

*EXISTENCE OF LARGE  $\varepsilon$ -KRONECKER AND  $FZI_0(U)$  SETS IN  
DISCRETE ABELIAN GROUPS*

BY

COLIN C. GRAHAM (Vancouver)  
and KATHRYN E. HARE (Waterloo)

**Abstract.** Let  $G$  be a compact abelian group with dual group  $\mathbf{\Gamma}$  and let  $\varepsilon > 0$ . A set  $\mathbf{E} \subset \mathbf{\Gamma}$  is a “weak  $\varepsilon$ -Kronecker set” if for every  $\varphi : \mathbf{E} \rightarrow \mathbb{T}$  there exists  $x$  in the dual of  $\mathbf{\Gamma}$  such that  $|\varphi(\gamma) - \gamma(x)| \leq \varepsilon$  for all  $\gamma \in \mathbf{E}$ . When  $\varepsilon < \sqrt{2}$ , every bounded function on  $\mathbf{E}$  is known to be the restriction of a Fourier–Stieltjes transform of a discrete measure. (Such sets are called  $I_0$ .)

We show that for every infinite set  $\mathbf{E}$  there exists a weak 1-Kronecker subset  $\mathbf{F}$ , of the same cardinality as  $\mathbf{E}$ , provided there are not “too many” elements of order 2 in the subgroup generated by  $\mathbf{E}$ . When there are “too many” elements of order 2, we show that there exists a subset  $\mathbf{F}$ , of the same cardinality as  $\mathbf{E}$ , on which every  $\{-1, 1\}$ -valued function can be interpolated exactly. Such sets are also  $I_0$ . In both cases, the set  $\mathbf{F}$  also has the property that the only continuous character at which  $\mathbf{F} \cdot \mathbf{F}^{-1}$  can cluster in the Bohr topology is  $\mathbf{1}$ . This improves upon previous results concerning the existence of  $I_0$  subsets of a given  $\mathbf{E}$ .

## 1. Introduction

**1.1. Terminology and background.** Let  $G$  denote a compact abelian group and  $\mathbf{\Gamma}$  its discrete dual group. For  $U \subseteq G$ , we let  $M_d(U)$  denote the bounded discrete measures concentrated on  $U$ , and  $M_d^+(U)$  the non-negative measures in  $M_d(U)$ . By  $\ell^\infty(\mathbf{E})$  we mean the bounded complex-valued functions on  $\mathbf{E}$  with supremum norm.

**DEFINITION 1.1.** The subset  $\mathbf{E} \subseteq \mathbf{\Gamma}$  is said to be  $I_0(U)$  if for every  $\varphi \in \ell^\infty(\mathbf{E})$  there is a measure  $\mu \in M_d(U)$  with  $\widehat{\mu}(\gamma) = \varphi(\gamma)$  for every  $\gamma \in \mathbf{E}$ . Furthermore,  $\mathbf{E}$  is said to be  $FZI_0(U)$  if whenever  $\varphi \in \ell^\infty(\mathbf{E})$  is Hermitian (i.e.,  $\varphi(\gamma^{-1}) = \overline{\varphi(\gamma)}$  for all  $\gamma \in \mathbf{E} \cap \mathbf{E}^{-1}$ ), then there exists  $\mu \in M_d^+(U)$  with  $\widehat{\mu} = \varphi$  on  $\mathbf{E}$ .

**DEFINITION 1.2.** Let  $\varepsilon > 0$ . The subset  $\mathbf{E}$  is said to be  $\varepsilon$ -Kronecker( $U$ ) if for every  $\varphi : \mathbf{E} \rightarrow \mathbb{T}$  there exists  $x \in U$  such that  $|\varphi(\gamma) - \gamma(x)| < \varepsilon$  for all

---

2010 *Mathematics Subject Classification*: Primary 42A55, 42A63, 43A25, 43A46; Secondary 43A05, 43A25.

*Key words and phrases*:  $\varepsilon$ -Kronecker sets, Fatou–Zygmund property,  $\varepsilon$ -free sets, Hadamard sets,  $I_0$  sets, Sidon sets.

$\gamma \in \mathbf{E}$ , and is called *weak  $\varepsilon$ -Kronecker( $U$ )* if the weaker inequality  $\leq$  can be achieved.

If  $U = G$ , we omit the writing of  $U$  from these definitions.

It is known that  $\varepsilon$ -Kronecker( $U$ ) sets are  $FZI_0(U)$  if  $\varepsilon < \sqrt{2}$ ,  $FZI_0(U)$  sets are  $I_0(U)$ , and these inclusions are proper [4]. Moreover, if  $\mathbf{E}$  is weak  $\varepsilon$ -Kronecker and  $U$  is an open set, then a Baire category theorem argument shows that there is a finite set  $\mathbf{F} \subset \mathbf{E}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is weak  $\varepsilon$ -Kronecker( $U$ ) [3, Thm. 3.2] and thus  $FZI_0(U)$ . Since not even singletons are  $FZI_0(U)$  for small enough  $U$  (cf. [4]), the exclusion of a finite set is essential.

Another interesting property of  $\varepsilon$ -Kronecker sets  $\mathbf{E}$  with  $\varepsilon < \sqrt{2}$  is that  $\mathbf{E} \cdot \mathbf{E}$  does not cluster in the Bohr topology at any continuous character, and the only continuous character at which  $\mathbf{E} \cdot \mathbf{E}^{-1}$  clusters is the identity character  $\mathbf{1}$ .

Examples of infinite  $I_0$ ,  $FZI_0$  and  $\varepsilon$ -Kronecker sets abound. For instance, an Hadamard set in  $\mathbb{Z}$  of ratio  $q$  is an  $\varepsilon$ -Kronecker set with  $\varepsilon = |1 - e^{i\pi/(q-1)}|$ . Hartman and Ryll-Nardzewski [10] were the first to prove that every infinite discrete group contains an  $I_0$  set of the same cardinality. Kunen and Rudin [13] showed the existence of an  $I_0$  subset  $\mathbf{E}$  with the same cardinality as  $\mathbf{\Gamma}$ , and with the additional properties that  $\mathbf{1}$  was the only continuous character which was a cluster point of  $\mathbf{E} \cdot \mathbf{E}^{-1}$ , and that  $\mathbf{E} \cdot \mathbf{E}$  had no cluster point in  $\mathbf{\Gamma}$  if  $\mathbf{\Gamma}$  did not contain “too many” elements of order 2. The current authors showed in [4] that every infinite  $\mathbf{\Gamma}$  contains an  $FZI_0$  set of the same cardinality, and proofs of the existence of large  $\varepsilon$ -Kronecker sets under various assumptions can be found in [1, 2, 8].

In this paper, our interest is in showing the existence of these and related thin sets in every infinite subset of  $\mathbf{\Gamma}$ . Our first result, Theorem 2.2, states that an infinite set  $\mathbf{E} \subseteq \mathbf{\Gamma}$  contains an infinite subset  $\mathbf{F}$  that is weak  $\varepsilon$ -Kronecker for some  $\varepsilon \leq 1$  and of the same cardinality as  $\mathbf{E}$ , provided that the subgroup generated by  $\mathbf{E}$  does not contain “too many” elements of order two. In the next section we make “too many” precise and relate the size of  $\varepsilon$  to the “amount” of torsion in  $\mathbf{\Gamma}$ .

Elements of order two cause complications. Indeed, if  $\mathbf{E}$  is  $\varepsilon$ -Kronecker for some  $\varepsilon < \sqrt{2}$ , then  $\mathbf{E}$  contains no elements of order two.

Recall that  $\mathbf{E}$  (not containing  $\mathbf{1}$ ) is called *independent* if whenever  $N \in \mathbb{N}$ ,  $\gamma_1, \dots, \gamma_N \in \mathbf{E}$ ,  $k_n \in \mathbb{Z}$  and  $\prod_{n=1}^N \gamma_n^{k_n} = \mathbf{1}$ , then  $\gamma_n^{k_n} = \mathbf{1}$  for all  $n$ . The set of Rademacher functions (i.e., the projections on single coordinates) in the dual group of  $\mathbb{Z}_2^{\aleph_0}$  is an example of an independent set of characters all of order two.

An independent set,  $\mathbf{E}$ , of characters of order two has the property that if  $\varphi : \mathbf{E} \rightarrow \{-1, 1\}$ , then there is an  $x \in G$  such that  $\varphi(\gamma) = \gamma(x)$  for all

$\gamma \in \mathbf{E}$ . Clearly, such sets are weak  $\sqrt{2}$ -Kronecker. We will see that they are  $FZI_0$  and suitable cofinite subsets are  $FZI_0(U)$ .

However, translates of a Rademacher set need not have such good properties.

**EXAMPLE 1.3.** Let  $\mathbf{E}$  be the set of Rademacher functions and suppose  $\gamma \in \mathbb{Z}_3$  has order three. Then  $\gamma\mathbf{E} \in \mathbb{Z}_3 \oplus \mathbb{Z}_2^{\mathbb{N}_0}$  is  $FZI_0$ , but no subset is  $FZI_0(U)$  for  $U = \{x \in G : \gamma(x) = 1\}$  since the Fourier transform of any positive measure concentrated on  $U$  will take on only real values on  $\gamma\mathbf{E}$ . This is an instance of “too many” elements of order two.

These comments and examples motivate the following new definitions.

**DEFINITION 1.4.** We say  $\mathbf{E}$  is *Rademacher* if every element of  $\mathbf{E}$  has order two and  $\mathbf{E}$  is independent. We say  $\mathbf{E}$  is *pseudo-Rademacher*( $U$ ) if for every  $\varphi : \mathbf{E} \rightarrow \{-1, 1\}$  there exists  $x \in U$  such that  $\varphi(\gamma) = \gamma(x)$  for all  $\gamma \in \mathbf{E}$ . A set is *Rademacher*( $U$ ) if it is both Rademacher and pseudo-Rademacher( $U$ ). When  $U = G$  we omit “( $U$ )” in these definitions.

A translate of a Rademacher set by an independent character, such as the set  $\gamma\mathbf{E}$  in Example 1.3, is an example of a pseudo-Rademacher set. An independent set of characters of even order is also pseudo-Rademacher.

Clearly, pseudo-Rademacher sets are weak  $\sqrt{2}$ -Kronecker. It will be shown that pseudo-Rademacher sets contain cofinite sets that are  $I_0(U)$ , but unlike Rademacher sets, they need not contain any  $FZI_0(U)$  subsets, as the example above illustrates. Like  $\varepsilon$ -Kronecker sets, pseudo-Rademacher sets also have the property that the only continuous character at which  $\mathbf{E} \cdot \mathbf{E}^{-1}$  can cluster is  $\mathbf{1}$ .

Our second main result, Theorem 2.4, states that if the subgroup generated by  $\mathbf{E}$  contains “too many” elements of order two, then  $\mathbf{E}$  contains a pseudo-Rademacher set of the same cardinality as  $\mathbf{E}$ . In view of Example 1.3 this is the best that can be done; not every set contains an infinite subset that is  $FZI_0(U)$ , but every infinite set contains an  $I_0(U)$  set of the same cardinality.

**1.2. Outline of paper.** In the next section we give precise statements of our results and explicitly define what is meant by “too many” elements of order two. In §3 we establish properties of Rademacher and pseudo-Rademacher sets. Those properties complete the results of this paper and complement known results for  $\varepsilon$ -Kronecker sets.

Preliminary lemmas for the proofs of our theorems are given in Section 4; the new key idea arises in the proof of Lemma 4.6. In the last section we give the proofs of our main theorems.

## 2. Statement of results

**2.1. “Large” sets.** Before we can state our first result, we need some notation and a definition. For a subset  $\mathbf{F} \subset \Gamma$  we let  $\langle \mathbf{F} \rangle$  be the group generated by  $\mathbf{F}$ , and  $q_{\mathbf{F}}$  be the quotient map  $\Gamma \rightarrow \Gamma/\langle \mathbf{F} \rangle$ . For  $2 \leq N \in \mathbb{N}$  we let  $\Gamma_N$  be the subgroup of elements of  $\Gamma$  whose order divides  $N$ . In place of  $q_{\Gamma_N}$  we write  $q_N$ . Finally,  $\Gamma_0 \subset \Gamma$  is the subgroup of elements of finite order, and  $q_0$  the associated quotient group homomorphism.

We write  $|S|$  for the cardinality of the set  $S$ .

**DEFINITION 2.1.** Let  $2 \leq N \in \mathbb{N}$  and  $\mathbf{E} \subset \Gamma$ . We say that  $\mathbf{E}$  is *N-large* if  $|q_N(\mathbf{E})| < |\mathbf{E}|$ . If  $\mathbf{E}$  is not *N-large*, we say that  $\mathbf{E}$  is *N-small*. We say  $\mathbf{E}$  is *tor-large* if  $|q_0(\mathbf{E})| < |\mathbf{E}|$ .

For example,  $\Gamma$  is 2-large if the index of  $\{\gamma \in \Gamma : \gamma^2 = 1\}$  in  $\Gamma$  is less than  $|\Gamma|$ . We note that if  $\mathbf{E}$  is *N-large* and  $k \geq 1$ , then  $\mathbf{E}$  is *kN-large*, and if  $\mathbf{E}$  is *N-large*, then it is *tor-large*. If  $\mathbf{E}$  generates  $\Gamma$  and  $|q_N(\Gamma)| < |\Gamma|$ , then  $\mathbf{E}$  is *N-large*.

But  $\mathbf{E}$  can be *N-large* without that last inequality holding. Take, for instance,  $\mathbf{E}$  to be the union of a countable Rademacher set with a singleton of infinite order in  $\Gamma = \mathbb{Z} \oplus \bigoplus \mathbb{Z}_2^{\aleph_0}$ . Then  $|\Gamma/\Gamma_2|$  is countably infinite, although  $q_2(\mathbf{E})$  contains only two elements.

We note that  $\mathbf{E}$  can be *tor-large* yet not be *N-large* for any  $N \geq 2$  (the method of Example 3.3 adapts easily).

## 2.2. Existence theorems

**THEOREM 2.2.** *Let  $\mathbf{E} \subset \Gamma$  be infinite.*

- (1) *If  $\mathbf{E}$  is *N-small* for all  $N \geq 2$ , then for every  $\varepsilon > 0$ ,  $\mathbf{E}$  contains an  $\varepsilon$ -Kronecker subset,  $\mathbf{F}$ , of the same cardinality.*
- (2) *Suppose that  $\mathbf{E}$  is *M-large* for some  $M$ . Let  $N$  be the smallest such  $M$ , and  $L$  any prime power that divides  $N$ . Then  $\mathbf{E}$  contains a weak  $\varepsilon$ -Kronecker subset of the same cardinality, where  $\varepsilon = |1 - e^{\pi i/L}|$ .*

**REMARKS 2.3.** (i) The example of  $\Gamma = \mathbb{Z}$  shows that one has to choose  $\varepsilon$  in Theorem 2.2(1) before finding  $\mathbf{F}$ .

(ii) The dual of  $(\mathbb{Z}_3 \times \mathbb{Z}_5)^{\aleph_0}$  contains a union  $\mathbf{E}$  of a countable independent set of elements of order 3 and a countable independent set of elements of order 5. That  $\mathbf{E}$  is 15-large (though not 3- or 5-large). The existence of such an  $\mathbf{E}$  shows that  $L$  in Theorem 2.2(2) is needed.

The 2-large case being of particular importance, we give a more explicit version of Theorem 2.2(2).

**THEOREM 2.4.** *Let  $\mathbf{E} \subset \mathbf{\Gamma}$  be infinite and 2-large. Then  $\mathbf{E}$  contains a subset  $\mathbf{F}$  that is pseudo-Rademacher and  $|\mathbf{F}| = |\mathbf{E}|$ . If  $\mathbf{E}$  is countable,  $\mathbf{F}$  may be taken to be a translate of a Rademacher set.*

This is best possible in the sense that a pseudo-Rademacher set  $\mathbf{E}$  does not always contain a translate of a Rademacher set of the same cardinality. See Example 3.3.

**2.3. Interpolation and the length of measures.** The preceding theorems describe existence of subsets. We now make the interpolation properties of those subsets clearer.

When  $G$  is connected, every  $I_0$  set is  $I_0(U)$  for all open, non-empty  $U$ . Furthermore, when  $G$  is connected, every  $I_0$  set is a finite union of sets that are  $I_0(U)$  with bounded constants, by which we mean that there exists a constant  $C$  such that for each open  $U \subseteq G$  there is a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is  $I_0(U)$  and the interpolation constant from  $M_d(U)$  to  $\ell^\infty(\mathbf{E} \setminus \mathbf{F})$  is at most  $C$  (see [7]).

A concept more useful than bounded constants is “bounded length”.

**DEFINITION 2.5.**  $\mathbf{E}$  is  $I_0(U, N)$  if there exists  $N \in \mathbb{N}$  such that for every  $\varphi \in \ell^\infty(\mathbf{E})$  with  $\|\varphi\|_\infty \leq 1$ , there exist  $c_n \in \mathbb{C}$ ,  $n = 1, \dots, N$ , with  $|c_n| \leq 1$  and  $x_n \in U$  such that  $\|\varphi - \sum_{n=1}^N c_n \widehat{\delta}_{x_n}\|_\infty \leq 1/2$ .

A Baire category theorem argument shows that every set that is  $I_0(U)$  is  $I_0(\overline{U}, N)$  for some  $N$ , and it is easy to see that  $\mathbf{E}$  is  $I_0(U, N)$  if and only if it is  $I_0(gU, N)$  for all  $g \in G$ . See [3, 12, 14, 16] for these and related results.

**DEFINITION 2.6.** We will say that  $\mathbf{E}$  is  $I_0(U)$  with bounded length if there exists a positive integer  $N$  with the property that for every open neighbourhood  $U$  of the identity of  $G$  there exists a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is  $I_0(U, N)$ .

Note that if  $\mathbf{E}$  is  $I_0(U)$  with bounded length, then so is any translate of  $\mathbf{E}$ . Clearly,  $I_0(U)$  with bounded length implies  $I_0(U)$  with bounded constants.

We analogously define  $FZI_0(U, N)$  and  $FZI_0(U)$  with bounded length, noting, however,  $FZI_0(U)$  is not preserved by translation of  $U$ . The proof of [4, Thm. 2.4], which states that  $FZI_0(U)$  implies  $I_0(U)$ , also shows  $FZI_0(U)$  with bounded length implies  $I_0(U)$  with bounded length. On the other hand, Proposition 3.4 below implies the set  $\gamma\mathbf{E}$  in Example 1.3 is  $I_0(U)$  with bounded length, but not  $FZI_0(U)$  with bounded length.

The proof of [7, Thm. 3.1] actually shows that every  $I_0$  set is a finite union of sets that are  $I_0(U)$  with bounded length. Furthermore, the proof given in [4, Thm. 3.1], that an  $\varepsilon$ -Kronecker( $U$ ) set with  $\varepsilon < \sqrt{2}$  is  $FZI_0(U)$ , shows that such a set is  $FZI_0(U, N)$  with  $N$  depending only on  $\varepsilon$ . Thus  $\varepsilon$ -Kronecker sets are  $FZI_0(U)$  with bounded length.

Similar arguments (see Prop. 3.4) show that every Rademacher set is  $FZI_0(U)$  with bounded length and every translate of a pseudo-Rademacher set is  $I_0(U)$  with bounded length.

To conclude, we have the following result. Here,  $M_0(\overline{\Gamma})$  is the set of measures on the Bohr compactification of  $\Gamma$  whose Fourier–Stieltjes transforms vanish at infinity on  $G$ , when  $G$  is given the discrete topology. Items (3) and (6) appear in [13] for the case  $\mathbf{E} = \mathbb{Z}$ .

**COROLLARY 2.7.** *Let  $\mathbf{E} \subset \Gamma$  be infinite. Then there exists  $\mathbf{F} \subset \mathbf{E}$  such that:*

- (1)  $|\mathbf{F}| = |\mathbf{E}|$ ;
- (2)  $\mathbf{F}$  is  $I_0(U)$  with bounded length;
- (3) the only continuous character at which  $\mathbf{F} \cdot \mathbf{F}^{-1}$  can cluster is  $\mathbf{1}$ ; and
- (4) the closure of  $\mathbf{F} \cdot \mathbf{F}^{-1}$  in  $\overline{\Gamma}$  supports no non-zero measure in  $M_0(\overline{\Gamma})$ .

If, in addition,  $\mathbf{E}$  is not 2-large, then:

- (5)  $\mathbf{F}$  is  $FZI_0(U)$  with bounded length;
- (6)  $\mathbf{F} \cdot \mathbf{F}$  does not cluster at any continuous character.

*Proof.* If  $\mathbf{E}$  is not 2-large, then Theorem 2.2 guarantees the existence of a weak 1-Kronecker subset of  $\mathbf{E}$  having the same cardinality as  $\mathbf{E}$ . Weak 1-Kronecker sets are known to have properties (2)–(6) [4, 5].

If  $\mathbf{E}$  is 2-large, then Theorem 2.4 gives a pseudo-Rademacher subset  $\mathbf{F}$  with  $|\mathbf{F}| = |\mathbf{E}|$ . Propositions 3.4–3.5 below show that  $\mathbf{F}$  has properties (2)–(4). ■

**3. Rademacher and pseudo-Rademacher sets.** In this section we prove basic facts about Rademacher and pseudo-Rademacher sets. A key observation is the following application of the Baire category theorem.

**LEMMA 3.1.** *Suppose  $\mathbf{E}$  is a (pseudo-)Rademacher set and  $U \subseteq G$  is a neighbourhood of the identity  $e \in G$ .*

- (1) *Then there is a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is (pseudo-)Rademacher( $U$ ).*
- (2) *Suppose  $\mathbf{E}$  is Rademacher and  $x \in G$ . Then there is a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is Rademacher( $xU$ ).*

*Proof.* First, we recall some information about the topology of the product space  $\{-1, 1\}^{\mathbf{E}}$ . For  $\gamma \in \mathbf{E}$ , let  $P_\gamma$  be the projection on coordinate  $\gamma$ . A basis for the open sets of  $\{-1, 1\}^{\mathbf{E}}$  is the collection of sets of the form  $U = U(\gamma_1, \dots, \gamma_L, \epsilon_1, \dots, \epsilon_L)$  where  $1 \leq L$ ,  $\gamma_l \in \mathbf{E}$ ,  $\epsilon_l \in \{-1, 1\}$  and

$$P_\gamma(U) = \begin{cases} \{\epsilon_l\} & \text{if } \gamma = \gamma_l, 1 \leq l \leq L, \\ \{-1, 1\} & \text{otherwise.} \end{cases}$$

(1) Choose a compact neighbourhood  $V$  such that  $V^2 \subseteq U$ . By the compactness of  $G$ , there are finitely many elements  $g_1, \dots, g_K \in G$  such that  $G = \bigcup_{k=1}^K g_k V$ . Let

$$X_k = \{\varphi \in \{-1, 1\}^{\mathbf{E}} : \exists x \in g_k V \text{ such that } \varphi(\gamma) = \gamma(x) \forall \gamma \in \mathbf{E}\}.$$

Those sets are closed in  $\{-1, 1\}^{\mathbf{E}}$ , and, since  $\mathbf{E}$  is (pseudo-)Rademacher, their union is all of the compact Hausdorff space  $\{-1, 1\}^{\mathbf{E}}$ . By the Baire category theorem, one of the sets  $X_k$  has non-empty interior, say some  $U = U(\gamma_1, \dots, \gamma_L, \epsilon_1, \dots, \epsilon_L)$  as above. That means there is a finite set  $\mathbf{F} = \{\gamma_1, \dots, \gamma_L\}$  such that for each  $\varphi \in \{-1, 1\}^{\mathbf{E}}$  there is an  $x \in g_k V$  such that  $\varphi(\gamma) = \gamma(x)$  for all  $\gamma \in \mathbf{E} \setminus \mathbf{F}$ . Because the constant function 1 is also in  $\{-1, 1\}^{\mathbf{E}}$ , there is some  $y \in g_k V$  such that  $\gamma(y) = 1$  for all  $\gamma \in \mathbf{E} \setminus \mathbf{F}$ . But then also  $\gamma(y^{-1}) = 1$  for all such  $\gamma$ . Take  $z = xy^{-1} \in V^2 \subseteq U$ . Then  $\gamma(z) = \varphi(\gamma)$  for all  $\gamma \in \mathbf{E} \setminus \mathbf{F}$ . That proves  $\mathbf{E} \setminus \mathbf{F}$  is pseudo-Rademacher( $U$ ).

(2) Now suppose  $\mathbf{E}$  is Rademacher. Then  $\mathbf{E} \setminus \mathbf{F}$  is pseudo-Rademacher( $U$ ) by the preceding argument. Because  $\mathbf{E}$  is Rademacher,  $\widehat{\delta}_x = \pm 1$  on  $\mathbf{E}$ . Let  $\varphi : \mathbf{E} \setminus \mathbf{F} \rightarrow \{-1, 1\}$ . Choose  $y \in U$  such that  $\widehat{\delta}_y = \widehat{\delta}_x \varphi$  on  $\mathbf{E} \setminus \mathbf{F}$ . Then  $\widehat{\delta}_{yx} = \varphi$  on  $\mathbf{E} \setminus \mathbf{F}$  and  $yx \in xU$ . Therefore  $\mathbf{E} \setminus \mathbf{F}$  is pseudo-Rademacher( $xU$ ), so  $\mathbf{E} \setminus \mathbf{F}$  is Rademacher( $xU$ ). ■

**COROLLARY 3.2.** *If  $\mathbf{E}$  is Rademacher,  $V \subset G$  is a neighbourhood of the identity and  $\gamma \in \Gamma$ , then there is a finite set  $\mathbf{F}$  so that  $\gamma(\mathbf{E} \setminus \mathbf{F})$  is pseudo-Rademacher( $V$ ).*

*Proof.* If  $\gamma$  has finite order, apply the lemma with  $U = \{x : \gamma(x) = 1\} \cap V$ . If  $\gamma$  has infinite order, then  $\gamma^k \neq \prod_{j=1}^K \gamma_j^{k_j}$  for all choices  $\gamma_1, \dots, \gamma_K \in \mathbf{E}$  and  $k_j \in \mathbb{N}$  since all  $\gamma_j \in \mathbf{E}$  have order 2. Thus  $\{\gamma\} \cup \mathbf{E}$  is still independent so, using Lemma 4.1 below, we may assume that  $\Gamma = \mathbb{Z} \times \langle \mathbf{E} \rangle$ . Let  $H$  be the dual of  $\langle \mathbf{E} \rangle$ . Now we see that  $\mathbf{E}$  restricted to  $H$  is pseudo-Rademacher( $U$ ), where  $U = (\{1\} \times H) \cap V$ , and so Lemma 3.1 gives us a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is pseudo-Rademacher( $U$ ). It is now immediate that  $\gamma(\mathbf{E} \setminus \mathbf{F})$  is pseudo-Rademacher( $V$ ). ■

However, a pseudo-Rademacher set need not be a translate of a single Rademacher set.

**EXAMPLE 3.3.** Inductively define cardinals  $a_j$  by  $a_1 = \aleph_0$  and  $a_{j+1} = 2^{a_j}$ . Put  $G = \mathbb{T} \times \prod_{j=1}^{\infty} \mathbb{Z}_2^{a_j}$  and let  $\gamma_j \in \Gamma = \widehat{G}$  be the character  $\gamma_j = (j, 1, 1, \dots)$ . Let  $\mathbf{E}_j$  be a Rademacher set of cardinality  $a_j$  in the dual <sup>(1)</sup> of  $\mathbb{Z}_2^{a_j}$  and set  $\mathbf{E} = \bigcup_{j=1}^{\infty} \gamma_j \mathbf{E}_j$ . It is easy to see that  $\mathbf{E}$  is pseudo-Rademacher, but contains no translate of a Rademacher subset of the same cardinality.

<sup>(1)</sup> We consider each such dual as a subgroup of  $\Gamma$ .

Lemma 3.1 is also useful in deducing that Rademacher and pseudo-Rademacher sets are  $FZI_0(U)$  and  $I_0(U)$  respectively.

PROPOSITION 3.4.

- (1) *Every translate of a pseudo-Rademacher set is  $I_0$  and  $I_0(U)$  with bounded length.*
- (2) *Every Rademacher set is  $FZI_0$  and  $FZI_0(U)$  with bounded length.*

*Proof.* (1) Suppose  $U$  is an  $\epsilon$ -neighbourhood in  $G$ . Because the property  $I_0(U)$  with length  $N$  is preserved under translation, there is no loss of generality in assuming  $\mathbf{E}$  is pseudo-Rademacher.

Obtain the finite set  $\mathbf{F}$  from Lemma 3.1 such that  $\mathbf{E} \setminus \mathbf{F}$  is pseudo-Rademacher( $U$ ) and assume  $\varphi \in \ell^\infty(\mathbf{E} \setminus \mathbf{F})$  has norm at most one. Let  $\mathbf{E}^+ = \{\gamma \in \mathbf{E} \setminus \mathbf{F} : \Re\varphi(\gamma) \geq 0\}$  and  $\mathbf{E}^- = (\mathbf{E} \setminus \mathbf{F}) \setminus \mathbf{E}^+$ . Obtain  $x \in U$  such that  $\gamma(x) = 1$  on  $\mathbf{E}^+$  and  $\gamma(x) = -1$  on  $\mathbf{E}^-$ . Then for all  $\gamma \in \mathbf{E} \setminus \mathbf{F}$ ,

$$\left| \Re\varphi(\gamma) - \frac{1}{2}\widehat{\delta}_x(\gamma) \right| \leq \frac{1}{2}.$$

Now put  $\varphi_1 = 2(\widehat{\delta}_x/2 - \Re\varphi)$ . This is a real-valued  $\mathbf{E} \setminus \mathbf{F}$ -function, with norm at most one, so we may repeat the argument to obtain  $x_1 \in U$  with

$$\nu_1 = \frac{1}{2}\delta_x + \frac{1}{4}\delta_{x_1} \quad \text{and satisfying} \quad |\Re\varphi(\gamma) - \widehat{\nu}_1(\gamma)| \leq \frac{1}{4}$$

for all  $\gamma \in \mathbf{E} \setminus \mathbf{F}$ . We argue similarly with  $\Im\varphi$  to obtain  $\nu_2$ . Then for all  $\gamma \in \mathbf{E} \setminus \mathbf{F}$ ,

$$|\varphi(\gamma) - (\widehat{\nu}_1 + i\widehat{\nu}_2)(\gamma)| \leq \frac{1}{2},$$

and thus  $\mathbf{E} \setminus \mathbf{F}$  is  $I_0(U, 4)$ .

(2) Since the Rademacher set  $\mathbf{E}$  consists of only characters of order two, to show  $\mathbf{E}$  is  $FZI_0(U)$  with bounded constants we need only interpolate real-valued  $\mathbf{E}$ -functions. The first step in the argument above shows that given  $U$  open, there is a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is  $FZI_0(U, 1)$ .

In either case, when  $U = G$  we can take  $\mathbf{F}$  empty, and hence  $\mathbf{E}$  is  $I_0$  in the first case and  $FZI_0$  in the second. ■

In particular, the set  $\gamma\mathbf{E}$  of Example 1.3 is  $I_0(U)$  with bounded length. Being  $I_0$ , a pseudo-Rademacher set will not cluster at a continuous character [15, 18], but more can be said.

PROPOSITION 3.5. *If  $\mathbf{E}$  is pseudo-Rademacher, then*

- (1)  $\mathbf{E} \cdot \mathbf{E}^{-1}$  *does not cluster at any continuous character other than the identity  $\mathbf{1}$ ;*
- (2) *for each integer  $k \geq 1$  the closure of  $(\mathbf{E} \cup \mathbf{E}^{-1})^k$  in  $\overline{\Gamma}$  supports no non-zero measure in  $M_0(\overline{\Gamma})$ .*



Unlike  $\varepsilon$ -Kronecker sets, the product of a pseudo-Rademacher set with itself can cluster at a character, the set  $\gamma\mathbf{E}$  of Example 1.3 being a case in point.

*Proof.* (1) If  $\mathbf{E} \cdot \mathbf{E}^{-1}$  clusters at the continuous character  $\gamma \neq \mathbf{1}$ , then there are disjoint  $\mathbf{E}_1, \mathbf{E}_2 \subset \mathbf{E}$  such that for any finite set  $\mathbf{F}$ ,  $(\mathbf{E}_1 \setminus \mathbf{F}) \cdot (\mathbf{E}_2 \setminus \mathbf{F})^{-1}$  clusters at  $\gamma$  [3, Thm. 4.3]. Let  $U = \{x : |\gamma(x) - 1| < \sqrt{2}\}$ . By Lemma 3.1, there is a finite set  $\mathbf{F}$  such that  $\mathbf{E} \setminus \mathbf{F}$  is pseudo-Rademacher( $U$ ). Choose  $u \in U$  such that  $\lambda(u) = 1$  for all  $\lambda \in \mathbf{E}_1 \setminus \mathbf{F}$  and  $\chi(u) = -1$  for all  $\chi \in \mathbf{E}_2 \setminus \mathbf{F}$ . Because  $\lambda\chi(u) = -1 \neq \gamma(u)$ , this is a contradiction.

(2) We will use induction on  $k$ , after some preliminaries.

We write  $M_0(X)$  for the elements of  $M_0(\overline{\mathbf{I}})$  supported on  $X \subset \overline{\mathbf{I}}$ . We use the easily proved facts that (a) if  $\mu \in M_0(\overline{\mathbf{I}})$  and  $\nu \ll \mu$ , then  $\nu \in M_0(\overline{\mathbf{I}})$  [9, 4.4.1], (b) if  $X \subset \overline{\mathbf{I}}$  is a Helson set, then  $M_0(X) = \{0\}$  [9, p. 110], and (c) the closure of an  $I_0$  set is Helson [11]; in particular, the closure of a pseudo-Rademacher set is Helson. (These results may also be found in [6].) That gives us the  $k = 1$  case.

Now consider  $k = 2$ . Items (a)–(c) tell us that  $M_0(\overline{(\mathbf{E} \cup \mathbf{E}^{-1}) \cdot \mathbf{F}}) = \{0\}$  for all pseudo-Rademacher sets  $\mathbf{E}$  and finite sets  $\mathbf{F}$ , and that if  $x, y \in G$  with  $\gamma(x) = \gamma(y)$  for all but a finite number of  $\gamma \in \mathbf{E}$ , then

$$(3.1) \quad \widehat{\mu}(x) - \widehat{\mu}(y) = \int (\widehat{\delta}_x - \widehat{\delta}_y) d\mu = 0 \quad \text{for } \mu \in M_0(\overline{(\mathbf{E} \cup \mathbf{E}^{-1})^2}).$$

Suppose  $\mu \in M_0(\overline{(\mathbf{E} \cup \mathbf{E}^{-1})^2})$ . Because of property (a), we may assume that  $\mu$  is a probability measure. Let  $\gamma_1, \gamma_2, \dots$  be distinct elements of  $\mathbf{E}$  and let  $x_1, x_2, \dots \in G$  be such that for  $\gamma \in \mathbf{E}$ ,

$$\gamma(x_j) = \begin{cases} -1 & \text{if } \gamma = \gamma_k \text{ with } k \leq j, \\ 1 & \text{otherwise.} \end{cases}$$

The  $x_j$  exist because  $\mathbf{E}$  is pseudo-Rademacher. They are of course distinct. By (3.1),  $\widehat{\mu}(x_j) = \widehat{\mu}(e) = 1$  for all  $j$ , so  $\mu \notin M_0(\overline{\mathbf{I}})$ , a contradiction.

For general  $k$ , assume that the assertion has been proved for all  $1 \leq k \leq K$ . If  $\mathbf{F} \subset \mathbf{E}$  is finite, then

$$(\mathbf{E} \cup \mathbf{E}^{-1})^{K+1} = [(\mathbf{E} \cup \mathbf{E}^{-1}) \setminus (\mathbf{F} \cup \mathbf{F}^{-1})]^{K+1} \cup X,$$

where  $X$  is a finite union of translates of sets of the form  $\mathbf{E}^m(\mathbf{E}^{-1})^n$  where  $1 \leq m + n \leq K$ . That finite union supports no non-zero measures in  $M_0(\overline{\mathbf{I}})$  by the induction hypothesis and (a). We now have the  $K + 1$  version of (3.1), and so we complete the argument as in the  $k = 2$  case. ■

**4. Preliminaries to the proofs of the theorems.** To prove our theorems, we will need to consider several kinds of discrete abelian groups. The facts we need are standard and can be found in [17].

We write  $\mathfrak{e} = |\mathbf{E}|$ , the cardinality of  $\mathbf{E}$ .

We start with several straightforward results, many of whose proofs we omit.

LEMMA 4.1. *Let  $\mathbf{E} \subset \mathbf{\Gamma}$ ,  $\varepsilon > 0$ ,  $\gamma \in \mathbf{\Gamma}$  and  $\mathbf{\Lambda} \subset \mathbf{\Gamma}$  a subgroup.*

- (1) *Let  $q : \mathbf{\Gamma} \rightarrow \mathbf{\Gamma}/\mathbf{\Lambda}$  be the natural homomorphism. If  $q$  is one-to-one on  $\mathbf{E}$  and  $q(\mathbf{E})$  is (weak)  $\varepsilon$ -Kronecker (resp., pseudo-Rademacher), then  $\mathbf{E}$  is (weak)  $\varepsilon$ -Kronecker (resp., pseudo-Rademacher).*
- (2) *Suppose  $\mathbf{E} \subset \mathbf{\Lambda}$ . Then  $\mathbf{E}$  is (weak)  $\varepsilon$ -Kronecker (resp., pseudo-Rademacher) as a subset of  $\mathbf{\Gamma}$  if and only if it is (weak)  $\varepsilon$ -Kronecker (resp., pseudo-Rademacher) as a subset of  $\mathbf{\Lambda}$ .*
- (3) *Let  $N \geq 2$ . Then  $\mathbf{E}$  is  $N$ -large (resp., tor-large) if and only if  $\gamma\mathbf{E}$  is  $N$ -large (resp., tor-large).*

LEMMA 4.2. *Suppose  $\mathbf{E}$  is infinite, tor-large and generates  $\mathbf{\Gamma}$ . Then there exists  $\mathbf{F} \subset \mathbf{E}$  such that the image  $q_{\mathbf{F}}(\mathbf{E})$  has the same cardinality as  $\mathbf{E}$  and  $\mathbf{\Gamma}/\mathbf{\Gamma}_{\mathbf{F}}$  is a torsion group.*

*Proof.* Let  $\mathbf{F}'$  be a maximal independent subset of  $\mathbf{\Gamma}/\mathbf{\Gamma}_0$ . Because  $q_0(\mathbf{E})$  generates  $\mathbf{\Gamma}/\mathbf{\Gamma}_0$  and  $\mathbf{E}$  is tor-large and generates  $\mathbf{\Gamma}$ ,

$$|\mathbf{F}'| \leq |\mathbf{\Gamma}/\mathbf{\Gamma}_0| = |q_0(\mathbf{E})| < |\mathbf{E}| = |\mathbf{\Gamma}|.$$

Use the axiom of choice to find  $\mathbf{F} \subset \mathbf{\Gamma}$  such that  $q_0 : \mathbf{F} \rightarrow \mathbf{F}'$  is one-to-one. Cardinal arithmetic says

$$|\mathbf{\Gamma}| = |\mathbf{\Gamma}/\langle \mathbf{F} \rangle| |\langle \mathbf{F} \rangle| = |q_{\mathbf{F}}(\mathbf{E})| |\mathbf{F}|,$$

so  $|q_{\mathbf{F}}(\mathbf{E})| = |\mathbf{E}|$ . ■

Let  $p$  be a prime and denote by  $\mathcal{C}(p^\infty)$  the discrete  $p$ -subgroup of  $\mathbb{T}$ , i.e., the group of all  $p^n$ -roots of unity. An important fact [17] is that every abelian group is isomorphic to a subgroup of

$$(4.1) \quad \bigoplus_{\alpha} \mathbb{Q}_{\alpha} \oplus \bigoplus_{\beta} \mathcal{C}(p_{\beta}^{\infty}),$$

where the  $\mathbb{Q}_{\alpha}$  are copies of the rationals and the  $p_{\beta}$  are primes.

We begin with  $\mathbb{Q}$  and  $\mathcal{C}(p_{\beta}^{\infty})$ .

PROPOSITION 4.3 ([4, Prop. 3.4]). *Let  $\varepsilon > 0$ . Each infinite subset of  $\mathcal{C}(p^\infty)$  (resp.  $\mathbb{Q}$ ) contains an infinite  $\varepsilon$ -Kronecker set.*

PROPOSITION 4.4 ([4, proof of Prop. 3.5]). *Suppose that  $\mathbf{E} \subset \mathbf{\Gamma}$  is independent and consists of elements of orders all greater than  $N \geq 2$ . Then  $\mathbf{E}$  is (at least) weak  $|1 - e^{i\pi/(N+1)}|$ -Kronecker.*

NOTATION 4.5. Let  $\mathbf{\Gamma} \subset \bigoplus_{\beta \in B} \mathcal{C}(p_{\beta}^{\infty})$  for some index set  $B$ . For each  $\beta \in B$  and  $\gamma \in \mathbf{\Gamma}$  we let  $\pi_{\beta}(\gamma)$  be the projection of  $\gamma$  on the  $\beta$ -coordinate

and let

$$B(\gamma) = \{\beta \in B : \pi_\beta(\gamma) \neq 1\}.$$

Note that each  $B(\gamma)$  is finite.

LEMMA 4.6. *Let  $\mathbf{\Gamma} \subset \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty)$ , where  $B$  is uncountable. Suppose  $N \geq 2$  and  $\mathbf{E} \subset \mathbf{\Gamma}$  is such that for every  $\beta$  there exists  $\gamma \in \mathbf{E}$  such that  $\pi_\beta(\gamma)$  has order at least  $N$ . Then there exists  $\mathbf{F} \subset \mathbf{E}$  such that  $|\mathbf{F}| = |B| = |\mathbf{E}|$  and:*

- (1) *If  $N > 2$ , then  $\mathbf{F}$  is weak  $|1 - e^{\pi i/N}|$ -Kronecker.*
- (2) *If  $N = 2$  and all the  $\pi_\beta(\gamma)$  have order exactly 2, then  $\mathbf{F}$  is pseudo-Rademacher.*

Cardinal arithmetic shows that  $|B| = |\mathbf{E}|$ , since otherwise  $|B| = |\mathbf{E}| \aleph_0 = |\mathbf{E}| < |B|$ .

*Proof of Lemma 4.6.* The finding of  $\mathbf{F}$  is the same in both (1)–(2). We will use transfinite induction. Let  $\mathcal{I}$  be a well-ordered index set of cardinality  $B$  with  $1, 2, \dots$  the first elements of  $\mathcal{I}$ .

Let  $\lambda_1 \in \mathbf{E}$  and  $\beta(1) \in B$  be such that the order of  $\pi_{\beta(1)}(\lambda_1)$  is at least  $N$ . That starts our induction.

Suppose  $i > 1$  and that we have found  $\lambda_{i'} \in \mathbf{E}$  for all  $1 \leq i' < i$  such that  $B(\lambda_{i'}) \not\subset \bigcup_{k < i'} B(\lambda_k)$ . If  $|\{\lambda_{i'} : 1 \leq i' < i\}| = \mathfrak{e}$ , we stop. Otherwise, we note that  $A = \bigcup_{i' < i} B(\lambda_{i'})$  also has cardinality less than  $\mathfrak{e}$  and there exist  $\lambda(i) \in \mathbf{E}$  and  $\beta(i) \in B$  such that  $\beta(i) \notin \bigcup_{i' < i} B(\lambda_{\beta(i')})$  and  $\pi_{\beta(i)}(\lambda_i)$  has order at least  $N$ . That completes the inductive step.

Because  $|B| = \mathfrak{e}$ , the set  $\mathbf{F} = \{\lambda_{\beta(i)} : 1 \leq i\}$  must have the same cardinality.

That completes the finding of  $\mathbf{F}$ . We now turn to the specific assertions about  $\mathbf{F}$ .

(1) We claim that  $\mathbf{F}$  is weak  $\varepsilon$ -Kronecker if  $N > 2$ , where  $\varepsilon = |1 - e^{\pi i/N}|$ . We use transfinite induction again. It will be convenient to assume that  $G = \prod_{\beta \in B} G_\beta$ , where  $G_\beta$  is the dual of  $\mathcal{C}(p_\beta^\infty)$ ,  $\beta \in B$ . That is justified by Lemma 4.1. We shall abuse notation by using  $\pi_\beta(x)$  to denote the  $\beta$ -coordinate of  $x \in G$ .

Let  $\varphi : \mathbf{F} \rightarrow \mathbb{T}$ . Choose  $x_1 \in G_{\beta(1)}$  such that  $|\varphi(\lambda_1) - \lambda_1(x_1)| \leq \varepsilon$ . That can be done because  $\pi_{\beta(1)}(\lambda_1)$  has order at least  $N$ .

Suppose that  $i \geq 1$  and  $x_{i'} \in \prod_{k \leq i'} G_{\beta(k)}$  have been chosen for  $1 \leq i' < i$  so that for  $1 \leq k \leq i' < i$ ,  $|\varphi(\lambda_k) - \lambda_k(x_{i'})| \leq \varepsilon$  and

$$(4.2) \quad \pi_{\beta(k)}(x_k) = \pi_{\beta(k)}(x_{i'}).$$

If  $i$  has an immediate predecessor,  $i'$ , we choose  $x \in G_{\beta(i)}$  such that  $|\varphi(\lambda_i) - \lambda_i(x_{i'}x)| \leq \varepsilon$ . Set  $x_i = x_{i'}x$ . Then (4.2) holds for  $1 \leq k \leq i' \leq i$ .

If  $i$  is a limit ordinal, let  $x_0$  be the limit point of the  $x_{i'}$  as  $i' \rightarrow i$ . Such a limit point exists because of (4.2).

That all ensures (4.2) holds with  $x_0$  in place of  $x_{i'}$  for all  $i' < i$ . Now choose  $x \in G_{\beta(i)}$  such that  $|\varphi(\lambda_i) - \lambda_i(x_0x)| \leq \varepsilon$  and set  $x_i = x_0x$ . It is clear that (4.2) now holds with  $i' \leq i$ . We now let  $z = \lim_i x_i$  and observe that  $|\varphi(\lambda) - \lambda(z)| < \varepsilon$  for all  $\lambda \in \mathbf{F}$ .

(2) Let  $\varphi : \mathbf{F} \rightarrow \{-1, 1\}$ . Choose  $x_1 \in G_{\beta(1)}$  such that  $\varphi(\lambda_1) = \lambda_1(x_1)$ . That can be done because  $\pi_{\beta(1)}(\lambda_1)$  has order 2. We continue as in the proof of (1). ■

LEMMA 4.7. *Suppose that  $\mathbf{E}$  generates  $\mathbf{\Gamma}$  and is uncountable and that  $\mathbf{\Gamma}$  is torsion-free. Then  $\mathbf{E}$  contains an independent subset  $\mathbf{F}$  with  $|\mathbf{F}| = |\mathbf{E}|$ .*

*Proof.* Let  $\mathcal{F}$  be the set of independent subsets of  $\mathbf{E}$ , ordered by inclusion. Zorn's lemma gives us a maximal element,  $\mathbf{F}$ , of  $\mathcal{F}$ . If  $|\mathbf{E}| > |\mathbf{F}|$ , then by cardinal arithmetic,

$$(4.3) \quad |q_{\mathbf{F}}(\mathbf{E})| = |\mathbf{\Gamma}/\langle \mathbf{F} \rangle| = \mathfrak{c}.$$

Suppose there exists  $\gamma \in \mathbf{E}$  such that  $q_{\mathbf{F}}(\gamma)$  has infinite order. Then  $\mathbf{F} \cup \{\gamma\}$  is independent and  $\mathbf{F}$  is not maximal, a contradiction. Therefore,  $q_{\mathbf{F}}(\gamma)$  has finite order for all  $\gamma \in \mathbf{E}$ . Since  $\mathfrak{c} > \aleph_0$ , there is an integer  $n \geq 2$  and  $\mathfrak{c}$  elements of  $q_{\mathbf{F}}(\mathbf{E})$  which are of order  $n$ . Thus,  $\gamma^n$  belongs to the group,  $\langle \mathbf{F} \rangle$ , generated by  $\mathbf{F}$  for at least  $\mathfrak{c}$  elements of  $\mathbf{E}$ . Because  $|\mathbf{F}| < \mathfrak{c}$ , there is some  $\rho \in \langle \mathbf{F} \rangle$  such that  $\mathbf{E}' = \{\gamma \in \mathbf{E} : \gamma^n = \rho\}$  has cardinality  $\mathfrak{c}$ . If  $\gamma_1 \neq \gamma_2 \in \mathbf{E}'$ , then  $(\gamma_1\gamma_2^{-1})^n = \mathbf{1}$ , contradicting the assumption that  $\mathbf{\Gamma}$  is torsion-free. ■

The hypotheses of no torsion and  $|\mathbf{E}| > \aleph_0$  cannot be relaxed, as Example 1.3 and  $\mathbb{Z}$ , respectively, show.

**5. Proof of Theorems 2.2 and 2.4.** When convenient, we will assume that  $\mathbf{E}$  generates  $\mathbf{\Gamma}$ . We will often assume  $\mathbf{\Gamma} \subseteq \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty)$  and in this case we will use Notation 4.5.

**5.1. Proof of Theorem 2.2(1).** We adapt [1, proof of Lem. 3.4]. Let  $\varepsilon > 0$  be given, and let  $N > 1$  with  $|1 - e^{i\pi/N}| = \varepsilon' < \varepsilon$ . For each finite  $\mathbf{F} \subset \mathbf{E}$  and  $\varphi : \mathbf{E} \rightarrow \mathbb{T}$  let

$$X_\varphi(\mathbf{F}) = \{x \in G : |\varphi(\gamma) - \gamma(x)| \leq \varepsilon' \text{ for all } \gamma \in \mathbf{F}\}.$$

Let  $\mathcal{A} = \{\mathbf{F} \subset \mathbf{E} : |\mathbf{F}| < \infty \text{ and } X_\varphi(\mathbf{F}) \neq \emptyset \text{ for all } \varphi : \mathbf{E} \rightarrow \mathbb{T}\}$ . It is evident that each  $\mathbf{F} \in \mathcal{A}$  is a weak  $\varepsilon'$ -Kronecker set and hence an  $\varepsilon$ -Kronecker set. Because  $\mathbf{E}$  is  $N!$ -small, there exists  $\gamma \in \mathbf{E}$  with  $\{\gamma\} \in \mathcal{A}$ . Order  $\mathcal{A}$  by inclusion <sup>(2)</sup>. An application of Zorn's lemma shows that  $\mathcal{A}$  has a maximal element,  $\mathbf{S}$ . If  $|\mathbf{S}| = |\mathbf{E}|$ , we are done.

<sup>(2)</sup> Because we depart from [1] by specifying that  $\varphi$  is defined on  $\mathbf{E}$  rather than just  $\mathbf{F}$ , inclusion is sufficient.

Assume that  $|\mathbf{S}| < |\mathbf{E}|$ . This will lead to a contradiction. First note that because  $|\mathbf{E}|$  is infinite and that because  $\mathbf{E}$  is  $\ell$ -small for all  $\ell \geq 2$ ,

$$(5.1) \quad |\mathbf{E}| = |\langle \mathbf{E} \rangle / \langle \mathbf{S} \rangle| = |q_\ell(\mathbf{E})| > |\mathbf{S}| \geq |q_\ell(\mathbf{S})|, \quad \text{and therefore}$$

$$(5.2) \quad |\mathbf{E}| = |\langle q_\ell(\mathbf{E}) \rangle / \langle q_\ell(\mathbf{S}) \rangle| > |\mathbf{S}|.$$

Use the axiom of choice to find a subset  $\mathbf{E}' \subseteq \mathbf{E}$  such that  $q_{N!}$  is one-to-one on  $\mathbf{E}'$  and  $q_{N!}(\mathbf{E}') = q_{N!}(\mathbf{E})$ . That is possible because  $\mathbf{E}$  is  $N!$ -small. We note that the mapping  $M_{N!} : \lambda \mapsto \lambda^{N!}$  identifies the quotient  $q_{N!}(\mathbf{\Gamma})$  with  $M_{N!}(\mathbf{\Gamma})$ . Therefore,  $M_{N!}$  is one-to-one on  $\mathbf{E}'$ .

We claim that (5.1)–(5.2) tell us that there exists  $\rho \in \mathbf{E} \setminus \mathbf{S}$  such that either

$$(5.3) \quad \begin{aligned} & \text{(a) } \langle \{\rho\} \rangle \cap \langle \mathbf{S} \rangle = \{\mathbf{1}\}, \text{ or} \\ & \text{(b) there exists } k \geq N \text{ with } \langle \{\rho\} \rangle \cap \langle \mathbf{S} \rangle = \langle \{\rho^k\} \rangle. \end{aligned}$$

To see the claim, we note that the cardinalities of  $M_{N!}(\mathbf{E}')$  and  $\langle \mathbf{S} \rangle$  tell us that  $\langle M_{N!}(\mathbf{E}') \rangle / (\langle M_{N!}(\mathbf{E}') \rangle \cap \langle \mathbf{S} \rangle)$  is a non-trivial group (we do not need to know more than that!). So there exists  $\gamma \in \mathbf{E}' \setminus \mathbf{S}$  such that  $\gamma^l \notin \langle \mathbf{S} \rangle$  for all  $1 \leq l \leq N$ .

Let  $\varphi : \mathbf{S} \cup \{\rho\} \rightarrow \mathbb{T}$  be given. Since  $\mathbf{S} \in \mathcal{A}$ , there exists  $x_1 \in G$  with  $|\varphi(\gamma) - \gamma(x_1)| \leq \varepsilon'$  for all  $\gamma \in \mathbf{S}$ . Use (5.3) to find  $x_2 \in \langle \mathbf{S} \rangle^\perp$  such that either (case (a))  $\rho(x_2) = \varphi(\rho)\rho(x_1)$ , or (case (b))  $|\rho(x_2) - \varphi(\rho)\rho(x_1)| \leq \varepsilon'$ . Then  $|\varphi(\gamma) - \gamma(x_1x_2)| \leq \varepsilon'$  for all  $\gamma \in \mathbf{S}$ . That shows  $\mathbf{S} \cup \{\rho\} \in \mathcal{A}$ , contradicting the maximality of  $\mathbf{S}$ . Therefore  $|\mathbf{S}| = |\mathbf{E}|$ . ■

**5.2. Proof of Theorem 2.2(2).** Let  $N$  be the minimal integer such that  $\mathbf{E}$  is  $N$ -large and  $K = p^k$  for a prime  $p$  with  $k$  the maximal exponent for which  $p^k | N$ . (Increasing the power of  $p$  will give a stronger version of (2).) Since  $\mathbf{E}$  is  $N$ -large, it is tor-large, and so, by Lemmas 4.1 and 4.2, we may assume that  $\mathbf{\Gamma}$  is a torsion group. We also assume that  $\mathbf{\Gamma} = \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty)$  and no longer suppose that  $\mathbf{E}$  generates  $\mathbf{\Gamma}$ . However, we can still assume that  $\pi_\beta(\mathbf{E}) \neq 1$  for all  $\beta \in B$ .

CASE I:  $\mathbf{E}$  is uncountable. Consider

$$B' = \{\beta \in B : p_\beta = p \text{ and } \exists \gamma \in \mathbf{E} \text{ with order } \pi_\beta(\gamma) \geq K\}.$$

Since  $\mathbf{E}$  is  $N$ -large, if  $N$  is prime, then  $B'$  has cardinality  $\mathfrak{e}$ . If  $N = p^k$  for some  $k \geq 2$ , or  $N$  is divisible by two or more primes, then the minimality of  $N$  ensures  $B'$  has cardinality  $\mathfrak{e}$ . By mapping  $\mathbf{\Gamma} \rightarrow \bigoplus_{\beta \in B'} \mathcal{C}(p_\beta^\infty)$ , we may apply Lemma 4.6 and conclude that  $\mathbf{E}$  has a weak  $\varepsilon$ -Kronecker set of cardinality  $\mathfrak{e}$ .

CASE II:  $\mathbf{E}$  is countable. Then we may assume that  $\mathbf{\Gamma} = \bigoplus_{j \in B} \mathcal{C}(p_j^\infty)$ , where  $B$  is countable or finite. But if  $B$  is finite, then the direct sum contains

only a finite number of elements of order  $\leq N$ , so  $\mathbf{E}$  cannot be  $N$ -large. Similarly, there must be an infinite number of  $j$  with  $p_j = p$ .

The maximality of  $k$  implies that

$$\{\beta \in B : p_\beta = p \text{ and } \exists \gamma \in \mathbf{E} \text{ with order } \pi_\beta(\gamma) = p^k\}$$

is countably infinite. We now appeal to Lemma 4.6. ■

**5.3. Proof of Theorem 2.4.** As in the proof of Theorem 2.2, we may assume that  $\mathbf{\Gamma} = \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty)$  for an appropriate index set  $B$ .

The set  $\mathbf{E}$  would not be 2-large if

$$\{\beta \in B : \exists \gamma \in \mathbf{E} \text{ with order } \pi_\beta(\gamma) > 2\}$$

had cardinality  $\mathfrak{e}$ . But then

$$\{\beta \in B : p_\beta = 2 \text{ and } \exists \gamma \in \mathbf{E} \text{ with order } \pi_\beta(\gamma) = 2\}$$

must have cardinality  $\mathfrak{e}$ , or  $|\mathbf{E}| < \mathfrak{e}$ .

CASE I:  $\mathbf{E}$  *uncountable*. We call on Lemma 4.6(2).

CASE II:  $\mathbf{E}$  *countable*. When  $\mathbf{E}$  is countable and 2-large,  $q_2(\mathbf{E})$  is finite, so there must exist  $\rho \in \mathbf{\Gamma}/\mathbf{\Gamma}_2$  such that  $\mathbf{E}' = q_2^{-1}(\rho) \cap \mathbf{E}$  is infinite. Let  $\rho' \in \mathbf{E}'$ . Then  $(\mathbf{E}' \setminus \{\rho'\})\rho'^{-1} \subset \mathbf{\Gamma}_2$ , that is, every element of the infinite set  $(\mathbf{E}' \setminus \{\rho'\})\rho'^{-1}$  has order 2. Induction gives us an infinite independent subset  $\mathbf{F}' \subset (\mathbf{E}' \setminus \{\rho'\})\rho'^{-1}$ . Then  $\mathbf{F} = \rho'\mathbf{F}'$  is a translate of the Rademacher set  $\mathbf{F}'$  and hence is pseudo-Rademacher. ■

**Acknowledgments.** The second author is partially supported by NSERC.

The first author thanks the Department of Pure Mathematics, University of Waterloo, for its generous hospitality when this paper was being prepared.

We thank the referee for his/her careful reading of our paper and for the suggestion to use the argument of [1] to prove Theorem 2.2(1), greatly shortening and clarifying the argument.

#### REFERENCES

- [1] J. Galindo and S. Hernández, *The concept of boundedness and the Bohr compactification of a MAP abelian group*, Fund. Math. 159 (1999), 195–218.
- [2] J. Galindo and S. Hernández, *Interpolation sets and the Bohr topology of locally compact groups*, Adv. Math. 188 (2004), 51–68.
- [3] C. C. Graham and K. E. Hare,  *$\varepsilon$ -Kronecker and  $I_0$  sets in abelian groups, III: Interpolation by measures on small sets*, Studia Math. 171 (2005), 15–32.
- [4] C. C. Graham and K. E. Hare,  *$\varepsilon$ -Kronecker and  $I_0$  sets in abelian groups, IV: Interpolation by non-negative measures*, Studia Math. 177 (2006), 9–24.
- [5] C. C. Graham and K. E. Hare,  *$\varepsilon$ -Kronecker and  $I_0$  sets in abelian groups, I: Arithmetic properties of  $\varepsilon$ -Kronecker sets*, Math. Proc. Cambridge Philos. Soc. 140 (2006), 475–489.

- [6] C. C. Graham and K. E. Hare, *Interpolation and Sidon Sets for Compact Abelian Groups*, CMS Books Math./Ouvrages Math. SMC, Springer, New York, to appear.
- [7] C. C. Graham, K. E. Hare, and L. T. Ramsey, *Union problems for  $I_0$  sets*, Acta Sci. Math. (Szeged) 75 (2009), 175–195.
- [8] C. C. Graham and A. T.-M. Lau, *Relative weak compactness of orbits in Banach spaces associated with locally compact groups*, Trans. Amer. Math. Soc. 359 (2007), 1129–1160.
- [9] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Grundlehren Math. Wiss. 238, Springer, New York, 1979.
- [10] S. Hartman and C. Ryll-Nardzewski, *Almost periodic extensions of functions*, Colloq. Math. 12 (1964), 23–39.
- [11] J.-P. Kahane, *Ensembles de Ryll-Nardzewski et ensembles de Helson*, Colloq. Math. 15 (1966), 87–92.
- [12] N. J. Kalton, *On vector-valued inequalities for Sidon sets and sets of interpolation*, Colloq. Math. 64 (1993), 233–244.
- [13] K. Kunen and W. Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc. 126 (1999), 117–137.
- [14] J.-F. Méla, *Approximation diophantienne et ensembles lacunaires*, Bull. Soc. Math. France Mém. 19 (1969), 26–54.
- [15] L. T. Ramsey, *A theorem of C. Ryll-Nardzewski and metrizable l.c.a. groups*, Proc. Amer. Math. Soc. 78 (1980), 221–224.
- [16] L. T. Ramsey, *Comparisons of Sidon and  $I_0$  sets*, Colloq. Math. 70 (1996), 103–132.
- [17] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed., Grad. Texts in Math. 148, Springer, New York, 1995.
- [18] C. Ryll-Nardzewski, *Concerning almost periodic extensions of functions*, Colloq. Math. 12 (1964), 235–237.

Colin C. Graham  
Department of Mathematics  
University of British Columbia  
Vancouver, BC, Canada  
Mailing address:  
P.O. Box 2031  
Haines Junction, YT, Canada, Y0B 1L0  
E-mail: c.c.graham@math.ubc.ca

Kathryn E. Hare  
Department of Pure Mathematics  
University of Waterloo  
Waterloo, ON, Canada N2L 3G1  
E-mail: kehare@uwaterloo.ca

Received 15 October 2011

(5558)

