overline{\Omega}(x_0, t_0; c) \subseteq E$, then by Theorem 1,

$$\mathcal{M}(u; x_0, t_0; c) - u(x_0, t_0) = \iint_{\Omega(x_0, t_0; c)} (W_0(y, s) - \tau(c)) \Theta u(y, s) \, dy \, ds.$$

By definition of the heat ball, $W_0(y, s) > \tau(c)$ if $(y, s) \in \Omega(x_0, t_0; c)$. Therefore if $\Theta u \geq 0$ on E, then $\mathcal{M}(u; x_0, t_0; c) \geq u(x_0, t_0)$. On the other hand, if $\Theta u < 0$ at some point of E, then the continuity of Θu implies that $\Theta u < 0$ on some open subset D of E. Then $\mathcal{M}(u; x_0, t_0; c) < u(x_0, t_0)$ for all c so small that $\overline{\Omega}(x_0, t_0; c) \subseteq D$, so that the conditions for the converse fail to hold. ∎

COROLLARY. *Let $u \in C^{2,1}(E)$. If u is a temperature on E, then the equality*

$$u(x_0, t_0) = \mathcal{M}(u; x_0, t_0; c) \tag{6}$$

holds whenever $\overline{\Omega}(x_0, t_0; c) \subseteq E$. Conversely if, given any point $(x_0, t_0) \in E$ and $\varepsilon > 0$, we can find $c < \varepsilon$ such that (6) holds, then u is a temperature on E.

Proof. Apply Theorem 2 to both u and $-u$. ∎

Theorem 2 was proved for $n = 1$ by Pini [13] (the converse under a stronger condition). The general case is due to Watson [15].

The first part of the corollary was proved for $n = 1$ by Pini [13], and for general n by the three authors Fulks [5], Smyrnélis [14], and Kuptsov [8], apparently independently of each other. The converse, with u assumed to be merely continuous but with (6) assumed to hold whenever $\overline{\Omega}(x_0, t_0; c) \subseteq E$, was proved by Pini [13] for $n = 1$, and by Fulks [5] for general n. Subsequently Kuptsov [8] proved a weaker version.

Before establishing further theorems, we need to introduce a notation for the region between two concentric heat spheres.

DEFINITION. Given a point $(x_0, t_0) \in \mathbb{R}^{n+1}$, and numbers b, c with $0 < b < c$, we put

$$\begin{aligned} A(b,c) = A(x_0,t_0;b,c) &= \{(y,s) : \tau(c) < W(x_0-y,t_0-s) < \tau(b)\} \\ &= \Omega(c) \setminus \overline{\Omega}(b), \end{aligned}$$

and call it the *heat annulus*, with *centre* (x_0, t_0), *inner radius* b, and *outer radius* c. Observe that $\partial A(b,c) = \partial\Omega(b) \cup \partial\Omega(c)$.

THEOREM 3. *Let* $u \in C^{2,1}(E)$, *let* $\overline{\Omega}(x_0,t_0;c) \subseteq E$ *and let* $W_0(x,t) = W(x_0-x, t_0-t)$. *Then, whenever* $0 < b < c$,

$$\begin{aligned} (7)\qquad \mathcal{M}(u;x_0,t_0;c) &- \mathcal{M}(u;x_0,t_0;b) \\ &= \iint\limits_{\Omega(x_0,t_0;c)} ((W_0 \wedge \tau(b)) - \tau(c))\Theta u\,dy\,ds. \end{aligned}$$

Furthermore, if $\Theta u \geq 0$ *on* $\Omega(x_0,t_0;c)$, *then the function* $\mathcal{M}(u;x_0,t_0;\cdot)$ *is increasing on* $]0,c]$.

Proof. If $0 < b < c$, then by Theorem 1,

$$\begin{aligned} &\mathcal{M}(u;x_0,t_0;c) - \mathcal{M}(u;x_0,t_0;b) \\ &\quad= \iint\limits_{\Omega(x_0,t_0;c)} (W_0 - \tau(c))\Theta u\,dy\,ds - \iint\limits_{\Omega(x_0,t_0;b)} (W_0 - \tau(b))\Theta u\,dy\,ds \\ &\quad= \iint\limits_{A(x_0,t_0;b,c)} (W_0 - \tau(c))\Theta u\,dy\,ds + \iint\limits_{\Omega(x_0,t_0;b)} (\tau(b) - \tau(c))\Theta u\,dy\,ds \\ &\quad= \iint\limits_{\Omega(x_0,t_0;c)} ((W_0 \wedge \tau(b)) - \tau(c))\Theta u\,dy\,ds, \end{aligned}$$

which proves (7). If now $0 < a < b \leq c$, and $\Theta u \geq 0$ on $\Omega(c)$, then by (7),

$$\mathcal{M}(u;x_0,t_0;b) - \mathcal{M}(u;x_0,t_0;a) = \iint\limits_{\Omega(b)} ((W_0 \wedge \tau(a)) - \tau(b))\Theta u\,dy\,ds \geq 0,$$

which proves the last part. ∎

COROLLARY. *If* $u \in C^{2,1}(E)$, $\overline{\Omega}(x_0,t_0;c) \subseteq E$, *and*

$$\lambda(\gamma) = \iint\limits_{\Omega(x_0,t_0;\gamma)} \Theta u\,dy\,ds,$$

then whenever $0 < b < c$ *we have*

$$(8)\qquad \mathcal{M}(u;x_0,t_0;c) = \mathcal{M}(u;x_0,t_0;b) - \int_b^c \tau'(\gamma)\lambda(\gamma)\,d\gamma.$$

Proof. By equality (7), the difference between the means is

$$\iint\limits_{\Omega(x_0,t_0;c)} ((W_0 \wedge \tau(b)) - \tau(c))\Theta u \, dy \, ds$$

$$= \int_0^c ((\tau(\gamma) \wedge \tau(b)) - \tau(c)) \, d\lambda(\gamma)$$

$$= ((\tau(\gamma) \wedge \tau(b)) - \tau(c))\lambda(\gamma)|_0^c - \int_b^c \tau'(\gamma)\lambda(\gamma) \, d\lambda$$

$$= -\int_b^c \tau'(\gamma)\lambda(\gamma) \, d\gamma,$$

which proves (8). ■

If we make $b \to 0$ in the identity (8), then the continuity of u ensures that $\mathcal{M}(u; x_0, t_0; b) \to u(x_0, t_0)$, so that we obtain

$$\mathcal{M}(u; x_0, t_0; c) = u(x_0, t_0) - \int_0^c \tau'(\gamma)\lambda(\gamma) \, d\gamma. \tag{9}$$

If $\Theta u \geq 0$, then (9) is a smooth analogue of Nevanlinna's First Fundamental Theorem on subharmonic functions. An exact analogue was proved by Watson [20].

The last part of Theorem 3 was proved, under milder smoothness conditions and for $n = 1$ only, by Pini [13]. The extension to general n, under even milder conditions, was made by Watson [15]. The monotonicity fails if we assume only that $\overline{A}(x_0, t_0; b, c) \subseteq E$. For example, if (x_1, t_1) is fixed with $t_1 < t_0$, $E = \mathbb{R}^{n+1} \setminus \{(x_1, t_1)\}$, and $u(x,t) = W(x - x_1, t - t_1)$ for all $(x, t) \in E$, then $\Theta u = 0$ on E and

$$\mathcal{M}(u; x_0, t_0; c) = \tau(c) \wedge W(x_0 - x_1, t_0 - t_1).$$

This identity has been proved by Garofalo and Lanconelli [6], Watson [17] and Brzezina [2, 3]; a new, elementary proof is given after Theorem 5 below. If b is chosen so that $(x_1, t_1) \in \Omega(x_0, t_0; b)$, then $\mathcal{M}(u; x_0, t_0; c) = \tau(c)$ for all $c > b$, and τ is strictly decreasing.

However, if we do assume only that $\overline{A}(x_0, t_0; b, c) \subseteq E$, something definite can still be said. Its proof requires the following result.

THEOREM 4. *Let $u \in C^{2,1}(E)$, let $\overline{A}(x_0, t_0; b, c) \subseteq E$, and let $W_0(x, t) = W(x_0 - x, t_0 - t)$. Then*

$$\mathcal{M}(u; x_0, t_0; c) - \mathcal{M}(u; x_0, t_0; b) = \iint\limits_{A(x_0,t_0;b,c)} W_0 \Theta u \, dy \, ds - \tau(c)\kappa(c) + \tau(b)\kappa(b),$$

where

$$\kappa(a) = \int\limits_{\partial\Omega(x_0,t_0;a)} (\langle \nabla_x u, \nu_x\rangle - u\nu_t)\, d\sigma$$

for $a \in \{b, c\}$.

Proof. We abbreviate the proof, as it is essentially similar to that of Theorem 1. For any t such that $t_0 - b < t < t_0$, we put

$$\Psi(t) = \{(y,s) \in A(x_0, t_0; b, c) : s < t\}.$$

We divide $\partial\Psi(t)$ into three parts:

$$T(t) = \partial\Psi(t) \cap (\mathbb{R}^n \times \{t\}),$$
$$U(a,t) = \{(y,s) \in \partial\Omega(x_0,t_0;a) : s < t\} \quad \text{for } a \in \{b,c\}.$$

Taking $D = \Psi(t)$, $v = W_0$ and $w = u$ in Green's formula (1), we get

$$\begin{aligned}\iint\limits_{\Psi(t)} W_0 \Theta u\, dy\, ds = & -\int\limits_{T(t)} uW_0\, d\sigma + \tau(c) \int\limits_{U(c,t)} (\langle \nabla_x u, \nu_x\rangle - u\nu_t)\, d\sigma \\ & - \tau(b) \int\limits_{U(b,t)} (\langle \nabla_x u, \nu_x\rangle - u\nu_t)\, d\sigma \\ & - \int\limits_{U(c,t)} \langle \nabla_x W_0, \nu_x\rangle u\, d\sigma + \int\limits_{U(b,t)} \langle \nabla_x W_0, \nu_x\rangle u\, d\sigma.\end{aligned}$$

The right-hand side tends to

$$\tau(c)\kappa(c) - \tau(b)\kappa(b) + \mathcal{M}(u; x_0, t_0; c) - \mathcal{M}(u; x_0, t_0; b)$$

as $t \to t_{0-}$, because $\int_{T(t)} uW_0\, dy \to 0$. The result follows. ∎

Theorem 4 does not seem to have been stated explicitly before, but its proof appears within the proof of [19, Theorem 4]. Two applications follow.

THEOREM 5. *Let* $u \in C^{2,1}(E)$, *let* $\overline{A}(x_0, t_0; b, c) \subseteq E$, *and suppose that* $\Theta u \geq 0$ *on* $A(x_0, t_0; b, c)$. *Then there is a convex function* ϕ *such that* $\mathcal{M}(u; x_0, t_0; a) = \phi(\tau(a))$ *for all* $a \in [b, c]$.

Proof. Let α, β be numbers such that $b \leq \alpha < \beta \leq c$. Applying Theorem 4 to u on $A(x_0, t_0; \alpha, \beta)$, we obtain

$$\mathcal{M}(\beta) - \mathcal{M}(\alpha) = \iint\limits_{A(\alpha,\beta)} W_0 \Theta u\, dy\, ds - \tau(\beta)\kappa(\beta) + \tau(\alpha)\kappa(\alpha),$$

where $\mathcal{M}(\gamma) = \mathcal{M}(u; x_0, t_0; \gamma)$ for $\gamma \in \{\alpha, \beta\}$. By Green's formula (2), with $D = A(\alpha, \beta)$ and $w = u$,

$$\kappa(\beta) - \kappa(\alpha) = \iint\limits_{A(\alpha,\beta)} \Theta u\, dy\, ds,$$

so that

$$\mathcal{M}(\beta) - \mathcal{M}(\alpha) = \iint_{A(\alpha,\beta)} W_0 \Theta u\,dy\,ds - \tau(\alpha)(\kappa(\beta) - \kappa(\alpha)) - \kappa(\beta)(\tau(\beta) - \tau(\alpha))$$
$$= \iint_{A(\alpha,\beta)} (W_0 - \tau(\alpha))\Theta u\,dy\,ds - \kappa(\beta)(\tau(\beta) - \tau(\alpha)).$$

It follows that, if $b \le p < q < r \le c$, then

$$\frac{\mathcal{M}(r) - \mathcal{M}(q)}{\tau(r) - \tau(q)} - \frac{\mathcal{M}(q) - \mathcal{M}(p)}{\tau(q) - \tau(p)}$$
$$= \iint_{A(q,r)} \left(\frac{W_0 - \tau(q)}{\tau(r) - \tau(q)}\right)\Theta u\,dy\,ds - \kappa(r) - \iint_{A(p,q)} \left(\frac{W_0 - \tau(p)}{\tau(q) - \tau(p)}\right)\Theta u\,dy\,ds + \kappa(q)$$
$$= \iint_{A(q,r)} \left(\frac{W_0 - \tau(q)}{\tau(r) - \tau(q)} - 1\right)\Theta u\,dy\,ds - \iint_{A(p,q)} \left(\frac{W_0 - \tau(p)}{\tau(q) - \tau(p)}\right)\Theta u\,dy\,ds$$
$$= \iint_{A(q,r)} \left(\frac{W_0 - \tau(r)}{\tau(r) - \tau(q)}\right)\Theta u\,dy\,ds + \iint_{A(p,q)} \left(\frac{\tau(p) - W_0}{\tau(q) - \tau(p)}\right)\Theta u\,dy\,ds.$$

By definition of the heat annulus, $\tau(\beta) < W_0 < \tau(\alpha)$ on $A(\alpha,\beta)$, so that both of the last two integrands are negative. Hence

$$\frac{\mathcal{M}(r) - \mathcal{M}(q)}{\tau(r) - \tau(q)} \le \frac{\mathcal{M}(q) - \mathcal{M}(p)}{\tau(q) - \tau(p)}$$

whenever $b \le p < q < r \le c$, which means that $\mathcal{M}$ is a convex function of τ. ■

Theorem 5 was proved (under milder conditions) by Watson in [17], [18] and [19], using three different methods. The method used here is the simplest of the three.

The case where u is a temperature is particularly nice.

COROLLARY. *Let* $u \in C^{2,1}(E)$, *let* $\overline{A}(x_0, t_0; b, c) \subseteq E$, *and let* u *be a temperature on* $A(x_0, t_0; b, c)$. *Then there are real numbers* α, β *such that*

$$\mathcal{M}(u; x_0, t_0; a) = \alpha\tau(a) + \beta$$

whenever $b \le a \le c$.

Proof. Apply Theorem 5 to both u and $-u$. ■

EXAMPLE. Given two points $(x_0, t_0), (x^*, t^*) \in \mathbb{R}^{n+1}$ with $t^* < t_0$, we evaluate the mean value of $W^*(x,t) = W(x - x^*, t - t^*)$ over $\partial\Omega(x_0, t_0; c)$, for every $c > 0$. Our methods are more elementary than any used before.

Let c_0 be the positive number such that $(x^*, t^*) \in \partial\Omega(x_0, t_0; c_0)$, or $W(x_0 - x^*, t_0 - t^*) = \tau(c_0)$. If $0 < c < c_0$, then W^* is a temperature on an open superset of $\overline{\Omega}(x_0, t_0; c)$, so that

$$\mathcal{M}(W^*; x_0, t_0; c) = W^*(x_0, t_0) = \tau(c_0)$$

by the Corollary to Theorem 2.

Now consider the case where $c > c_0$. The Corollary to Theorem 5 shows that $\mathcal{M}(W^*; x_0, t_0; c) = \alpha\tau(c) + \beta$ for some $\alpha, \beta \in \mathbb{R}$. Since $\tau(c) \to 0$ as $c \to \infty$, and the means are nonnegative, we have $\beta \geq 0$. We can use Theorem 4 to show that $\alpha = 1$, a process which involves the evaluation of

$$\kappa(a) = \int\limits_{\partial\Omega(x_0, t_0; a)} (\langle \nabla_x W^*, \nu_x \rangle - W^* \nu_t)\, d\sigma$$

for $a > c_0$. To achieve this, we go back to Green's formula. Given $a > c_0$, we choose $r, \varrho > 0$ such that the closed cylinder $C = \overline{B}(x^*, r) \times [t^*, t^* + \varrho] \subseteq \Omega(a)$. We apply Green's formula (2) with $D = \Omega(x_0, t_0; a) \setminus C = \Omega(a) \setminus C$ and $w = W^*$. This gives

$$0 = \int\limits_{\partial\Omega(a)} (\langle \nabla_x W^*, \nu_x \rangle - W^* \nu_t)\, d\sigma - \int\limits_{\partial C} (\langle \nabla_x W^*, \nu_x \rangle - W^* \nu_t)\, d\sigma.$$

We evaluate the limit of the latter integral as $\varrho \to 0$. Let

$$T = \overline{B}(x^*, r) \times \{t^* + \varrho\} \quad \text{and} \quad L = \partial B(x^*, r) \times [t^*, t^* + \varrho],$$

so that this integral can be written as

$$-\int\limits_T W^*\, d\sigma + \int\limits_L \langle \nabla_x W^*, \nu_x \rangle\, d\sigma.$$

Now,

$$\lim_{\varrho \to 0} \int\limits_L \langle \nabla_x W^*, \nu_x \rangle\, d\sigma = 0$$

because the integrand is bounded on L. Furthermore,

$$\int\limits_T W^*\, d\sigma = \int\limits_{\|y - x^*\| \leq r} W(y - x^*, \varrho)\, dy \to 1 \quad \text{as } \varrho \to 0$$

([16, p. 3]). It follows that

$$\kappa(a) = \lim_{\varrho \to 0} \int\limits_{\partial C} (\langle \nabla_x W^*, \nu_x \rangle - W^* \nu_t)\, d\sigma = -1.$$

We now take $E = \mathbb{R}^{n+1} \setminus \{(x^*, t^*)\}$ and $u = W^*$ in Theorem 4, with $c > b > c_0$. This gives

$$\mathcal{M}(W^*; x_0, t_0; c) - \mathcal{M}(W^*; x_0, t_0; b) = -\tau(c)\kappa(c) + \tau(b)\kappa(b) = \tau(c) - \tau(b),$$

which implies that $\alpha = 1$. Thus

$$\mathcal{M}(W^*; x_0, t_0; c) = \begin{cases} \tau(c_0) & \text{if } 0 < c < c_0, \\ \tau(c) + \beta & \text{if } c > c_0, \end{cases}$$

where $\beta \geq 0$. We shall prove that $\mathcal{M}(W^*;x_0,t_0;\cdot)$ is a decreasing function on $]0,\infty[$, which implies that $\beta = 0$ and that $\mathcal{M}(W^*;x_0,t_0;c_0) = \tau(c_0)$. To do this, we approximate W^* with an increasing sequence $\{w_k\}$ of functions in $C^{2,1}(\mathbb{R}^{n+1})$ such that $\Theta w_k \leq 0$ for every k. This sequence is constructed in the following way. Let ψ be a continuously differentiable function on $\mathbb{R}$ such that $\psi(t) = 0$ for all $t \leq 1/2$, $\psi(t) = 1$ for all $t \geq 1$, and $\psi'(t) \geq 0$ for all $t \in \mathbb{R}$. Put

$$w_k(x,t) = W^*(x,t)\psi(k(t-t^*))$$

whenever $(x,t) \in \mathbb{R}^{n+1}$ and $k \in \mathbb{N}$. Since $\psi(k(t-t^*)) = 0$ for all $t \leq t^* + 1/(2k)$, each function w_k is in $C^{2,1}(\mathbb{R}^{n+1})$. Furthermore,

$$\Theta w_k(x,t) = -W^*(x,t)k\psi'(k(t-t^*)) \leq 0$$

for all $(x,t) \in \mathbb{R}^{n+1}$. Theorem 3 shows that each function $\mathcal{M}(w_k;x_0,t_0;\cdot)$ is decreasing. Furthermore, since ψ is an increasing function, the sequence $\{w_k\}$ is increasing; and since $\psi(k(t-t^*)) = 1$ for all $t \geq t^* + 1/k$, we have

$$\lim_{k\to\infty} w_k = W^*.$$

Hence, by the monotone convergence theorem,

$$\mathcal{M}(W^*;x_0,t_0;\cdot) = \lim_{k\to\infty} \mathcal{M}(w_k;x_0,t_0;\cdot),$$

and the latter function is decreasing. We deduce that $\beta = 0$ and that $\mathcal{M}(W^*;x_0,t_0;c_0) = \tau(c_0) = W^*(x_0,t_0)$. Hence

$$\mathcal{M}(W^*;x_0,t_0;c_0) = W^*(x_0,t_0) \wedge \tau(c)$$

for all $c > 0$.

Mean values of smooth subtemperatures over heat balls. Henceforth, if $u \in C^{2,1}(E)$ and $\Theta u \geq 0$ on E, we shall call u a *smooth subtemperature* on E.

The essential properties of the mean values of smooth subtemperatures over heat spheres, given in Theorems 2, 3 and 5 above, carry over to mean values over heat balls. We get from one to the other by integration, and there are uncountably many different possibilities for the kernel in the heat ball case. For a discussion of an infinity of the options, see [21]. Unfortunately, the most natural choice of kernel is not the simplest kernel, and there is no choice that yields a bounded kernel. For most purposes the choice of kernel is not important, so we choose the simplest.

Given a function u on the heat ball $\Omega(x_0,t_0;c)$ for which the integral exists, we define the volume mean value of u by

$$\mathcal{V}(u;x_0,t_0;c) = \frac{n}{2}\, c^{-n/2} \int_0^c r^{(n/2)-1} \mathcal{M}(u;x_0,t_0;r)\, dr. \tag{10}$$

To express the integral in rectangular coordinates, we need to know the Jacobian. If J is defined on $\mathbb{R}^n \times]0,\infty[$ by

$$J(x,t) = 2nt\exp\left(-\frac{\|x\|^2}{2nt}\right)(4\|x\|^2t^2 + (\|x\|^2 - 2nt)^2)^{-1/2},$$

then

$$\iint\limits_{\Omega(x_0,t_0;c)} u(y,s)\,dy\,ds = \int\limits_0^c \Big(\int\limits_{\partial\Omega(x_0,t_0;r)} u(y,s)J(x_0-y,t_0-s)\,d\sigma(y,s)\Big)\,dr,$$

by [15, Lemma 3]. Since the kernel Q for the mean value over the heat sphere satisfies

$$Q(x,t) = \frac{\|x\|^2}{2nt}\exp\left(\frac{\|x\|^2}{2nt}\right)J(x,t),$$

and $r = (t_0-s)\exp(\|x_0-y\|^2/(2n(t_0-s)))$ whenever $(y,s) \in \partial\Omega(x_0,t_0;r) \setminus \{(x_0,t_0)\}$, we obtain

$$\mathcal{V}(u;x_0,t_0;c) = \tau(c) \iint\limits_{\Omega(x_0,t_0;c)} \frac{\|x_0-y\|^2}{4(t_0-s)^2}\,u(y,s)\,dy\,ds. \tag{11}$$

We now present variants of Theorems 2, 3, and 5 for the volume means.

THEOREM 6. *Let $u \in C^{2,1}(E)$. If $\Theta u \ge 0$ on E, then the inequality*

$$u(x_0,t_0) \le \mathcal{V}(u;x_0,t_0;c) \tag{12}$$

holds whenever $\overline{\Omega}(x_0,t_0;c) \subseteq E$. Conversely if, given any point $(x_0,t_0) \in E$ and $\varepsilon > 0$, we can find $c < \varepsilon$ such that (12) holds, then $\Theta u \ge 0$ on E.

Proof. Suppose that $\overline{\Omega}(x_0,t_0;c) \subseteq E$. If u is a smooth subtemperature on E, then $u(x_0,t_0) \le \mathcal{M}(u;x_0,t_0;b)$ whenever $0 < b \le c$, by Theorem 2. It therefore follows from (10) that

$$\mathcal{V}(u;x_0,t_0;c) \ge \frac{n}{2}\,c^{-n/2}\int\limits_0^c r^{(n/2)-1}u(x_0,t_0)\,dr = u(x_0,t_0),$$

so that (12) holds. For the converse, if $\Theta u < 0$ at some point of E, then the continuity of Θu implies that $\Theta u < 0$ on an open subset D of E. If $\overline{\Omega}(x_0,t_0;c) \subseteq D$, then $u(x_0,t_0) > \mathcal{M}(u;x_0,t_0;b)$ whenever $0 < b \le c$, by Theorem 2. It now follows from (10) that $\mathcal{V}(u;x_0,t_0;b) < u(x_0,t_0)$ whenever $0 < b \le c$, so that the condition in the converse fails to hold. ∎

COROLLARY. *Let $u \in C^{2,1}(E)$. If u is a temperature on E, then (12) holds with equality whenever $\overline{\Omega}(x_0,t_0;c) \subseteq E$. Conversely if, given any point $(x_0,t_0) \in E$ and $\varepsilon > 0$, we can find $c < \varepsilon$ such that (12) holds with equality, then u is a temperature on E.*

The authors who first calculated the mean values of smooth subtemperatures over heat balls, all chose different kernels. This is of no consequence in the context of Theorem 6, and is therefore ignored in the following remarks.

Theorem 6 was first proved, with the converse condition strengthened to have (12) hold for all $\overline{\Omega}(x_0, t_0; c) \subseteq E$, by Pini [13] for $n = 1$. Then the corollary alone was proved for general n, with a similar strengthening of the converse condition, by Smyrnélis [14]. Then Watson proved the theorem as above (but with a milder smoothness condition) in [15]. Subsequently Kuptsov proved the same version that Smyrnélis had done, in [8].

THEOREM 7. *If u is a smooth subtemperature on E, and $(x_0, t_0) \in E$, then the function $\mathcal{V}(u; x_0, t_0; \cdot)$ is increasing on the set of $c > 0$ such that $\overline{\Omega}(x_0, t_0; c) \subseteq E$.*

Proof. If $0 < b < c$ and $\overline{\Omega}(x_0, t_0; c) \subseteq E$, then by (10),

$$\mathcal{V}(u; x_0, t_0; c) - \mathcal{V}(u; x_0, t_0; b) = \frac{n}{2} c^{-n/2} \int_0^c r^{(n/2)-1} (\mathcal{M}(u; x_0, t_0; r) - \mathcal{M}(u; x_0, t_0; br/c))\, dr,$$

and the integrand is positive by Theorem 3. ∎

Theorem 7 was proved in the case $n = 1$ by Pini [13], under a milder smoothness condition. For a general n, it was proved by Watson, in [17] for a different kernel, in [21] for the present one, under yet milder smoothness conditions.

THEOREM 8. *Let $u \in C^{2,1}(E)$, let $\overline{\Omega}(x_0, t_0; c) \subseteq E$, and suppose $\Theta u \geq 0$ on $\Omega(x_0, t_0; c)$. Then there is a convex function ϕ such that $\mathcal{V}(u; x_0, t_0; a) = \phi(\tau(a))$ for all $a \in]0, c]$.*

Proof. The result is a consequence of Theorem 5 and (10). First observe that, if α, β, γ, r are strictly positive, then

$$\tau(\beta) - \tau(\alpha) = (r/\gamma)^{n/2} (\tau(\beta r/\gamma) - \tau(\alpha r/\gamma)).$$

It follows that, if $0 < \alpha < \beta < \gamma \leq c$, $\mathcal{V}(a) = \mathcal{V}(u; x_0, t_0; a)$ and $\mathcal{M}(a) = \mathcal{M}(u; x_0, t_0; a)$, then by (10),

$$\frac{\mathcal{V}(\gamma) - \mathcal{V}(\beta)}{\tau(\gamma) - \tau(\beta)} - \frac{\mathcal{V}(\beta) - \mathcal{V}(\alpha)}{\tau(\beta) - \tau(\alpha)}$$
$$= \frac{n}{2} \gamma^{-n/2} \int_0^\gamma \left(\frac{\mathcal{M}(r) - \mathcal{M}(\beta r/\gamma)}{\tau(\gamma) - \tau(\beta)} - \frac{\mathcal{M}(\beta r/\gamma) - \mathcal{M}(\alpha r/\gamma)}{\tau(\beta) - \tau(\alpha)} \right) r^{(n/2)-1}\, dr$$
$$= \frac{n}{2} \int_0^\gamma \left(\frac{\mathcal{M}(r) - \mathcal{M}(\beta r/\gamma)}{\tau(r) - \tau(\beta r/\gamma)} - \frac{\mathcal{M}(\beta r/\gamma) - \mathcal{M}(\alpha r/\gamma)}{\tau(\beta r/\gamma) - \tau(\alpha r/\gamma)} \right) r^{-1}\, dr.$$

By Theorem 5, the last integrand is negative. Therefore,

$$\frac{\mathcal{V}(\gamma)-\mathcal{V}(\beta)}{\tau(\gamma)-\tau(\beta)} \le \frac{\mathcal{V}(\beta)-\mathcal{V}(\alpha)}{\tau(\beta)-\tau(\alpha)}$$

whenever $0 < \alpha < \beta < \gamma \le c$, which means that $\mathcal{V}$ is a convex function of τ. ■

Theorem 8 was proved, under milder conditions, by Watson in [17] for a different kernel, in [21] for the present one. The method used here is the same as in [21].

Specializing to the case where u is a temperature, we obtain a result analogous to the Corollary to Theorem 5.

COROLLARY. *Let $u \in C^{2,1}(E)$, let $\overline{\Omega}(x_0, t_0; c) \subseteq E$, and let u be a temperature on $\Omega(x_0, t_0; c)$. Then there are real numbers α and β such that*

$$\mathcal{V}(u; x_0, t_0; a) = \alpha\tau(a) + \beta$$

whenever $0 < a \le c$.

Modified heat balls. The kernel for the heat ball, given in formula (11), is unbounded near the centre of the ball. For most purposes this does not cause problems, but for some it causes substantial ones. We now consider a family of modified heat balls indexed by an integer $m \ge 1$. As m increases, the kernel's behaviour improves. For $m \ge 3$ the kernel is bounded, while for $m \ge 5$ it has a smooth extension by zero to the whole of $\mathbb{R}^{n+1}$. Theorem 10 below gives the key results, and is new. An application is given to the characterization of temperatures.

Let m be an integer, $m \ge 1$. Given $(x_0, t_0) \in \mathbb{R}^{n+1}$ and $c > 0$, we put

$$\begin{aligned}\Omega_m(x_0, t_0; c) &= \left\{(y,s) : (t_0-s)^{-(m+n)/2}\exp\left(-\frac{\|x_0-y\|^2}{4(t_0-s)}\right) > c^{-(m+n)/2}\right\}\\ &= \left\{(y,s) : \|x_0-y\|^2 < 2(m+n)(t_0-s)\log\left(\frac{c}{t_0-s}\right),\ 0 < t_0-s < c\right\}.\end{aligned}$$

Thus $\Omega_m(x_0, t_0; c)$ is the projection onto $\mathbb{R}^{n+1}$ of a heat ball in $\mathbb{R}^{m+n+1}$. Let $u \in C^{2,1}(E)$, and put

$$\widehat{u}(\xi, x, t) = u(x,t) \quad \text{for all } \xi \in \mathbb{R}^m \text{ and } (x,t) \in E. \tag{13}$$

Then $\widehat{u} \in C^{2,1}(\mathbb{R}^m \times E)$, and we can apply the above theorems to $\widehat{u}$. Note that the volume mean formula (11), when applied to $\widehat{u}$, becomes

$$\begin{aligned}&\mathcal{V}(\widehat{u}; \xi_0, x_0, t_0; c)\\ &\quad = (4\pi c)^{-(m+n)/2} \iiint\limits_{\Omega(\xi_0, x_0, t_0; c)} \frac{\|\xi_0-\eta\|^2 + \|x_0-y\|^2}{4(t_0-s)^2}\,\widehat{u}(\eta, y, s)\, d\eta\, dy\, ds.\end{aligned}$$

Since $\widehat{u}(\eta, y, s) = u(y, s)$ does not depend on η, we can integrate out η and obtain a volume mean for u, which we denote by $\mathcal{V}_m(u; x_0, t_0; c)$. Thus

$$\mathcal{V}_m(u; x_0, t_0; c) = (4\pi c)^{-(m+n)/2} \iint_{\Omega_m(x_0,t_0;c)} \left(\int_{\|\xi_0-\eta\|<R} \frac{\|\xi_0 - \eta\|^2 + \|x_0 - y\|^2}{4(t_0 - s)^2} \, d\eta \right) u(y, s) \, dy \, ds,$$

where

$$R = R(x_0 - y, t_0 - s) = (2(m + n)(t_0 - s) \log(c/(t_0 - s)) - \|x_0 - y\|^2)^{1/2}.$$

The innermost integral can be evaluated explicitly, and the simplicity of the chosen kernel facilitates the calculation. We have

$$\int_{\|\xi_0-\eta\|<R} \frac{\|\xi_0 - \eta\|^2 + \|x_0 - y\|^2}{4(t_0 - s)^2} \, d\eta = \sigma_m \int_0^R \frac{r^2 + \|x_0 - y\|^2}{4(t_0 - s)^2} \, r^{m-1} \, dr = \frac{\omega_m R^m}{4(t_0 - s)^2} \left(\frac{m}{m+2} R^2 + \|x_0 - y\|^2 \right),$$

where σ_m is the surface area of the unit sphere in $\mathbb{R}^m$, and ω_m is the volume it encloses. Hence

$$\mathcal{V}_m(u; x_0, t_0; c) = \iint_{\Omega_m(x_0,t_0;c)} K_{m,c}(x_0 - y, t_0 - s) u(y, s) \, dy \, ds,$$

where

$$K_{m,c}(x_0 - y, t_0 - s) = \frac{\omega_m (4\pi c)^{-(m+n)/2}}{2(m+2)} R(x_0-y, t_0-s)^m \left(\frac{m(m+n)}{(t_0 - s)} \log\left(\frac{c}{t_0 - s}\right) + \frac{\|x_0 - y\|^2}{(t_0 - s)^2} \right)$$

is a continuous, positive function of (y, s) on $\overline{\Omega}_m(x_0, t_0; c) \setminus \{(x_0, t_0)\}$, and is zero on $\partial\Omega_m(x_0, t_0; c) \setminus \{(x_0, t_0)\}$.

If m is sufficiently large, then the function $(y, s) \mapsto K_{m,c}(x_0 - y, t_0 - s)$ has a continuous extension by zero at (x_0, t_0). For if

$$\|x_0 - y\|^2 < 2(m + n)(t_0 - s) \log(c/(t_0 - s)),$$

then

$$0 < R(x_0 - y, t_0 - s)^2 < 2(m + n)(t_0 - s) \log(c/(t_0 - s)),$$

so that

$$K_{m,c}(x_0 - y, t_0 - s) \le A(t_0 - s)^{(m-2)/2} (\log c/(t_0 - s))^{(m+2)/2}$$

for some positive constant A which depends only on c, m and n. It follows that if $m \ge 3$ then $K_{m,c}(x_0 - y, t_0 - s) \to 0$ as $s \to t_0$. In particular, the kernel is bounded if $m \ge 3$.

The idea for modifying the heat ball to get a bounded kernel is due to Kuptsov [8]. It has also been used by Garofalo and Lanconelli [6]. In both papers a Harnack inequality was proved using the boundedness of the kernel.

Given any function $u \in C^{2,1}(E)$, we can define $\widehat{u}$ by (13) and apply Theorems 6–8 to $\widehat{u}$. We thus obtain the following results about the modified volume means of u.

THEOREM 9. *Let $u \in C^{2,1}(E)$, and let m be an integer, $m \geq 1$.*

(i) *If u is a smooth subtemperature on E, then the inequality*

$$u(x_0, t_0) \leq \mathcal{V}_m(u; x_0, t_0; c) \tag{14}$$

holds whenever $\overline{\Omega}_m(x_0, t_0; c) \subseteq E$. Conversely if, given any point $(x_0, t_0) \in E$ and $\varepsilon > 0$, we can find $c < \varepsilon$ such that (14) holds, then u is a smooth subtemperature on E.

(ii) *If u is a smooth subtemperature on E, and $(x_0, t_0) \in E$, then the function $\mathcal{V}_m(u; x_0, t_0; \cdot)$ is increasing on the set of $c > 0$ such that $\overline{\Omega}_m(x_0, t_0; c) \subseteq E$.*

(iii) *If u is a smooth subtemperature on $\Omega_m(x_0, t_0; c)$ and $\overline{\Omega}_m(x_0, t_0; c) \subseteq E$, then there is a convex function ϕ_m such that $\mathcal{V}_m(u; x_0, t_0; a) = \phi_m(\tau_m(a))$ whenever $0 < a \leq c$, where $\tau_m(a) = (4\pi a)^{-(m+n)/2}$.*

Proof. Given $u \in C^{2,1}(E)$, we define $\widehat{u}$ by (13), and denote by $\widehat{\Theta}$ the heat operator in $\mathbb{R}^{m+n+1}$.

(i) Suppose that $\Theta u \geq 0$ on E, and that $\overline{\Omega}_m(x_0, t_0; c) \subseteq E$. Then $\widehat{\Theta}\widehat{u} = \Theta u \geq 0$ on $\mathbb{R}^m \times E$, and $\overline{\Omega}(\xi_0, x_0, t_0; c) \subseteq \mathbb{R}^m \times E$ for every $\xi_0 \in \mathbb{R}^m$. Applying Theorem 6 to $\widehat{u} \in C^{2,1}(\mathbb{R}^m \times E)$, we obtain

$$u(x_0, t_0) = \widehat{u}(\xi_0, x_0, t_0) \leq \mathcal{V}(\widehat{u}; \xi_0, x_0, t_0; c) = \mathcal{V}_m(u; x_0, t_0; c).$$

Conversely, suppose that we are given $(x_0, t_0) \in E$ and $\varepsilon > 0$, and that $c < \varepsilon$ is chosen so that (14) holds. Then, whenever $\xi_0 \in \mathbb{R}^m$, we have

$$\widehat{u}(\xi_0, x_0, t_0) = u(x_0, t_0) \leq \mathcal{V}_m(u; x_0, t_0; c) = \mathcal{V}(\widehat{u}; \xi_0, x_0, t_0; c),$$

and another application of Theorem 6 to $\widehat{u}$ shows that $\widehat{\Theta}\widehat{u} \geq 0$ on $\mathbb{R}^m \times E$. Hence $\Theta u \geq 0$ on E.

(ii) If $\Theta u \geq 0$ on E and $(x_0, t_0) \in E$, then $\widehat{\Theta}\widehat{u} \geq 0$ on $\mathbb{R}^m \times E$. Therefore, by Theorem 7, for any $\xi_0 \in \mathbb{R}^m$ the function $\mathcal{V}(\widehat{u}; \xi_0, x_0, t_0; \cdot) = \mathcal{V}_m(u; x_0, t_0; \cdot)$ is increasing on the set of $c > 0$ such that $\overline{\Omega}(\xi_0, x_0, t_0; c) \subseteq \mathbb{R}^m \times E$, equivalently $\overline{\Omega}_m(x_0, t_0; c) \subseteq E$.

(iii) If $\overline{\Omega}_m(x_0, t_0; c) \subseteq E$ and $\Theta u \geq 0$ on $\Omega_m(x_0, t_0; c)$, then for any $\xi_0 \in \mathbb{R}^m$ we have $\overline{\Omega}(\xi_0, x_0, t_0; c) \subseteq \mathbb{R}^m \times E$ and $\widehat{\Theta}\widehat{u} \geq 0$ on $\Omega(\xi_0, x_0, t_0; c)$. Therefore, by Theorem 8 applied to $\widehat{u}$, there is a convex function ϕ_m such that $\mathcal{V}_m(u; x_0, t_0; a) = \mathcal{V}(\widehat{u}; \xi_0, x_0, t_0; a) = \phi_m(\tau_m(a))$ whenever $0 < a \leq c$. ∎

THEOREM 10. *Let u be a locally integrable function on E, let m be an integer with $m \geq 5$, let $c > 0$, and let*

$$E_c = \{(y,s) : \overline{\Omega}_m(y,s;c) \subseteq E\}.$$

If u_c is defined for all $(x,t) \in E_c$ by

$$u_c(x,t) = \mathcal{V}_m(u;x,t;c),$$

then $u_c \in C^{2,1}(E_c)$ and $u_c \to u$ locally in L^1 as $c \to 0$. Furthermore, if there is an integer $p \geq 0$ such that

$$u(x,t) \leq \mathcal{V}_p(u;x,t;b) \tag{15}$$

whenever $\overline{\Omega}_p(x,t;b) \subseteq E$, then u_c is a smooth subtemperature. (*Here, and below, $\mathcal{V}_0 = \mathcal{V}$ and $\Omega_0 = \Omega$.*)

Proof. Let D be a bounded open set with $\overline{D} \subseteq E$, and put

$$D_c = \{(y,s) : \overline{\Omega}_m(y,s;c) \subseteq D\}.$$

Then u is integrable on D, and the first part of the theorem will follow if we prove that $u_c \in C^{2,1}(D_c)$. Put

$$\varrho(r,t) = (2(m+n)t\log(c/t) - r^2)^{1/2}$$

whenever $r \in \mathbb{R}$ and $0 < t < c$. In addition, put

$$\lambda_c(r,t) = \begin{cases} \alpha_c \varrho(r,t)^m \left(m(m+n)\,\dfrac{1}{t}\log\dfrac{c}{t} + \dfrac{r^2}{t^2}\right) & \text{if } r^2 < 2(m+n)t\log\dfrac{c}{t}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_c = \omega_m(4\pi c)^{-(m+n)/2}/(2(m+2))$. Then

$$\mathcal{V}_m(u;x,t;c) = \iint_{\mathbb{R}^{n+1}} \lambda_c(\|x-y\|, t-s)u(y,s)\,dy\,ds,$$

so that u_c will belong to $C^{2,1}(D_c)$ if $\lambda \in C^{2,1}(\mathbb{R}^2)$. Routine calculations show that there are constants $a_1, \ldots, a_9$ such that

$$D_1\lambda_c(r,t) = \alpha_c\varrho(r,t)^{m-2}\left(a_1\frac{r}{t}\log\frac{c}{t} + a_2\frac{r^3}{t^2}\right),$$

$$D_1^2\lambda_c(r,t) = \alpha_c\varrho(r,t)^{m-4}\left(a_3\frac{r^2}{t}\log\frac{c}{t} + a_4\left(\log\frac{c}{t}\right)^2 + a_5\frac{r^4}{t^2}\right),$$

$$D_2\lambda_c(r,t) = \alpha_c\varrho(r,t)^{m-2}\left(a_6\frac{1}{t}\left(\log\frac{c}{t}\right)^2 + a_7\frac{1}{t}\log\frac{c}{t} + a_8\frac{r^2}{t^2}\log\frac{c}{t} + a_9\frac{r^4}{t^3}\right)$$

whenever $r^2 < 2(m+n)t\log(c/t)$. Therefore, because $m \geq 5$, all these derivatives tend to zero as (r,t) approaches any point (R,T) where $R^2 = 2(m+n) \times T\log(c/T)$ and $T > 0$. Furthermore, whenever $r^2 < 2(m+n)t\log(c/t)$ we have

$$\lambda_c(r,t) \leq At^{(m-2)/2}\left(\log\frac{c}{t}\right)^{(m+2)/2},$$

$$|D_1\lambda_c(r,t)| \leq At^{(m-3)/2}\left(\log\frac{c}{t}\right)^{(m+1)/2},$$

$$|D_1^2\lambda_c(r,t)| \leq At^{(m-4)/2}\left(\log\frac{c}{t}\right)^{m/2},$$

$$|D_2\lambda_c(r,t)| \leq At^{(m-4)/2}\left(\log\frac{c}{t}\right)^{m/2}\left(1+\log\frac{c}{t}\right),$$

for some constant A. Since $m \geq 5$, it follows that all these functions tend to zero as $t \to 0^+$, so that $\lambda \in C^{2,1}(\mathbb{R}^2)$. The first part of the theorem follows.

To prove that $u_c \to u$ locally in L^1 as $c \to 0$, we need only show that the family $\{\phi_c : c > 0\}$ of functions given by

$$\phi_c(y,s) = \lambda_c(\|y\|, s)$$

is an approximate identity. Since $\phi_c \geq 0$, and

$$\iint_{\mathbb{R}^{n+1}} \phi_c(y,s)\,dy\,ds = \mathcal{V}_m(1;0,0;c) = 1$$

by Theorem 9(i), and the support of ϕ_c is contained in the set

$$\overline{B}(0,(2(m+n)c/e)^{1/2}) \times [0,c],$$

the result follows.

Finally, suppose that there is an integer $p \geq 0$ such that (15) holds whenever $\overline{\Omega}_p(x,t;b) \subseteq E$. To prove that u_c is a smooth subtemperature, it suffices to show that, given any point $(x,t) \in E_c$ and any $\varepsilon > 0$, we can find $b < \varepsilon$ such that

$$u_c(x,t) \leq \mathcal{V}_p(u_c;x,t;b),$$

in view of Theorem 6 (if $p = 0$) or Theorem 9(i) (if $p \geq 1$). Given D as above, suppose that $\overline{\Omega}_p(x,t;a) \subseteq D_c$ and that $0 < b < a$. Then

$$\begin{aligned}
&\mathcal{V}_p(u_c;x,t;b)\\
&\quad = \iint_{\Omega_p(x,t;b)} K_{p,b}(x-y,t-s)\Big(\iint_{\Omega_m(y,s;c)} K_{m,c}(y-z,s-r)u(z,r)\,dz\,dr\Big)\,dy\,ds\\
&\quad = \iint_{\Omega_p(x,t;b)} K_{p,b}(x-y,t-s)\\
&\qquad \times\Big(\iint_{\Omega_m(0,0;c)} K_{m,c}(-z,-r)u(z+y,r+s)\,dz\,dr\Big)\,dy\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_{\Omega_m(0,0;c)} K_{m,c}(-z,-r) \\
&\quad \times \Big(\iint\limits_{\Omega_p(x,t;b)} K_{p,b}(x-y,t-s)u(y+z,s+r)\,dy\,ds \Big)\,dz\,dr \\
&= \iint\limits_{\Omega_m(0,0;c)} K_{m,c}(-z,-r) \\
&\quad \times \Big(\iint\limits_{\Omega_p(x+z,t+r;b)} K_{p,b}(x+z-y,t+r-s)u(y,s)\,dy\,ds \Big)\,dz\,dr \\
&\ge \iint\limits_{\Omega_m(0,0;c)} K_{m,c}(-z,-r)u(x+z,t+r)\,dz\,dr \\
&= \iint\limits_{\Omega_m(x,t;c)} K_{m,c}(x-z,t-r)u(z,r)\,dz\,dr = u_c(x,t).
\end{aligned}
$$

The change in the order of the integrals is justified by Fubini's theorem. For, if $M = \max K_{m,c}$ then

$$
\begin{aligned}
&\iint\limits_{\Omega_p(x,t;b)} K_{p,b}(x-y,t-s)\Big(\iint\limits_{\Omega_m(y,s;c)} K_{m,c}(y-z,s-r)|u(z,r)|\,dz\,dr \Big)\,dy\,ds \\
&\qquad \le M \iint\limits_{\Omega_p(x,t;b)} K_{p,b}(x-y,t-s)\Big(\iint\limits_{\Omega_m(y,s;c)} |u(z,r)|\,dz\,dr \Big)\,dy\,ds \\
&\qquad \le M \iint\limits_{D} |u(z,r)|\,dz\,dr\,\mathcal{V}_p(1;x,t;b) < \infty. \ \blacksquare
\end{aligned}
$$

As an immediate application of Theorem 10, we give a new characterization of temperatures within the class of locally integrable functions.

THEOREM 11. *Let u be a locally integrable function on E, let m be an integer with $m \ge 5$, and suppose that*

$$u(x,t) = \mathcal{V}_m(u;x,t;b) \tag{16}$$

whenever $\overline{\Omega}_m(x,t;b) \subseteq E$. Then u is a temperature on E.

Proof. For each $c > 0$, define the set E_c and function u_c as in Theorem 10. Then that theorem implies that u_c is a smooth subtemperature on E_c, in view of (16). Since $-u$ satisfies the same conditions as u, the function $-u_c = (-u)_c = -(u_c)$ is also a smooth subtemperature on E_c. Hence u_c is a temperature on E_c. Since $u_c = u$ on E_c, by (16), it follows that u is a temperature on $\bigcup_{c>0} E_c = E$. $\blacksquare$

An analogous result using averages over $(n+1)$-dimensional intervals was claimed by Doob [4, p. 276].

General subtemperatures. We now consider non-smooth subtemperatures. For each integer $m \geq 0$, we define "m-subtemperatures" associated with the modified heat balls Ω_m. The case $m = 0$ corresponds to the subtemperatures defined in [15]. We prove some basic properties for all $m \geq 0$, and use them in conjunction with Theorem 11 to prove Harnack theorems for temperatures.

First we recall an important class of subsets of E.

NOTATION. Given a point $(x_0, t_0) \in E$, we denote by $\Lambda(x_0, t_0; E)$ the set of all points $(x_1, t_1) \in E$ with $t_1 < t_0$ which have the following property: There exists a polygonal path $\gamma : [0, 1] \to E$ such that $\gamma(0) = (x_0, t_0)$, $\gamma(1) = (x_1, t_1)$, and if ord denotes the projection mapping from $\gamma([0, 1])$ onto $\{0\} \times \mathbb{R}$, then $\mathrm{ord} \circ \gamma$ is strictly decreasing. In other words, (x_0, t_0) can be joined to (x_1, t_1) by a polygonal path in E along which the time variable is strictly decreasing.

DEFINITION. Let m be an integer with $m \geq 0$, and let u be a function on E. We call u an *m-subtemperature* on E if it satisfies the following conditions:

(δ_1) $-\infty \leq u(x, t) < +\infty$ for all $(x, t) \in E$;

(δ_2) u is upper semicontinuous on E;

(δ_3) given any point $(x_1, t_1) \in E$, there is a point $(x_0, t_0) \in E$ such that $(x_1, t_1) \in \Lambda(x_0, t_0; E)$ and $u(x_0, t_0) > -\infty$;

(δ_4) $u(x, t) \leq \mathcal{V}_m(u; x, t; c)$ whenever $\overline{\Omega}_m(x, t; c) \subseteq E$.

Note that (δ_1) and (δ_2) together imply that u is locally bounded above on E, so that the integrals in (δ_4) always exist and are never $+\infty$.

Theorem 6 (if $m = 0$) or Theorem 9(i) (if $m \geq 1$) tells us that a function $u \in C^{2,1}(E)$ is an m-subtemperature if and only if it is a smooth subtemperature.

The next theorem exhibits a fundamental property of subtemperatures. The case $m = 0$ was proved in [15], using a similar technique.

THEOREM 12. *Let m be an integer with $m \geq 0$. Let $\{u_k\}$ be a decreasing sequence of m-subtemperatures on E, with limit u. If $u(x_0, t_0) > -\infty$ for some $(x_0, t_0) \in E$, then u is an m-subtemperature on $\Lambda(x_0, t_0; E)$.*

Proof. It is immediate that u satisfies (δ_1) and (δ_2). For (δ_4), if $\overline{\Omega}_m(x, t; c) \subseteq E$ then

$$u_k(x, t) \leq \mathcal{V}_m(u_k; x, t; c)$$

for all k, so that the monotone convergence theorem yields

$$u(x, t) \leq \mathcal{V}_m(u; x, t; c).$$

In particular, we can choose c_0 such that $\overline{\Omega}_m(x_0,t_0;c_0) \subseteq E$ and

$$-\infty < u(x_0,t_0) \le \mathcal{V}_m(u;x_0,t_0;c_0).$$

Hence $u(y,s)$ is finite for almost all $(y,s) \in \Omega_m(x_0,t_0;c_0)$.

Put $\Lambda = \Lambda(x_0,t_0;E)$. Given any point $(x_1,t_1) \in \Lambda$, let γ denote a polygonal path from (x_0,t_0) to (x_1,t_1) in E whose projection onto $\{0\}\times\mathbb{R}$ is strictly decreasing. Consider that connected part of the trace of γ in $\Omega_m(x_0,t_0;c_0)$ which has (x_0,t_0) as one endpoint, and label the other endpoint (x_2,t_2). Choose a point $(x_3,t_3) \in \Omega_m(x_0,t_0;c_0)$ such that $u(x_3,t_3) > -\infty$ and $(x_2,t_2) \in \Lambda(x_3,t_3;\Lambda)$. Then $(x_1,t_1) \in \Lambda(x_3,t_3;\Lambda)$, and (δ_3) holds. Hence u is an m-subtemperature on Λ. ∎

It follows from Theorem 12 that, if u is the limit of a decreasing sequence of smooth subtemperatures on E, then u is an m-subtemperature on E, for any $m \ge 0$, provided that the finiteness condition (δ_3) is satisfied. If $m \ge 5$, we can prove a converse result by elementary methods; see Theorem 16 below. The proof requires a preliminary result of interest in itself.

THEOREM 13. *Let m be an integer with $m \ge 0$. If u is an m-subtemperature on E, then u is locally integrable on E.*

Proof. Take any point $(x_1,t_1) \in E$. By (δ_3), there is a point (x_0,t_0) such that $(x_1,t_1) \in \Lambda(x_0,t_0;E)$ and $u(x_0,t_0) > -\infty$. Let $\gamma : [0,1] \to E$ be a polygonal path such that $\gamma(0) = (x_0,t_0), \gamma(1) = (x_1,t_1)$ and $\mathrm{ord}\circ\gamma$ is strictly decreasing. Since the trace γ^* of γ is compact, it is a positive distance from $\mathbb{R}^{n+1} \setminus E$. Therefore we can find c_0 such that $\overline{\Omega}_m(x,t;c_0) \subseteq E$ for all $(x,t) \in \gamma^*$.

For any point $(x,t) = \gamma(\tau) \in \gamma^*$, we put

$$P(x,t) = \{(y,s) : \|x-y\|^2 < 2(m+n)(s-t)\}.$$

Then, if $(y,s) \in P(x,t)$ and $s-t < c_0/e$, we have $(x,t) \in \Omega_m(y,s;c_0)$. Suppose that u is not integrable on any neighbourhood of (x,t). If $m \ge 1$, then the kernel $K_{m,c_0} > 0$ except on $\partial\Omega_m(y,s;c_0)$, so that $\mathcal{V}_m(u;y,s;c_0) = -\infty$. If $m = 0$, then $K_{0,c_0} > 0$ except on $\{y\}\times\mathbb{R}$, so that if $y \ne x$ then $\mathcal{V}_m(u;y,s;c_0) = -\infty$. In either case, it follows from (δ_4) that $u(y,s) = -\infty$ for all $(y,s) \in P(x,t)$ such that $s-t < c_0/e$ and $s \ne t$. Because γ^* consists of finitely many line segments, there is therefore $\eta > 0$, independent of $\tau \in [0,1]$, such that u is $-\infty$ almost everywhere on some neighbourhood of $\gamma(\omega)$ for every ω with $(\tau-\eta)\vee 0 \le \omega < \tau$. In particular, if $\tau-\eta > 0$ then u is not integrable on any neighbourhood of $\gamma(\tau-\eta)$, so that a repeat of the above argument shows that u is $-\infty$ almost everywhere on some neighbourhood of $\gamma(\omega)$ for every ω with $(\tau-2\eta)\vee 0 \le \omega < \tau$. Iterating this argument finitely many times, we conclude that u is $-\infty$ almost everywhere on a neighbourhood of $\gamma(\omega)$ for every ω with $0 \le \omega < \tau$. It follows from (δ_4) that $u(\gamma(\omega)) = -\infty$ for such ω.

If (x_1, t_1) could be chosen so that u is not integrable on any neighbourhood of (x_1, t_1), it would then follow that $u(x_0, t_0) = -\infty$, a contradiction. It follows that each point of E has a neighbourhood on which u is integrable, so that u is locally integrable on E. ∎

The case $m = 0$ of Theorem 13 was proved in [15]. The method here is new, and much shorter.

We are now able to give an easy proof of the Harnack monotone convergence theorem.

THEOREM 14. *Let $\{u_k\}$ be an increasing sequence of temperatures on E, and let $u = \lim_{k\to\infty} u_k$. If $u(x_0, t_0) < \infty$ for some point $(x_0, t_0) \in E$, then u is a temperature on $\Lambda(x_0, t_0; E)$, and the convergence is locally uniform there.*

Proof. Put $\Lambda = \Lambda(x_0, t_0; E)$, and let m be an integer with $m \ge 5$. Since $\{-u_k\}$ is a decreasing sequence of m-subtemperatures on E, Theorem 12 shows that $-u$ is an m-subtemperature on Λ. Therefore u is locally integrable on Λ, by Theorem 13. Furthermore, for every k we have

$$u_k(x,t) = \mathcal{V}_m(u_k; x, t; b)$$

whenever $\overline{\Omega}_m(x,t;b) \subseteq \Lambda$ (in particular), by Theorem 9(i). It therefore follows from the monotone convergence theorem that

$$u(x,t) = \mathcal{V}_m(u; x, t; b)$$

whenever $\overline{\Omega}_m(x,t;b) \subseteq \Lambda$. Now Theorem 11 shows that u is a temperature on Λ. Dini's Theorem implies that the convergence is locally uniform on Λ. ∎

Following Doob [4, p. 277], we can deduce a form of the Harnack inequality from the Harnack monotone convergence theorem.

THEOREM 15. *Let μ be a measure on E, and let S be the intersection with E of the support of μ. Let A be a compact subset of E such that for each point $(x_1, t_1) \in A$ there is a point $(x_0, t_0) \in S$ with $(x_1, t_1) \in \Lambda(x_0, t_0; E)$. Then there is a constant κ, which depends only on E, μ and A, such that*

$$\max_A u \le \kappa \int_S u \, d\mu$$

for every non-negative temperature u on E.

Proof. Suppose that, given E, μ and A, there is no such constant. Then for each integer $k \ge 0$, there is a non-negative temperature u_k on E such that

$$\max_A u_k \ge 2^k \int_S u_k \, d\mu.$$

We may assume that

$$\max_A u_k \geq 1 \quad \text{and} \quad \int_S u_k \, d\mu \leq 2^{-k}.$$

Consider the series $\sum_{k=0}^{\infty} u_k$. Since

$$\int_S \sum_{k=0}^{\infty} u_k \, d\mu = \sum_{k=0}^{\infty} \int_S u_k \, d\mu \leq 2,$$

the series is convergent μ-a.e. on S, and hence on a dense subset of S. Applying Theorem 14 to the sequence of partial sums of the series, we see that if the series is convergent at a point (x_0, t_0) then it is convergent on $\Lambda(x_0, t_0; E)$ to a temperature. It follows that the series converges to a temperature on the set $\bigcup_{(x,t)\in S} \Lambda(x, t; E)$, which contains A. Dini's theorem implies that the convergence is uniform on A. Therefore the sequence $\{u_k\}$ is uniformly convergent to zero on A, contrary to the fact that $\max_A u_k \geq 1$ for all k. ■

Corollary. *Let $(x_0, t_0) \in E$, and let A be a compact subset of the set $\Lambda(x_0, t_0; E)$. Then there is a constant κ, which depends only on $E, (x_0, t_0)$ and A, such that*

$$\max_A u \leq \kappa u(x_0, t_0)$$

for every non-negative temperature u on E.

Proof. In Theorem 15, take μ to be the unit mass at (x_0, t_0). ■

Harnack theorems for temperatures were proved, in various forms and using various methods, by Hadamard [7], Pini [12], Moser [10, 11], Bauer [1], and Kuptsov [8]. The general form in Theorem 15 is due to Bauer [1, Satz 1.4.4].

The classes of m-subtemperatures for $m \geq 5$. We prove that, on any bounded open set D with $\overline{D} \subseteq E$, any m-subtemperature is the limit of a decreasing sequence of smooth subtemperatures, if $m \geq 5$. For smaller values of m, we have to add a continuity condition to get the result by similar methods. We deduce that the class of m-subtemperatures is independent of m for $m \geq 5$. We also deduce that the mean value properties of smooth subtemperatures carry over to m-subtemperatures, if $m \geq 5$.

Using advanced techniques, Garofalo and Lanconelli [6] have shown that the class of m-subtemperatures is independent of m for all $m \geq 0$ in the case $E = \mathbb{R}^{n+1}$.

We shall use the following simple result.

LEMMA 2. *Let u be an upper semicontinuous function on E such that $u(x,t) < \infty$ for all $(x,t) \in E$. Then*

$$u(x_0,t_0) \geq \limsup_{b\to 0} \mathcal{V}_m(u;x_0,t_0;b)$$

for each $(x_0,t_0) \in E$ and each $m \geq 0$. Furthermore, if u is a p-subtemperature for some integer $p \geq 0$, then

$$u(x_0,t_0) = \lim_{b\to 0} \mathcal{V}_p(u;x_0,t_0;b)$$

for all $(x_0,t_0) \in E$.

Proof. Given any real number $A > u(x_0,t_0)$, we can find a neighbourhood N of (x_0,t_0) such that $u(y,s) < A$ for all $(y,s) \in N$. Therefore, whenever $\overline{\Omega}_m(x_0,t_0;b) \subseteq N$, we have

$$\mathcal{V}_m(u;x_0,t_0;b) \leq \mathcal{V}_m(A;x_0,t_0;b) = A.$$

This proves the first part. For the second, condition (δ_4) implies that

$$u(x_0,t_0) \leq \liminf_{b\to 0} \mathcal{V}_p(u;x_0,t_0;b),$$

so that the result follows from the first part. ■

We now present the aforementioned continuity condition.

DEFINITION. A function u on E is said to be Λ-*continuous* at a point $(x_0,t_0) \in E$ if $u(x_0,t_0)$ is the limit of $u(y,s)$ as $(y,s) \to (x_0,t_0)$ with $(y,s) \in \Lambda(x_0,t_0;E)$.

For example, the function W is Λ-continuous at $(0,0)$.

The next result is a theorem on the monotone approximation of m-subtemperatures by smooth ones, which is crucial to our approach. Similar results were proved earlier, using less elementary methods, by Garofalo and Lanconelli [6], and by Lanconelli and Pascucci [9].

THEOREM 16. *Let p be an integer with $p \geq 0$, let u be a p-subtemperature on E, and let D be a bounded open set such that $\overline{D} \subseteq E$. Then there is a decreasing sequence $\{v_k\}$ of smooth subtemperatures on D such that*

$$u(x_0,t_0) = \lim_{k\to\infty} v_k(x_0,t_0)$$

holds (i) *for all $(x_0,t_0) \in D$ if $p \geq 5$, and* (ii) *at every point (x_0,t_0) of D where u is Λ-continuous if $p \leq 4$.*

Proof. If $0 \leq p \leq 4$, choose any integer $m \geq 5$. If $p \geq 5$, choose $m = p$. Given $c > 0$, put

$$E_c = \{(y,s) : \overline{\Omega}_m(y,s;c) \subseteq E\},$$

and define u_c on E_c by putting

$$u_c(x,t) = \mathcal{V}_m(u;x,t;c).$$

By Theorem 10, $u_c \in C^{2,1}(E_c)$ and $u_c \to u$ locally in L^1 as $c \to 0$. Therefore, given a closed modified heat ball $\overline{\Omega}_m(x_0,t_0;b) \subseteq E$, and any c such that $E_c \supseteq \overline{\Omega}_m(x_0,t_0;b)$, we have

$$\begin{aligned}
&|\mathcal{V}_m(u_c;x_0,t_0;b) - \mathcal{V}_m(u;x_0,t_0;b)| \\
&\qquad = \Big| \iint_{\Omega_m(x_0,t_0;b)} K_{m,b}(x_0-y,t_0-s)(u_c(y,s)-u(y,s))\,dy\,ds \Big| \\
&\qquad \le \max_{\mathbb{R}^{n+1}} K_{m,b} \iint_{\Omega_m(x_0,t_0;b)} |u_c(y,s)-u(y,s)|\,dy\,ds \\
&\qquad \to 0 \quad \text{as } c \to 0.
\end{aligned}$$

Thus the function $\mathcal{V}_m(u_c;x_0,t_0;\cdot)$ converges pointwise to $\mathcal{V}_m(u;x_0,t_0;\cdot)$ as $c \to 0$. Since u satisfies (δ_4), the inequality (15) holds whenever $\overline{\Omega}_p(x,t;b) \subseteq E$, so that each function u_c is a smooth subtemperature on E_c, by Theorem 10. Therefore, by Theorem 9(ii), for each $(x_0,t_0) \in E_c$ the function $\mathcal{V}_m(u_c;x_0,t_0;\cdot)$ is increasing on the set of b such that $\overline{\Omega}_m(x_0,t_0;b) \subseteq E_c$. It follows that the function

$$\mathcal{V}_m(u;x_0,t_0;\cdot) = \lim_{c\to 0} \mathcal{V}_m(u_c;x_0,t_0;\cdot)$$

is increasing on the set of b such that $\overline{\Omega}_m(x_0,t_0;b) \subseteq \bigcup_{c>0} E_c = E$. Therefore $\mathcal{V}_m(u;x_0,t_0;b)$ tends to a limit as $b \to 0$. The conditions in (i) and (ii) ensure that this limit is $u(x_0,t_0)$. In case (i) we have $m = p$, so that the second part of Lemma 2 shows that the limit is $u(x_0,t_0)$. In case (ii) we have $m \neq p$, so that Lemma 2 shows only that

$$\lim_{b\to 0} \mathcal{V}_m(u;x_0,t_0;b) \le u(x_0,t_0). \tag{17}$$

Equality follows if $u(x_0,t_0) = -\infty$. Otherwise, if u is Λ-continuous at (x_0,t_0), then for any $\varepsilon > 0$ we can find a half-ball $H = \{(y,s) : \|x_0-y\|^2 + (t_0-s)^2 < r^2,\ s < t_0\}$ in which $u(y,s) > u(x_0,t_0) - \varepsilon$. Whenever $\overline{\Omega}_m(x_0,t_0;b) \subseteq H$ we have $\mathcal{V}_m(u;x_0,t_0;b) \ge u(x_0,t_0) - \varepsilon$, and hence equality in (17) follows.

Given a bounded open set D such that $\overline{D} \subseteq E$, choose $a > 0$ such that $D \subseteq E_a$. Then u_c is a smooth subtemperature on D whenever $0 < c < a$. Put $v_k = u_{1/k}$ for all $k > 1/a$. Then for each point $(x,t) \in D$, the function $k \mapsto v_k(x,t) = \mathcal{V}_m(u;x,t;1/k)$ is decreasing, and $\lim_{k\to\infty} v_k(x,t) = u(x,t)$. ∎

Theorem 17. (i) *If u is a p-subtemperature on E for some $p \ge 5$, then u is an m-subtemperature on E for all $m \ge 0$.*

(ii) *If u is a p-subtemperature on E for some $p \ge 0$, and u is Λ-continuous at almost every point of E, then u is an m-subtemperature for all $m \ge 0$.*

Proof. (i) Let D be a bounded open set with $\overline{D} \subseteq E$. Then u is the limit of a decreasing sequence of smooth subtemperatures on D, by Theorem 16. Since u is finite almost everywhere on D by Theorem 13, it follows that u is an m-subtemperature on D for all $m \geq 0$, by Theorem 12. The result follows.

(ii) In this case, for D as before, there is a decreasing sequence $\{v_k\}$ of smooth subtemperatures on D which converges to u almost everywhere, by Theorem 16. Let $v = \lim_{k\to\infty} v_k$ on D. Since u is finite almost everywhere, so is v. Therefore v is an m-subtemperature on D for all $m \geq 0$, by Theorem 12. Furthermore, for any point $(x_0, t_0) \in D$ and any p-subtemperature w on D, we have

$$w(x_0, t_0) = \lim_{b\to 0} \mathcal{V}_p(w; x_0, t_0; b),$$

by Lemma 2. Therefore, because u and v are p-subtemperatures that are equal almost everywhere on D, we have

$$u(x_0, t_0) = \lim_{b\to 0} \mathcal{V}_p(u; x_0, t_0; b) = \lim_{b\to 0} \mathcal{V}_p(v; x_0, t_0; b) = v(x_0, t_0).$$

Thus $u = v$ everwhere on D, so that u is an m-subtemperature on D for all $m \geq 0$. The result follows. ■

We can now extend several earlier results about smooth subtemperatures to the case of m-subtemperatures with $m \geq 5$.

THEOREM 18. *Let m and p be integers such that $m \geq 5$ and $p \geq 0$. Let u be an m-subtemperature on E, and let $(x_0, t_0) \in E$.*

(i) *If $\overline{\Omega}(x_0, t_0; c) \subseteq E$, then*

$$u(x_0, t_0) \leq \mathcal{M}(u; x_0, t_0; c), \tag{18}$$

the function $\mathcal{M}(u; x_0, t_0; \cdot)$ is increasing on $]0, c]$, and

$$u(x_0, t_0) = \lim_{b\to 0} \mathcal{M}(u; x_0, t_0; b).$$

(ii) *If $\overline{A}(x_0, t_0; b, c) \subseteq E$, then there is a function ϕ, which is either convex or identically $-\infty$, such that*

$$\mathcal{M}(u; x_0, t_0; a) = \phi(\tau(a))$$

for all $a \in [b, c]$.

(iii) *If $\overline{\Omega}_p(x_0, t_0; c) \subseteq E$, then*

$$u(x_0, t_0) \leq \mathcal{V}_p(u; x_0, t_0; c),$$

the function $\mathcal{V}_p(u; x_0, t_0; \cdot)$ is increasing on $]0, c]$,

$$u(x_0, t_0) = \lim_{b\to 0} \mathcal{V}_p(u; x_0, t_0; b),$$

and there is a function ϕ_p, either convex or identically $-\infty$, such that

$$\mathcal{V}_p(u; x_0, t_0; a) = \phi_p(\tau_p(a))$$

for all $a \in]0,c]$, *where* $\tau_p(a) = (4\pi a)^{-(p+n)/2}$. *If* $p \geq 3$, *then* ϕ_p *is finite-valued.*

Proof. (i) Let D be a bounded open superset of $\overline{\Omega}(x_0,t_0;c)$ with $\overline{D} \subseteq E$. By Theorem 16, there is a decreasing sequence $\{v_k\}$ of smooth subtemperatures on D with limit u. By Theorem 2,

$$v_k(x_0,t_0) \leq \mathcal{M}(v_k;x_0,t_0;c)$$

for all k. Making $k \to \infty$, and using the monotone convergence theorem, we obtain (18). Furthermore, by Theorem 3, for each k the function $\mathcal{M}(v_k;x_0,t_0;\cdot)$ is increasing on $]0,c]$, so that the same is true of

$$\mathcal{M}(u;x_0,t_0;\cdot) = \lim_{k\to\infty} \mathcal{M}(v_k;x_0,t_0;\cdot).$$

Finally, since u is upper semicontinuous on E, given any real number $A > u(x_0,t_0)$ we can find a neighbourhood N of (x_0,t_0) such that $u(y,s) < A$ for all $(y,s) \in N$. So whenever $\overline{\Omega}(x_0,t_0;b) \subseteq N$ we have $\mathcal{M}(u;x_0,t_0;b) \leq A$. Hence

$$\lim_{b\to 0} \mathcal{M}(u;x_0,t_0;b) \leq u(x_0,t_0),$$

so that equality holds in view of (18).

(ii) Choose b' and c' such that $0 < b' < b$, $c < c'$, and $\overline{A}(x_0,t_0;b',c') \subseteq E$. Let D be a bounded open superset of $\overline{A}(x_0,t_0;b',c')$ such that $\overline{D} \subseteq E$. By Theorem 16, there is a decreasing sequence $\{v_k\}$ of smooth subtemperatures on D with limit u. By Theorem 5, for each k there is a convex function ϕ_k such that

$$\mathcal{M}(v_k;x_0,t_0;a) = \phi_k(\tau(a))$$

for all $a \in [b',c']$. Making $k \to \infty$ and using the monotone convergence theorem, we obtain

$$\mathcal{M}(u;x_0,t_0;a) = \lim_{k\to\infty} \phi_k(\tau(a))$$

for all $a \in [b',c']$. Since each ϕ_k is convex, $\lim_{k\to\infty} \phi_k$ is convex on $]b',c'[\supseteq [b,c]$ if it is finite at one point of $]b',c'[$.

(iii) Most of the proof is similar to that of (i) and (ii). For the last statement, if $p \geq 3$ then the kernel $K_{p,a}$ is bounded, so that

$$\mathcal{V}_p(u^-;x_0,t_0;a) \leq \max K_{p,a} \iint_{\Omega_p(x_0,t_0;a)} u^-(y,s)\,dy\,ds,$$

where u^- denotes the negative part of u. The latter integral is finite by Theorem 13, so that the function $\phi_p \circ \tau_p = \mathcal{V}_p(u;x_0,t_0;\cdot)$ is finite-valued. ■

The function $\mathcal{M}(u;x_0,t_0;\cdot)$ in Theorem 18 is also finite-valued, but the only known proofs depend on the Riesz decomposition theorem. See Watson [17] and Brzezina [2, 3]. The volume mean $\mathcal{V}_0(u; x_0, t_0; \cdot)$ is not always finite-valued, as was shown by Watson [21].

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