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ON THE NONLINEAR NEUMANN PROBLEM AT RESONANCE WITH CRITICAL SOBOLEV NONLINEARITY

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Abstract. We consider the Neumann problem for the equation $-\Delta u - \lambda u = Q(x)|u|^{2^*-2}u$, $u \in H^1(\Omega)$, where Q is a positive and continuous coefficient on $\overline{\Omega}$ and λ is a parameter between two consecutive eigenvalues λ_{k-1} and λ_k. Applying a min-max principle based on topological linking we prove the existence of a solution.

1. Introduction. In this paper we are concerned with the semilinear Neumann problem

$$\begin{cases} -\Delta u - \lambda u = Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and ν is the unit outward normal at the boundary $\partial\Omega$. The coefficient Q is continuous and positive on $\overline{\Omega}$ and $2^* = 2N/(N-2)$, $N \geq 3$, denotes the critical Sobolev exponent. The parameter λ satisfies the inequality

$$\lambda_{k-1} < \lambda < \lambda_k \tag{1.2}$$

for some $k \geq 2$. Here $\{\lambda_k\}$, $k = 1, 2, \ldots$, denotes the sequence of eigenvalues for the Neumann problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Each eigenvalue is repeated according to its multiplicity. It is well known that $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \ldots$ and the eigenspace corresponding to $\lambda_1 = 0$ consists of constant functions.

If the parameter λ does not interfere with the spectrum of the operator $-\Delta$, then problem (1.1) can be written in the form

$$\begin{cases} -\Delta u + \lambda u = Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

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where $\lambda > 0$. Problem (1.3) has an extensive literature, specially in the case $Q(x) \equiv 1$ on Ω; we refer to papers [1]–[6], [12], [19], [20]–[24], [16]–[18]. Solutions of (1.3) were obtained as minimizers of the variational problem

$$(1.4) \qquad m_\lambda = \inf\Big\{ \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx;\ u \in H^1(\Omega),\ \int_\Omega Q(x)|u|^{2^*}\,dx = 1\Big\}.$$

A suitable multiple of a minimizer for m_λ is a solution of problem (1.3). These solutions are called the *least energy solutions*. The least energy solutions can be chosen to be positive and have a tendency to concentrate at the most curved part of the boundary of $\partial\Omega$ as $\lambda \to \infty$. Some extensions of these results to problem (1.3) with $Q(x) \not\equiv \text{const}$ can be found in [8]–[10].

To describe these results and supply some motivation for our paper we need some notations. Let $Q_M = \max_{x\in\bar\Omega} Q(x)$ and $Q_m = \max_{x\in\partial\Omega} Q(x)$. By $H(y)$ we denote the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$ with respect to the inner normal to $\partial\Omega$. The existence of least energy solutions has been examined in papers [10] and [8]. In particular, if $Q_M \le 2^{2/(N-2)} Q_m$ and $Q_m = Q(y)$ with $y \in \partial\Omega$ satisfying

$$(1.5) \qquad |Q(x) - Q(y)| = o(|x-y|) \quad \text{for } x \text{ near } y,$$

then problem (1.1) has a least energy solution for every $\lambda > 0$. If $Q_M > 2^{2/(N-2)} Q_m$, then there exists $\Lambda > 0$ such that problem (1.1) has a least energy solution for each $0 < \lambda \le \Lambda$ and no least energy solution for $\lambda > \Lambda$. A similar situation occurs if

$$\{y;\ y \in \partial\Omega,\ Q(y) = Q_m\} \subset \{y;\ y \in \partial\Omega, H(y) < 0\}.$$

In this case, if $Q_M \le 2^{2/(N-2)} Q_m$, there exists a constant $\bar\Lambda > 0$ such that problem (1.1) has a least energy solution for each $0 < \lambda \le \bar\Lambda$ and no least energy solution for each $\lambda > \bar\Lambda$. The existence of positive solutions in the case $\lambda = 0$ has been established in the paper [9]. In this case positive solutions exist provided Q changes sign and $\int_\Omega Q(x)\,dx < 0$. If λ interferes with the spectrum of $-\Delta$, then the method of the constrained minimization (1.4) breaks down as the quadratic functional appearing in m_λ changes sign. To obtain the existence of solutions in this case we apply a min-max method based on topological linking [25]. The main existence results of this paper are contained in Section 3: Theorems 3.3 and 3.4. To apply the topological linking we need to investigate Palais–Smale sequences of the variational functional for problem (1.1).

We recall that a C^1 functional $\phi: X \to \mathbb{R}$ on a Banach space X satisfies the *Palais–Smale condition* at a level c ($(\mathrm{PS})_c$ *condition* for short) if each sequence $\{x_n\} \subset X$ such that

(∗) $\phi(x_n) \to c$,
(∗∗) $\phi'(x_n) \to 0$ in X^*

is relatively compact in X.

Finally, any sequence $\{x_n\}$ satisfying $(*)$ and $(**)$ is called a *Palais–Smale sequence* at level c (a $(\mathrm{PS})_c$ *sequence* for short).

Throughout this paper we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^q(\Omega)$ are denoted by $\|\cdot\|_q$. By $H^1(\Omega)$ we denote the standard Sobolev space on Ω equipped with the norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\,dx.$$

The paper is organized as follows. In Section 2 we determine the energy level of the variational functional for (1.1) below which the Palais–Smale condition holds. The approach is based on the P. L. Lions concentration-compactness principle. Section 3 is devoted to the existence results for (1.1). First we verify that the variational functional for (1.1) has the geometry of topological linking. We use instantons to show that at a min-max level the Palais–Smale condition holds. This restricts the validity of the existence results to dimensions $N \ge 5$ in Theorem 3.3 and $N \ge 7$ in Theorem 3.4.

2. The Palais–Smale condition. Solutions to problem (1.1) will be found as critical points of the variational functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx$$

for $u \in H^1(\Omega)$.

LEMMA 2.1. *Let $\{u_m\} \subset H^1(\Omega)$ be such that $J_\lambda(u_m) \to c$ and $J'_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$. Then the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$.*

Proof. We argue by contradiction. Assume that $\|u_m\| \to \infty$. We set $v_m = u_m/\|u_m\|$. Then

$$\int_\Omega (\nabla u_m \nabla \phi - \lambda u_m \phi)\,dx - \int_\Omega Q(x)|u_m|^{2^*-2} u_m \phi\,dx \to 0 \tag{2.1}$$

as $m \to \infty$ for each $\phi \in H^1(\Omega)$. Since $\|v_m\| = 1$ for each m, we may assume that $v_m \rightharpoonup v$ in $H^1(\Omega)$ and $v_m \to v$ in $L^p(\Omega)$ for each $2 \le p < 2^*$. Consequently, we deduce from (2.1) that

$$\int_\Omega Q(x)|v|^{2^*-2} v \phi\,dx = 0 \tag{2.2}$$

for each $\phi \in H^1(\Omega)$. This implies that $v = 0$ a.e. on Ω. Since $\{u_m\}$ is a Palais–Smale sequence we see that

$$\frac{1}{2}\int_\Omega (|\nabla v_m|^2 - \lambda v_m^2)\,dx - \frac{1}{2^*}\|u_m\|^{2^*-2}\int_\Omega Q(x)|v_m|^{2^*}\,dx \to 0 \tag{2.3}$$

and

$$\int_\Omega (|\nabla v_m|^2 - \lambda v_m^2)\, dx - \|u_m\|^{2^*-2} \int_\Omega Q(x)|v_m|^{2^*}\, dx \to 0 \tag{2.4}$$

as $m \to \infty$. Since $v_m \to 0$ in $L^2(\Omega)$, (2.3) and (2.4) can be rewritten as

$$\frac{1}{2}\int_\Omega |\nabla v_m|^2\, dx - \frac{1}{2^*}\|u_m\|^{2^*-2}\int_\Omega Q(x)|v_m|^{2^*}\, dx \to 0$$

and

$$\int_\Omega |\nabla v_m|^2\, dx - \|u_m\|^{2^*-2}\int_\Omega Q(x)|v_m|^{2^*}\, dx \to 0.$$

This is only possible when $\int_\Omega |\nabla v_m|^2\, dx \to 0$ and $\|u_m\|^{2^*-2}\int_\Omega Q(x)|v_m|^{2^*}\, dx \to 0$, which is impossible. ∎

PROPOSITION 2.2. (i) *Let $Q_M \le 2^{2/(N-2)} Q_m$. Then J_λ satisfies the* $(\mathrm{PS})_c$ *condition with*

$$c < \frac{S^{N/2}}{2NQ_m^{(N-2)/2}}.$$

(ii) *Let $Q_M > 2^{2/(N-2)} Q_m$. Then J_λ satisfies the* $(\mathrm{PS})_c$ *condition with*

$$c < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

Proof. (i) Let $\{u_m\}$ be a $(\mathrm{PS})_c$ sequence with

$$c < \frac{S^{N/2}}{2NQ_m^{(N-2)/2}}$$

and $J'_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$. By Lemma 2.1, $\{u_m\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $u_m \to u$ in $L^p(\Omega)$, $2 \le p < 2^*$. By the concentration-compactness principle [14], we may assume that

$$|u_m|^{2^*} \rightharpoonup |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j} \quad \text{and} \quad |\nabla u_m|^2 \rightharpoonup |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}$$

in the sense of measure, where $\nu_j > 0$, $\mu_j > 0$ are constants and the set J is at most countable. Moreover,

$$\text{if} \quad x_j \in \Omega, \quad \text{then} \quad S\nu_j^{2/2^*} \le \mu_j,$$
$$\text{if} \quad x_j \in \partial\Omega, \quad \text{then} \quad \frac{S\nu_j^{2/2^*}}{2^{2/N}} \le \mu_j.$$

Fix x_j. Using a family of test functions concentrating at x_j we check that $Q(x_j)\nu_j = \mu_j$, $j \in J$. Hence, if $\nu_j > 0$, then

$$\frac{S^{N/2}}{Q(x_j)^{N/2}} \le \nu_j \quad \text{if } x_j \in \Omega, \tag{2.5}$$

$$\frac{S^{N/2}}{2Q(x_j)^{N/2}} \le \nu_j \quad \text{if } x_j \in \partial\Omega. \tag{2.6}$$

We now write

$$J_\lambda(u_m) - \frac{1}{2}\langle J'_\lambda(u_m), u_m\rangle = \frac{1}{N}\int_\Omega Q(x)|u_m|^{2^*}\,dx$$

and letting $m \to \infty$ we get

$$c \ge \frac{1}{N}\int_\Omega Q(x)|u|^{2^*}\,dx + \frac{1}{N}\sum_{j\in J} Q(x_j)\nu_j.$$

If $\nu_j > 0$ for some $j \in J$, then

$$c \ge \frac{S^{N/2}}{NQ(x_j)^{N/2}}\,Q(x_j) \ge \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \ge \frac{S^{N/2}}{2NQ_m^{(N-2)/2}} \quad \text{if } x_j \in \Omega,$$

$$c \ge \frac{S^{N/2}}{2NQ(x_j)^{N/2}}\,Q(x_j) \ge \frac{S^{N/2}}{2NQ_m^{(N-2)/2}} \quad \text{if } x_j \in \partial\Omega.$$

We see that in both cases we obtain a contradiction. This yields $u_m \to u$ in $L^{2^*}(\Omega)$ and in $L^2(\Omega)$. Using the fact that $J'_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$, it is easy to show that $\nabla u_m \to \nabla u$ in $L^2(\Omega)$ and the result follows.

In a similar manner we prove (ii). ■

3. Existence of solutions of problem (1.1). Throughout this section we assume that λ satisfies (1.2). Let $\{e_j\}$ be the sequence of eigenfunctions corresponding to $\{\lambda_j\}$ and set $E^- = \operatorname{span}\{e_1, \ldots, e_{k-1}\}$. We have the orthogonal decomposition of $H^1(\Omega)$,

$$H^1(\Omega) = E^- \oplus E^+.$$

Let $z_\circ \in E^+ - \{0\}$ and define the set

$$M = \{u \in H^1(\Omega);\ u = v + sz_\circ,\ v \in E^-,\ s \ge 0 \text{ and } \|u\| \le R\}$$

(see [25, Section 2.7]).

The proof of the following result is standard.

PROPOSITION 3.1. *There exist $\alpha > 0$, $\varrho > 0$ and $R > \varrho$ (R depending on $z_\circ$) such that*

$$J_\lambda(u) \begin{cases} \ge \alpha & \text{for all } u \in E^+ \cap \partial B(0,\varrho), \\ \le 0 & \text{for all } u \in \partial M. \end{cases}$$

Let

$$U(x) = c_N/(1+|x|^2)^{(N-2)/2},$$

where $c_N = (N(N-2))^{(N-2)/4}$. It is known that $\|\nabla U\|_2^2 = \|U\|_{2^*}^{2^*} = S^{N/2}$.

For $\varepsilon > 0$ and $y \in \mathbb{R}^N$ we set

$$U_{\varepsilon,y}(x) = U\left(\frac{x-y}{\varepsilon}\right) = \frac{c_N \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}.$$

Our argument is based on topological linking. Towards this end we define

$$Z_\varepsilon = E^- \oplus \mathbb{R}U_{\varepsilon,y} = E^- \oplus \mathbb{R}U^+_{\varepsilon,y},$$

where $U^+_{\varepsilon,y}$ denotes the projection of $U_{\varepsilon,y}$ onto E^+. From now on we use $z_\circ = U^+_{\varepsilon,y}$ in the definition of M.

PROPOSITION 3.2. (i) *Let $N \geq 5$. Suppose that $Q_M \leq 2^{2/(N-2)} Q_m$ and that $Q(y) = Q_m$ for some $y \in \partial\Omega$ with $H(y) > 0$ and*

$$|Q(x) - Q(y)| = o(|x-y|) \quad \text{for } x \text{ near } y.$$

Then

$$\sup_{u \in M} J_\lambda(u) < \frac{S^{N/2}}{2NQ_m^{(N-2)/2}} \tag{3.1}$$

for $\varepsilon > 0$ sufficiently small.

(ii) *Let $N \geq 7$. Suppose that $Q_M > 2^{2/(N-2)} Q_m$ and that $D_i Q(y) = 0$, $D^2_{ij} Q(y) = 0$, $i, j = 1, \ldots, N$, for some $y \in \{x;\, Q(x) = Q_M\}$. Then*

$$\sup_{u \in M} J_\lambda(u) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}. \tag{3.2}$$

Proof. (i) We follow, with some modifications, the argument on pp. 52–53 in [25]. If $u \neq 0$, then

$$\max_{t \geq 0} J_\lambda(tu) = \frac{1}{N} \cdot \frac{\{\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx\}^{N/2}}{\{\int_\Omega Q(x)|u|^{2^*}\, dx\}^{(N-2)/2}}$$

whenever the integral in the numerator is positive, and the maximum is 0 otherwise. In what follows we always denote by C_i positive constants independent of ε. It is clear that if

$$m_\varepsilon = \sup_{u \in Z_\varepsilon,\, \|u\|_{2^*,Q} = 1} \int_{\mathbb{R}^N} (|\nabla u|^2 - \lambda u^2)\, dx < \frac{S}{2^{2/N} Q_m^{(N-2)/N}}, \tag{3.3}$$

then

$$\sup_{Z_\varepsilon} J_\lambda(u) < \frac{S^{N/2}}{2NQ_m^{(N-2)/2}}$$

and this obviously implies (i). For simplicity we assume that $y = 0$ and set $U_\varepsilon = U_{\varepsilon,0}$. If $u \in Z_\varepsilon$ and $\|u\|_{2^*,Q} = 1$, then

$$u = u^- + sU_\varepsilon = (u^- + sU_\varepsilon^-) + sU_\varepsilon^+,$$

where U_ε^- denotes the projection of U_ε onto E^-. We now observe that

$$\int_\Omega (|\nabla U_\varepsilon^-|^2 - \lambda (U_\varepsilon^-)^2)\,dx \le 0,$$

so

$$\int_\Omega |\nabla U_\varepsilon^-|^2\,dx \le \lambda \int_\Omega (U_\varepsilon^-)^2\,dx \le \lambda \int_\Omega U_\varepsilon^2\,dx = O(\varepsilon^2).$$

Therefore

$$\|U_\varepsilon^-\|_{2^*} \le C_2(\|\nabla U_\varepsilon^-\|_2 + \|U_\varepsilon^-\|_2) \to 0.$$

From this we deduce that there exists a constant $C_3 > 0$ such that $0 < s \le C_3$ and $\|u^-\|_{2^*} \le C_3$. Since all norms in E^- are equivalent, we have $\|u^-\|_\infty \le C\|u^-\|_{2^*} \le C'$. It follows from the convexity of $\|\cdot\|_{2^*,Q}^{2^*}$ that

$$\begin{aligned}1 = \|u\|_{2^*,Q}^{2^*} &\ge \|sU_\varepsilon\|_{2^*,Q}^{2^*} + 2^* \int_\Omega Q(x) u^- (sU_\varepsilon)^{2^*-1}\,dx \\ &\ge \|sU_\varepsilon\|_{2^*,Q}^{2^*} - C_4 \|U_\varepsilon\|_{2^*-1}^{2^*-1}.\end{aligned}$$

This implies that

$$\|sU_\varepsilon\|_{2^*,Q}^{2^*} \le 1 + C_5 \varepsilon^{(N-2)/2}. \tag{3.4}$$

Since all norms in E^- are equivalent we see that

$$\begin{aligned}\int_\Omega (\nabla u^- \nabla U_\varepsilon - \lambda u^- U_\varepsilon)\,dx &\le C_5(\|\nabla U_\varepsilon\|_1 + \|U_\varepsilon\|_1)\|u^-\|_2 \\ &= O(\varepsilon^{(N-2)/2})\|u^-\|_2.\end{aligned} \tag{3.5}$$

It follows from the regularity of Q at 0 that

$$\|U_\varepsilon\|_{2^*,Q}^{2^*} = Q_m \int_\Omega U_\varepsilon^{2^*}\,dx + o(\varepsilon). \tag{3.6}$$

By (3.5) we have

$$\begin{aligned}\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx &\le (\lambda_{k-1} - \lambda) \int_\Omega |u^-|^2\,dx + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\quad + s^2 \int_\Omega (|\nabla U_\varepsilon|^2 - \lambda U_\varepsilon^2)\,dx \\ &= -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\quad + s^2 \int_\Omega (|\nabla U_\varepsilon|^2 - \lambda U_\varepsilon^2)\,dx \\ &= -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\quad + \frac{\int_\Omega (|\nabla U_\varepsilon|^2 - \lambda U_\varepsilon^2)\,dx}{(\int_\Omega Q(x) U_\varepsilon^{2^*}\,dx)^{2/2^*}} \Big(s^{2^*} \int_\Omega Q(x) U_\varepsilon^{2^*}\,dx\Big)^{2/2^*}.\end{aligned} \tag{3.7}$$

To proceed further, we use the following asymptotic formula: if we let

$$E_\lambda(u) = \frac{\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx}{(\int_\Omega Q(x)|u|^{2^*}\,dx)^{2/2^*}},$$

then

$$E_\lambda(U_\varepsilon) = \frac{S}{2^{2/N}} - A_N H(y)\varepsilon - a_N\lambda\varepsilon^2 + O(\varepsilon^2) + o(\lambda\varepsilon^2) \quad \text{if } N \geq 5, \tag{3.8}$$

where $A_N > 0$ and $a_N > 0$ are constants depending on N. It follows from (3.6)–(3.8) that if $N \geq 5$ then

$$\begin{aligned} m_\varepsilon \leq\ & -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2}\|u^-\|_2) \\ & + \left[\frac{S}{2^{2/N}} Q_m^{(N-2)/N} - A_N Q_m^{-(N-2)/N} H(y)\varepsilon + o(\varepsilon)\right](1 + C_4\varepsilon^{(N-2)/2}) \\ < & \frac{S}{2^{2/N} Q_m^{(N-2)/N}} \end{aligned}$$

for ε sufficiently small.

(ii) The only change is in the estimation of m_ε. We have

$$\begin{aligned} m_\varepsilon \leq\ & -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ & + \frac{\int_\Omega (|\nabla U_\varepsilon|^2 - \lambda U_\varepsilon^2)\,dx}{(\int_\Omega Q(x) U_\varepsilon^{2^*}\,dx)^{2/2^*}} \Big(\int_\Omega s^{2^*} Q(x) U_\varepsilon^{2^*}\,dx\Big)^{2/2^*} \\ \leq\ & -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ & + \frac{K_1 + O(\varepsilon^{N-2}) - \lambda c\varepsilon^2}{(K_2 Q_M + o(\varepsilon^2))^{(N-2)/N}} \Big(\int_\Omega s^{2^*} Q(x) U_\varepsilon^{2^*}\,dx\Big)^{2/2^*} \\ =\ & -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ & + (K_1 + O(\varepsilon^{N-2}) - \lambda c\varepsilon^2)((K_2 Q_m)^{-(N-2)/N} + o(\varepsilon^2))(1 + C_4\varepsilon^{(N-2)/2}) \\ \leq\ & \frac{S}{Q_M^{(N-2)/N}} + O(\varepsilon^{(N-2)/2}) - c\lambda\varepsilon^2, \end{aligned}$$

where $c > 0$ is a constant independent of ε, $K_1 = \int_{\mathbb{R}^N} |\nabla U|^2\,dx$ and $K_2 = \int_{\mathbb{R}^N} U^{2^*}\,dx$. Since $S = K_1/K_2^{(N-2)/2}$, by taking ε sufficiently small the result follows. ∎

Applying a min-max theorem based on topological linking [25], we derive the following existence result:

THEOREM 3.3. *Under assumptions* (i) *and* (ii) *of Proposition* 3.2 *problem* (1.1) *admits a nontrivial solution.*

By a similar argument we can establish the existence result in the case when $\partial\Omega$ has a flat part. We need the following assumption:

(F) $D(a,0) \subset \partial\Omega$ for some $a > 0$, where $D(a,0) = B(0,a) \cap \{x_N = 0\}$ and $\{x;\, x \in \partial\Omega,\ Q(x) = Q_m\} \subset D(a,0)$.

THEOREM 3.4. *Let $N \geq 5$. Suppose that* (F) *holds and that $D_iQ(y) = 0$, $D_{ij}Q(y) = 0$, $i,j = 1,\ldots,N$, for some $y \in \partial\Omega$ with $Q_m = Q(y)$. Then problem* (1.1) *admits a nontrivial solution.*

Proof. Without loss of generality we may assume that $y = 0$. It is sufficient to notice that

$$\frac{\int_\Omega (|\nabla U_\varepsilon|^2 - \lambda U_\varepsilon^2)\,dx}{(\int_\Omega Q(x)U_\varepsilon^{2^*}\,dx)^{2/2^*}} = \frac{K_2/2 + O(\varepsilon^{N-2}) - \lambda\int_\Omega U_\varepsilon^2\,dx}{((K_2/2)Q_m + O(\varepsilon^N) + o(\varepsilon^2))^{(N-2)/N}}.$$

As is easy to see, the above expression is strictly less than $S/(2^{2/N}Q_m^{(N-2)/N})$ for ε sufficiently small. The remaining part of the proof is the same as in Theorem 3.3. ∎

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