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Abstract. Let G be an infinite locally compact abelian group and X be a Banach space. We show that if every bounded Fourier multiplier T on $L^2(G)$ has the property that $T \otimes \mathrm{Id}_X$ is bounded on $L^2(G,X)$ then X is isomorphic to a Hilbert space. Moreover, we prove that if $1 < p < \infty$, $p \neq 2$, then there exists a bounded Fourier multiplier on $L^p(G)$ which is not completely bounded. Finally, we examine unconditionality from the point of view of Schur multipliers. More precisely, we give several necessary and sufficient conditions for an operator space to be completely isomorphic to an operator Hilbert space.

1. Introduction. In [DJ, Theorem 1], M. Defant and M. Junge proved the following (see also [AB, Theorem 1.5] and [PW, Theorem 8.4.11]).

THEOREM 1.1. *Let X be a Banach space. Suppose that there exists a positive constant C such that for any $n \in \mathbb{N}$, any complex numbers $t_{-n}, \dots, t_n$ and any $x_{-n}, \dots, x_n \in X$ we have*

$$\Big\| \sum_{k=-n}^{n} t_k e^{2\pi i k\,\cdot} \otimes x_k \Big\|_{L^2(\mathbb{T},X)} \leq C \sup_{-n \leq k \leq n} |t_k| \Big\| \sum_{k=-n}^{n} e^{2\pi i k\,\cdot} \otimes x_k \Big\|_{L^2(\mathbb{T},X)}. \tag{1.1}$$

Then the Banach space X is isomorphic to a Hilbert space.

This result says that if every bounded Fourier multiplier T on $L^2(\mathbb{T})$ has the property that $T \otimes \mathrm{Id}_X$ is bounded on $L^2(\mathbb{T},X)$ then the Banach space X is isomorphic to a Hilbert space. The paper [DJ] contains a generalization to infinite compact abelian groups. Our first main result is an extension of this theorem to infinite arbitrary locally compact abelian groups.

THEOREM 1.2. *Let G be an infinite locally compact abelian group and X be a Banach space. If every bounded Fourier multiplier T on $L^2(G)$ has the property that $T \otimes \mathrm{Id}_X$ is bounded on $L^2(G,X)$ then the Banach space X is isomorphic to a Hilbert space.*

Our proof is independent of [DJ].

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Suppose $1 \leq p \leq \infty$. We denote by $S^p = S^p(\ell^2)$ the Schatten space. Let Ω be a measure space. Recall that a linear map $T\colon L^p(\Omega) \to L^p(\Omega)$ is *completely bounded* if $T \otimes \mathrm{Id}_{S^p}$ extends to a bounded operator $T \otimes \mathrm{Id}_{S^p} : L^p(\Omega, S^p) \to L^p(\Omega, S^p)$ (see [P2]). In this case, the completely bounded norm $\|T\|_{\mathrm{cb}, L^p(\Omega) \to L^p(\Omega)}$ is defined by

$$\|T\|_{\mathrm{cb}, L^p(\Omega) \to L^p(\Omega)} = \|T \otimes \mathrm{Id}_{S^p}\|_{L^p(\Omega, S^p) \to L^p(\Omega, S^p)}. \tag{1.2}$$

Let G be a locally compact abelian group. If $p = 1, 2$ or ∞, it is easy to see that every bounded Fourier multiplier is completely bounded on $L^p(G)$. If $1 < p < \infty$, $p \neq 2$, the situation is different. Indeed, G. Pisier showed the following theorem (see [P2, Proposition 8.1.3], [P3, p. 181] and also [H, Proposition 3.1]).

Theorem 1.3. *Suppose $1 < p < \infty$, $p \neq 2$. Let G be an infinite compact abelian group. There exists a bounded Fourier multiplier on $L^p(G)$ which is not completely bounded.*

The author [A, Theorems 3.4 and 3.5] has given variants of this result by proving the next theorem:

Theorem 1.4. *Suppose $1 < p < \infty$, $p \neq 2$. If $G = \mathbb{R}$ or $G = \mathbb{Z}$, there exists a bounded Fourier multiplier on $L^p(G)$ which is not completely bounded.*

In this paper, we give an extension of both these theorems to arbitrary infinite locally compact abelian groups. Our second principal result is the following.

Theorem 1.5. *Suppose $1 < p < \infty$, $p \neq 2$. Let G be an infinite locally compact abelian group. There exists a bounded Fourier multiplier on $L^p(G)$ which is not completely bounded.*

The proof of this theorem and the one of Theorem 1.3 use a form of conditionality (i.e. non-unconditionality).

If $1 < p < \infty$ and E is an operator space, let $S^p(E)$ denote the vector-valued noncommutative L^p-space defined in [P2]. The readers are referred to [P2] and [P3] for details on operator spaces and completely bounded maps. For any index set I, we denote by $\mathrm{OH}(I)$ the associated operator Hilbert space introduced by G. Pisier; see [P3] and [P4] for more information. For any integers $i, j \geq 1$, let e_{ij} be the element of S^p corresponding to the matrix with coefficients equal to one at the (i, j) entry and zero elsewhere. In the last section, we show some results linked with unconditionality in the spirit of Theorem 1.1. In particular, the following result is proved.

Theorem 1.6. *Let E be an operator space. The following assertions are equivalent:*

- *There exists a positive constant C such that*
$$\Big\| \sum_{i,j=1}^{n} t_{ij} e_{ij} \otimes x_{ij} \Big\|_{S^2(E)} \leq C \sup_{1\leq i,j\leq n} |t_{ij}| \Big\| \sum_{i,j=1}^{n} e_{ij} \otimes x_{ij} \Big\|_{S^2(E)}$$
for any $n \in \mathbb{N}$, $t_{ij} \in \mathbb{C}$ and $x_{ij} \in E$.
- *The operator space E is completely isomorphic to an operator Hilbert space* $\mathrm{OH}(I)$ *for some index set I.*

The paper is organized as follows. Section 2 gives preliminaries on probability theory, Fourier multipliers and groups. Section 3 contains the proof of Theorem 1.2. In Section 4, we give a proof of Theorem 1.5. Section 5 is devoted to unconditionality from the point of view of Schur multipliers. We present a proof of Theorem 1.6.

We will use $\lesssim$ to indicate an inequality up to a constant which does not depend on the particular elements to which it applies. Moreover $A(x) \approx B(x)$ will mean that both $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

2. Preliminaries. Let us recall some basic notation. If A is a subset of a set E, we let 1_A be the characteristic function of A. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\Omega_0 = \{-1, 1\}^\infty$ be the Cantor group equipped with their normalized Haar measure. For any integer $i \geq 1$, we define ε_i by $\varepsilon_i(\omega) = \omega_i$ if $\omega = (\omega_k)_{k\geq 1} \in \Omega_0$. We can see the ε_i's as independent Rademacher variables on the probability space Ω_0. Let X be a Banach space. Suppose $1 < p < \infty$. We let $\mathrm{Rad}_p(X) \subset L^p(\Omega_0, X)$ be the closure of $\mathrm{Span}\{\varepsilon_i \otimes x \mid i \geq 1,\, x \in X\}$ in the Bochner space $L^p(\Omega_0, X)$. Thus, for any finite family $x_1, \ldots, x_n$ in X, we have
$$\Big\|\sum_{i=1}^{n} \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}_p(X)} = \Big(\int_{\Omega_0} \Big\|\sum_{i=1}^{n} \varepsilon_i(\omega) x_i\Big\|_X^p \, d\omega\Big)^{1/p}.$$
We let $\mathrm{Rad}(X) = \mathrm{Rad}_2(X)$. By Kahane's inequalities (see e.g. [DJT, Theorem 11.1]), the Banach spaces $\mathrm{Rad}(X)$ and $\mathrm{Rad}_p(X)$ are canonically isomorphic.

We say that a set $F \subset B(X)$ is *R-bounded* provided that there exists a constant $C \geq 0$ such that for any finite families $T_1, \ldots, T_n$ in F and $x_1, \ldots, x_n$ in X, we have
$$\Big\|\sum_{i=1}^{n} \varepsilon_i \otimes T_i(x_i)\Big\|_{\mathrm{Rad}(X)} \leq C \Big\|\sum_{i=1}^{n} \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}(X)}.$$
The notion of R-boundedness was introduced in [BG] and then developed in the fundamental paper [CPSW]. We refer to the latter paper and to [KW, Section 2] for a detailed presentation.

Recall that a Banach space X has *property* (α) if there exists a positive constant C such that for any integer n, any $t_{ij} \in \mathbb{C}$ and any $x_{ij} \in X$ we have

$$\Big\|\sum_{i,j=1}^{n} t_{ij}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \leq C \sup_{1\leq i,j\leq n}|t_{ij}|\Big\|\sum_{i,j=1}^{n}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.$$

If $1 < p < \infty$, $p \neq 2$, it is well-known that the space S^p does not have property (α). If the Banach space X has property (α) and if Ω is a σ-finite measure space then, for any $1 < p < \infty$, the space $L^p(\Omega, X)$ also has property (α). See [P1], [CPSW, p. 148] and [KW, p. 127] for more information on this property.

Let Y be a Banach space and let $u\colon Y \to B(X)$ be a bounded map. We say that u is *R-bounded* if the set $\{u(y) \mid \|y\|_Y \leq 1\}$ is R-bounded. We recall a fact which is highly relevant for our paper. This result is [PR, Corollary 2.19] (see also [KLM, Corollary 4.5]).

THEOREM 2.1. *Let K be a compact topological space and X be a Banach space with property (α). Any bounded homomorphism $u\colon C(K) \to B(X)$ is R-bounded.*

Now, we record the following elementary lemma for later use. The easy proof is left to the reader.

LEMMA 2.2. *Suppose $1 < p < \infty$. Let E be an operator space. For any $n \in \mathbb{N}$ and $x_{ij} \in E$, we have*

$$\Big\|\sum_{i,j=1}^{n} e_{ij}\otimes x_{ij}\Big\|_{S^p(E)} = \Big\|\sum_{i,j=1}^{n}\varepsilon_i\otimes\varepsilon_j\otimes e_{ij}\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(S^p(E)))}.$$

Let G be a locally compact abelian group with dual group $\widehat{G}$. If H is a subgroup of G, we denote by $H^{\perp}$ the annihilator of H. The group $(H^{\perp})^{\perp}$ is equal to the closure $\overline{H}$ of H in G. If H is a closed subgroup of G and if $\pi : G \to G/H$ denotes the canonical map, the mapping $\chi \mapsto \chi \circ \pi$ is an isomorphism of $\widehat{G/H}$ onto $H^{\perp}$. Note that if G is a locally compact abelian group and if H is a closed subgroup of G, we have an isomorphism $\widehat{G}/H^{\perp} = \widehat{H}$ given by $\overline{\chi} \mapsto \chi|H$ (see [HR, Theorem 24.11]). See [F] and [HR] for background on abstract harmonic analysis.

Let G be a compact abelian group. A sequence $(\gamma_i)_{i\geq 1}$ of $\widehat{G}$ is a *Sidon set* if there exists a positive constant C such that

$$\sum_{i=1}^{n}|\alpha_i| \leq C\Big\|\sum_{i=1}^{n}\alpha_i\gamma_i\Big\|_{L^\infty(G)}, \qquad n \in \mathbb{N},\ \alpha_1,\ldots,\alpha_n \in \mathbb{C}.$$

A typical example for $G = \mathbb{T}$ is an Hadamard set, e.g. $\{2^i : i \geq 1\}$. See [HR] and [LR] for more information on Sidon sets. Recall the following theorem [P5, Theorem 2.1].

THEOREM 2.3. *Let G be a compact abelian group and $(\gamma_i)_{i\geq 1}$ a Sidon set in $\widehat{G}$. Let X be a Banach space. Suppose $1<p<\infty$. Then*

$$\Big\|\sum_{i=1}^{n}\varepsilon_i\otimes x_i\Big\|_{\mathrm{Rad}(X)}\approx\Big\|\sum_{i=1}^{n}\gamma_i\otimes x_i\Big\|_{L^p(G,X)},\qquad n\in\mathbb{N},\,x_1,\ldots,x_n\in X.$$

Let $(\gamma_i)_{i\geq 1}$ be a Sidon set in $\widehat{G}$ where G is a compact abelian group. Let P be the orthogonal projection from $L^2(G)$ onto the closed span of $\{\gamma_i \mid i\geq 1\}$ in the Hilbert space $L^2(G)$. Suppose $1<p<\infty$. It is well-known that the restriction of P to $L^2(G)\cap L^p(G)$ extends to a bounded projection from $L^p(G)$ on the closure of $\mathrm{Span}\{\gamma_i \mid i\geq 1\}$ in the space $L^p(G)$.

For any integer q, we consider the abelian group $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$ equipped with the discrete topology. By [HR, Theorem 23.22 and p. 367], the dual group of $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$ is isomorphic to the compact group $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$.

For any integer $i\geq 1$, we define the character $\varepsilon_{i,q}$ of the group $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$ by $\varepsilon_{i,q}(\overline{k_1},\ldots,\overline{k_j},\ldots)=e^{2\pi\sqrt{-1}\,k_i/q}$ where $(k_j)_{j\geq 1}$ is a sequence of integers. The compact group $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$ is an example of a Vilenkin group and the set of all characters of this group is called the associated *Vilenkin system*. For more information, we refer the reader to [SWS, Appendix 0.7] and the references contained therein.

We will use the following lemma, with proof left to the reader.

LEMMA 2.4. *Let $q\geq 2$ be an integer. The sequence $(\varepsilon_{i,q})_{i\geq 1}$ of characters of the group $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$ is a Sidon set.*

We remark that $(\varepsilon_{i,q})_{i\geq 1}$ can be regarded as a sequence of independent complex random variables on the probability space $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$. For any integers n and q, we introduce the compact finite group $\Omega_q^n=\mathbb{Z}/q\mathbb{Z}\times\cdots\times\mathbb{Z}/q\mathbb{Z}$. Note that Ω_q^n is a subgroup of $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$. The restrictions $\varepsilon_{i,q}|\Omega_q^n$, where $1\leq i\leq n$, are characters of the group Ω_q^n (see [HR, Theorem 23.21]), and can also be regarded as a finite sequence of independent complex random variables on the probability space Ω_q^n.

We only require the use of averages of these random variables. Moreover, if X is a Banach space and $1<p<\infty$, these averages are identical: for any $n\in\mathbb{N}$, we have

$$\Big\|\sum_{i=1}^{n}\varepsilon_{i,q}|\Omega_q^n\otimes x_i\Big\|_{L^p(\Omega_q^n,X)}=\Big\|\sum_{i=1}^{n}\varepsilon_{i,q}\otimes x_i\Big\|_{L^p(\prod_1^\infty \mathbb{Z}/q\mathbb{Z},X)},\qquad x_1,\ldots,x_n\in X.$$

Thus, if n and q are integers and $1\leq i\leq n$, we will also use the notation $\varepsilon_{i,q}$ for the restriction $\varepsilon_{i,q}|\Omega_q^n$.

Suppose $1<p<\infty$. An operator $T\colon L^p(G)\to L^p(G)$ is a *Fourier multiplier* if there exists a function $\varphi\in L^\infty(\widehat{G})$ such that for any $f\in L^p(G)\cap L^2(G)$ we have $\mathcal{F}(T(f))=\varphi\mathcal{F}(f)$ where $\mathcal{F}$ denotes the Fourier transform. In this case, we let $T = M_\varphi$. We denote by $M_p(G)$ the space of bounded Fourier multipliers on $L^p(G)$. See [LA] and [D] for more information. Let X be a Banach space. Then $M_p(G,X)$ is the space of bounded Fourier multipliers M_φ such that $M_\varphi \otimes \mathrm{Id}_X$ extends to a bounded operator $M_\varphi \otimes \mathrm{Id}_X : L^p(G,X) \to L^p(G,X)$. With these definitions and by (1.2), we see that $M_p(G,S^p)$ coincides with the space of completely bounded Fourier multipliers.

If $b \in L^1(G)$, we define the convolution operator C_b by

$$C_b : L^p(G) \to L^p(G), \qquad f \mapsto b * f.$$

This operator is a completely bounded Fourier multiplier and we have $C_b = M_{\mathcal{F}(b)}$. We will use the following approximation result [LA, Theorem 5.6.1] (see also [D, Corollary 4, p. 98]).

Theorem 2.5. *Suppose $1 < p < \infty$. Let G be a locally compact abelian group. Let $M_\varphi \colon L^p(G) \to L^p(G)$ be a bounded Fourier multiplier. Then there exists a net of continuous functions $(b_i)_{i\in I}$ with compact support such that*

$$\|C_{b_i}\|_{L^p(G)\to L^p(G)} \leq \|M_\varphi\|_{L^p(G)\to L^p(G)} \quad \text{and} \quad C_{b_i} \xrightarrow[i]{so} M_\varphi$$

(*convergence for the strong operator topology*).

We need the following vectorial extension of [D, Theorem 2, p. 113] (see also [S, Theorem 3.3]). We can prove this result in a similar way.

Theorem 2.6. *Let G be a locally compact abelian group, H be a closed subgroup of G and X be a Banach space. We denote by $\pi : \widehat{G} \to \widehat{G}/H^\perp$ the canonical map. Then the linear map*

$$M_p(H,X) \to M_p(G,X), \qquad M_\varphi \mapsto M_{\varphi\circ\pi},$$

is an isometry.

The following proposition is well-known (see e.g. [F, p. 57]).

Proposition 2.7 (Weil's formula). *Let G be a locally compact abelian group and H be a closed subgroup of G. For any Haar measures μ_G and μ_H on G and H, respectively, there exists a Haar measure $\mu_{G/H}$ on the group G/H such that for every continuous function $f \colon G \to \mathbb{C}$ with compact support,*

$$\int_G f(x)\,d\mu_G(x) = \int_{G/H}\int_H f(xh)\,d\mu_H(h)\,d\mu_{G/H}(xH).$$

With this result, we can prove the next proposition.

Proposition 2.8. *Suppose $1 < p < \infty$. Let G be a locally compact abelian group, H be a compact subgroup of G and X be a Banach space. If $\varphi \colon H^\perp \to \mathbb{C}$ is a complex function, we denote by $\widetilde{\varphi} \colon \widehat{G} \to \mathbb{C}$ the extension of*

φ on $\widehat{G}$ which is zero off $H^\perp$. Then the linear map

$$M_p(G/H, X) \to M_p(G, X), \qquad M_\varphi \mapsto M_{\widetilde{\varphi}},$$

is an isometry.

Proof. We denote by $\pi\colon G \to G/H$ the canonical map. We use the Haar measures μ_H, given by Proposition 2.7, and suppose that $\mu_H(H) = 1$. Using Weil's formula, it is not difficult to prove that the linear map

$$\Phi_p : L^p(G/H) \to L^p(G), \qquad f \mapsto f \circ \pi$$

and its tensorisation $\Phi_p \otimes \mathrm{Id}_X\colon L^p(G/H, X) \to L^p(G, X)$ are isometries. Note that the adjoint map $\Phi_{p^*}^*$ and the orthogonal projection of $L^2(G)$ onto $\Phi_2(L^2(G/H))$ coincide on $L^2(G) \cap L^p(G)$. Moreover, it is easy to see that the linear map $\Phi_{p^*}^* \otimes \mathrm{Id}_X$ is well-defined and contractive. The end of the proof is straightforward and left to the reader. ■

Recall the following structure theorem for locally compact abelian groups (see e.g. [HR, Theorem 24.30]).

THEOREM 2.9. *Any locally compact abelian group is isomorphic to a product $\mathbb{R}^n \times G_0$ where $n \geq 0$ is an integer and G_0 is a locally compact abelian group containing a compact subgroup K such that G_0/K is discrete.*

Let $(G_i)_{i \in I}$ be a family of groups and let $\prod_{i\in I} G_i$ be their cartesian product. Recall that the direct sum $\bigoplus_{i\in I} G_i$ is the set of all $(x_i)_{i\in I} \in \prod_{i\in I} G_i$ such that $x_i = e_i$ for all but a finite set of indices, where e_i is the neutral element of G_i. The group $\bigoplus_{i\in I} G_i$ is a subgroup of $\prod_{i\in I} G_i$. Recall that a group is *of bounded order* if every element has finite order and the order of each element is less than some fixed positive integer. Note the following result [HR, p. 449].

THEOREM 2.10. *Every abelian group G (without topology) of bounded order is isomorphic to a direct sum $\bigoplus_{i\in I} \mathbb{Z}/q_i^{r_i}\mathbb{Z}$ of cyclic groups, where only finitely many distinct primes q_i and positive integers r_i occur.*

This theorem implies that an infinite abelian group G of bounded order contains a direct sum $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$ where q is a fixed prime.

3. Unconditionality and Fourier multipliers. Suppose $1 < p < \infty$. Let G be a locally compact group and X a Banach space. If $t \in G$, we denote by τ_t the translation operator on $L^p(G)$ defined by $\tau_t(f)(s) = f(t^{-1}s)$ for $f \in L^p(G)$ and $s \in G$. We start with the following result.

LEMMA 3.1. *Let G be an infinite locally compact group and X a Banach space. If the set $\{\tau_t \otimes \mathrm{Id}_X \mid t \in G\}$ is R-bounded in $B(L^2(G, X))$ then X is isomorphic to a Hilbert space.*

Proof. Let $n \geq 1$ be an integer and $t_1, \ldots, t_n$ be distinct elements of G. There exists a compact neighborhood V of the neutral element e_G of G such that the sets $t_1 V, \ldots, t_n V$ are disjoint. We have $\mu_G(V) > 0$. For any integer $1 \leq i \leq n$, we let $V_i = t_i V$. First note that, for any $x_1, \ldots, x_n \in X$, we have (since the V_i's are disjoint)

$$\begin{aligned} \Big(\sum_{i=1}^n \|1_{V_i}\|_{L^2(G)}^2 \|x_i\|_X^2\Big)^{1/2} &= \Big(\int_{\Omega_0} \sum_{i=1}^n \|\varepsilon_i(\omega) 1_{V_i} \otimes x_i\|_{L^2(G,X)}^2 \, d\omega\Big)^{1/2} \\ &= \Big(\int_{\Omega_0} \Big\|\sum_{i=1}^n \varepsilon_i(\omega) 1_{V_i} \otimes x_i\Big\|_{L^2(G,X)}^2 d\omega\Big)^{1/2} \\ &= \Big\|\sum_{i=1}^n \varepsilon_i \otimes 1_{V_i} \otimes x_i\Big\|_{\mathrm{Rad}(L^2(G,X))}. \end{aligned} \tag{3.1}$$

We deduce that

$$\begin{aligned} \Big(\sum_{i=1}^n \|1_{V_i}\|_{L^2(G)}^2 \|x_i\|_X^2\Big)^{1/2} &= \Big\|\sum_{i=1}^n \varepsilon_i \otimes (\tau_{t_i} \otimes \mathrm{Id}_X)(1_V \otimes x_i)\Big\|_{\mathrm{Rad}(L^2(G,X))} \\ &\lesssim \Big\|\sum_{i=1}^n \varepsilon_i \otimes 1_V \otimes x_i\Big\|_{\mathrm{Rad}(L^2(G,X))} \\ &= \|1_V\|_{L^2(G)} \Big\|\sum_{i=1}^n \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}(X)}. \end{aligned}$$

For $1 \leq i \leq n$, we have $\|1_V\|_{L^2(G)} = \|1_{V_i}\|_{L^2(G)}$. We infer that

$$\Big(\sum_{i=1}^n \|x_i\|_X^2\Big)^{1/2} \lesssim \Big\|\sum_{i=1}^n \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}(X)}.$$

We deduce that X has cotype 2. Now, for any $x_1, \ldots, x_n \in X$, we have

$$\begin{aligned} &\|1_V\|_{L^2(G)} \Big\|\sum_{i=1}^n \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}(X)} \\ &\quad= \Big\|\sum_{i=1}^n \varepsilon_i \otimes 1_V \otimes x_i\Big\|_{\mathrm{Rad}(L^2(G,X))} \\ &\quad= \Big\|\sum_{i=1}^n \varepsilon_i \otimes (\tau_{t_i^{-1}} \otimes \mathrm{Id}_X)(\tau_{t_i} \otimes \mathrm{Id}_X)(1_V \otimes x_i)\Big\|_{\mathrm{Rad}(L^2(G,X))} \\ &\quad\lesssim \Big\|\sum_{i=1}^n \varepsilon_i \otimes 1_{V_i} \otimes x_i\Big\|_{\mathrm{Rad}(L^2(G,X))} = \Big(\sum_{i=1}^n \|1_{V_i}\|_{L^2(G)}^2 \|x_i\|_X^2\Big)^{1/2} \quad \text{by (3.1)}. \end{aligned}$$

Using, one more time, the equality $\|1_V\|_{L^2(G)} = \|1_{V_i}\|_{L^2(G)}$ for $1 \leq i \leq n$, we deduce that

$$\Big\|\sum_{i=1}^{n} \varepsilon_i \otimes x_i\Big\|_{\mathrm{Rad}(X)} \lesssim \Big(\sum_{i=1}^{n} \|x_i\|_X^2\Big)^{1/2}.$$

Thus X has type 2. Hence, by Kwapień's theorem [K1, Proposition 3.1] (or [DJT, Corollary 12.20]), the Banach space X is isomorphic to a Hilbert space. ∎

Let G be a locally compact abelian group and X be a Banach space. If X is isomorphic to a Hilbert space, it is clear that we have a canonical isomorphism $M_2(G, X) = M_2(G)$. We will show the reverse implication for *infinite* locally compact abelian groups.

We begin with the case of $\mathbb{T}$. We give a proof which does not use [DF]. We will use the following elementary lemma, with proof left to the reader.

LEMMA 3.2. *Let $g : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ be a continuous complex function. We have*

$$\int_{\mathbb{T}} g(z, z^k)\, dz \xrightarrow[k\to\infty]{} \int_{\mathbb{T}\times\mathbb{T}} g(z, z')\, dz\, dz'.$$

Now, we can prove the following proposition.

PROPOSITION 3.3. *Let X be a Banach space. There is a canonical isomorphism $M_2(G, \mathbb{T}) = M_2(\mathbb{T})$ if and only if X is isomorphic to a Hilbert space.*

Proof. Suppose that $M_2(\mathbb{T}, X) = M_2(\mathbb{T})$. For any integer $i \geq 1$, we let $n_i = 2^{2i}$ and $m_i = 2^{2i+1}$. The sequences $(n_i)_{i\geq 1}$ and $(m_j)_{j\geq 1}$ are Sidon sets for the group $\mathbb{T}$. We will use the fact that there exist arbitrarily large integers $k \geq 1$ such the map $(i, j) \mapsto n_i + km_j$ is one-to-one. Note that, by Theorem 2.3, we have

(3.2)
$$\Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \approx \Big\|\sum_{i,j=1}^{n} e^{2\pi\sqrt{-1}\, n_i\, \cdot} \otimes e^{2\pi\sqrt{-1}\, m_j\, \cdot} \otimes x_{ij}\Big\|_{L^2(\mathbb{T}\times\mathbb{T}, X)}$$

for all $n \in \mathbb{N}$ and $x_{ij} \in X$.

Now, suppose that the Banach space X does not have property (α). Let C be a positive constant. Then there exist $n \geq 1$, $t_{ij} \in \mathbb{C}$ with $|t_{ij}| = 1$ and $x_{ij} \in X$ such that

$$\Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \leq 1$$

with arbitrarily large $\| \sum_{i,j=1}^{n} t_{ij} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \|_{\mathrm{Rad}(\mathrm{Rad}(X))}$. Using the equivalence (3.2), we deduce that there exist $n \geq 1$, $t_{ij} \in \mathbb{C}$ with $|t_{ij}| = 1$ and

$x_{ij} \in X$ such that

$$\Big\| \sum_{i,j=1}^{n} e^{2\pi\sqrt{-1}\, n_i \,\cdot} \otimes e^{2\pi\sqrt{-1}\, m_j \,\cdot} \otimes x_{ij} \Big\|_{L^2(\mathbb{T}\times\mathbb{T},X)} \leq \frac{1}{2}$$

and

$$\Big\| \sum_{i,j=1}^{n} e^{2\pi\sqrt{-1}\, n_i \,\cdot} \otimes e^{2\pi\sqrt{-1}\, m_j \,\cdot} \otimes x_{ij} \Big\|_{L^2(\mathbb{T}\times\mathbb{T},X)} \geq 2C.$$

Moreover, by Lemma 3.2, we have

$$\Big\| \sum_{i,j=1}^{n} e^{2\pi\sqrt{-1}(n_i+km_j)\,\cdot} \otimes x_{ij} \Big\|_{L^2(\mathbb{T},X)} \xrightarrow[k\to+\infty]{} \Big\| \sum_{i,j=1}^{n} e^{2\pi\sqrt{-1}\, n_i \,\cdot} \otimes e^{2\pi\sqrt{-1}\, m_j \,\cdot} \otimes x_{ij} \Big\|_{L^2(\mathbb{T}\times\mathbb{T},X)}.$$

For some k large enough, we deduce that

$$\Big\| \sum_{i,j=1}^{n} e^{2\pi\sqrt{-1}\,(n_i+km_j)\,\cdot} \otimes x_{ij} \Big\|_{L^2(\mathbb{T},X)} \leq 1$$

and

$$\Big\| \sum_{i,j=1}^{n} t_{ij} e^{2\pi\sqrt{-1}\,(n_i+km_j)\,\cdot} \otimes x_{ij} \Big\|_{L^2(\mathbb{T},X)} > C.$$

We infer that (1.1) is not satisfied, a contradiction. Thus, X has property (α). Hence $L^2(\mathbb{T}, X)$ also has property (α).

Now, note that $L^\infty(\mathbb{T})$ is a commutative unital C^*-algebra. By Gelfand's Theorem (see e.g. [F, Theorem 1.20]), the Banach algebra $L^\infty(\mathbb{T})$ is isometrically isomorphic to $C(K)$ where K is a compact topological space. Moreover, we have a bounded homomorphism

$$L^\infty(\mathbb{T}) \to B(L^2(\mathbb{T}, X)), \quad \varphi \mapsto M_\varphi.$$

By Theorem 2.1, this linear map is R-bounded. For any $t \in G$, the map τ_t is an isometric Fourier multiplier. Hence the set $\{\tau_t \otimes \mathrm{Id}_X \mid t \in \mathbb{T}\}$ is R-bounded. By Lemma 3.1, we conclude that X is isomorphic to a Hilbert space. ∎

Now, we extend Proposition 3.3 to the groups $\mathbb{R}$ and $\mathbb{Z}$. We use a method similar to the one of [A, Theorems 3.4 and 3.5]. Since we need variants of this method later (and also for the convenience of the reader), we include some details. We need the following vectorial extension of [L, Proposition 3.3]; it can be proved as [CW, Theorem 3.4].

Theorem 3.4. *Let X be a Banach space. Suppose $1 < p < \infty$. Let ψ be a continuous function on $\mathbb{R}$ which defines a bounded Fourier multiplier*

M_ψ on $L^p(\mathbb{R},X)$. Then $\psi|\mathbb{Z}$ defines a bounded Fourier multiplier $M_{\psi|\mathbb{Z}}$ on $L^p(\mathbb{T},X)$.

Moreover, we need the following result of Jodeit [J, Theorem 3.5]. We introduce the function $\Lambda\colon \mathbb{R}\to\mathbb{R}$ defined by

$$\Lambda(x)=\begin{cases}1-|x| & x\in[-1,1],\\ 0 & |x|>1.\end{cases}$$

THEOREM 3.5. *Suppose $1<p<\infty$. Let φ be a complex function defined on $\mathbb{Z}$ such that M_φ is a bounded Fourier multiplier on $L^p(\mathbb{T})$. Then the complex function $\psi\colon\mathbb{R}\to\mathbb{C}$ given by*

$$\psi(x)=\sum_{k\in\mathbb{Z}}\varphi(k)\Lambda(x-k),\qquad x\in\mathbb{R}, \tag{3.3}$$

defines a bounded Fourier multiplier M_ψ on $L^p(\mathbb{R})$.

Now, we can prove the following proposition.

PROPOSITION 3.6. *Let X be a Banach space, and $G=\mathbb{R}$ or $G=\mathbb{Z}$. There is a canonical isomorphism $M_2(G,X)=M_2(G)$ if and only if X is isomorphic to a Hilbert space.*

Proof. Suppose that X is not isomorphic to a Hilbert space. By Proposition 3.3, there exists a bounded Fourier multiplier $M_\varphi\colon L^2(\mathbb{T})\to L^2(\mathbb{T})$ such that $M_\varphi\otimes \mathrm{Id}_X$ is not bounded on $L^2(\mathbb{T},X)$. Consider the function ψ given by (3.3). By Theorem 3.5, ψ defines a bounded Fourier multiplier $M_\psi\colon L^2(\mathbb{R})\to L^2(\mathbb{R})$. Now, suppose that $M_\psi\otimes\mathrm{Id}_X\colon L^2(\mathbb{R},X)\to L^2(\mathbb{R},X)$ is bounded. Since $\psi\colon\mathbb{R}\to\mathbb{C}$ is continuous, by Theorem 3.4, the restriction $\psi|\mathbb{Z}$ defines a bounded Fourier multiplier $M_{\psi|\mathbb{Z}}$ on $L^2(\mathbb{T},X)$. Moreover, we have $\psi(k)=\varphi(k)$ for any $k\in\mathbb{Z}$. Hence we deduce that M_φ is bounded on $L^2(\mathbb{T},X)$, a contradiction. Consequently, the Fourier multiplier M_ψ is bounded on $L^2(\mathbb{R})$ and $M_\psi\otimes\mathrm{Id}_X$ is not bounded on $L^2(\mathbb{R},X)$. Hence, the case $G=\mathbb{R}$ is completed.

We can suppose that the above multiplier M_ψ satisfies $\|M_\psi\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}=1$. By Theorem 2.5, there exists a net $(b_i)_{i\in I}$ of continuous functions with compact support such that

$$\|C_{b_i}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\leq 1\quad\text{and}\quad C_{b_i}\xrightarrow[i]{\text{so}} M_\psi.$$

Let $C>1$. Then it is not difficult to deduce that there exists a continuous function $b\colon\mathbb{R}\to\mathbb{C}$ with compact support such that $\|C_b\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\leq 1$ and $\|C_b\otimes\mathrm{Id}_X\|_{L^2(\mathbb{R},X)\to L^2(\mathbb{R},X)}\geq 2C$. Now, we consider the sequence $(a_n)_{n\geq 1}$ of complex sequences indexed by $\mathbb{Z}$, defined by

$$a_{n,k}=\int_0^1\!\!\int_0^1 \frac{1}{n}b\Big(\frac{t-s+k}{n}\Big)\,ds\,dt,\qquad k\in\mathbb{Z}. \tag{3.4}$$

For any $n \geq 1$, we introduce the conditional expectation $\mathbb{E}_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with respect to the σ-algebra generated by the intervals $[k/n,(k+1)/n[$, $k \in \mathbb{Z}$. For $f \in L^2(\mathbb{R})$ and $n \geq 1$, we have

$$\mathbb{E}_n f = n \sum_{k \in \mathbb{Z}} \Big(\int_{k/n}^{(k+1)/n} f(t)\,dt \Big) 1_{[k/n,(k+1)/n[}$$

(see [AA, p. 227]). Now, we define the linear map $J_n : \ell^2_{\mathbb{Z}} \to \mathbb{E}_n(L^2(\mathbb{R}))$ by

$$J_n(u) = n^{1/2} \sum_{k \in \mathbb{Z}} u_k 1_{[k/n,(k+1)/n[}, \qquad u \in \ell^2_{\mathbb{Z}}.$$

It is easy to check that J_n is an isometry of $\ell^2_{\mathbb{Z}}$ onto $\mathbb{E}_n(L^2(\mathbb{R}))$. For any $u \in \ell^2_{\mathbb{Z}}$, mimicking the computation in the proof of [A, Theorem 3.5], we obtain

$$\mathbb{E}_n C_b J_n(u) = J_n C_{a_n}(u).$$

Then it is easy to prove that there exists $n \geq 1$ such that $\|C_{a_n}\|_{\ell^2_{\mathbb{Z}} \to \ell^2_{\mathbb{Z}}} \leq 1$ and $\|C_{a_n} \otimes \mathrm{Id}_X\|_{\ell^2_{\mathbb{Z}}(X) \to \ell^2_{\mathbb{Z}}(X)} \geq C$. Finally, we conclude the case $G = \mathbb{Z}$ by applying the closed graph theorem. ■

Now, we pass to discrete groups. We first prove the following result with a method similar to that of Proposition 3.3.

PROPOSITION 3.7. *Let X be a Banach space and $q \geq 2$ be an integer. There is a canonical isomorphism $M_2(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}, X) = M_2(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z})$ if and only if X is isomorphic to a Hilbert space.*

Proof. Assume that $M_2(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}, X) = M_2(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z})$. Then there exists a positive constant C such that for any $\varphi \in L^\infty(\prod_1^\infty \mathbb{Z}/q\mathbb{Z})$,

$$\|M_\varphi\|_{L^2(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z},X) \to L^2(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z},X)} \leq C \|\varphi\|_{L^\infty(\prod_1^\infty \mathbb{Z}/q\mathbb{Z})}.$$

Moreover, since $\Omega_q^n \times \Omega_q^n$ is a closed subgroup of $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$, for any $n \geq 1$, $t_{ij} \in \mathbb{C}$ and $x_{ij} \in X$, we deduce that

$$\Big\| \sum_{i,j=1}^n t_{ij} \varepsilon_{i,q} \otimes \varepsilon_{j,q} \otimes x_{ij} \Big\|_{L^2(\Omega_q^n \times \Omega_q^n, X)} \leq C \sup_{1 \leq i,j \leq n} |t_{ij}| \Big\| \sum_{i,j=1}^n \varepsilon_{i,q} \otimes \varepsilon_{j,q} \otimes x_{ij} \Big\|.$$

Now, by Theorem 2.3 and Lemma 2.4,

$$\Big\| \sum_{i,j=1}^n \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \approx \Big\| \sum_{i,j=1}^n \varepsilon_{i,q} \otimes \varepsilon_{j,q} \otimes x_{ij} \Big\|_{L^2(\Omega_q^n \times \Omega_q^n, X)}$$

for $n \in \mathbb{N}$ and $x_{ij} \in X$. Thus the Banach space X has property (α). The end of the proof is similar to the end of the proof of Proposition 3.3. ■

PROPOSITION 3.8. *Let G be an infinite discrete abelian group and X a Banach space. There is a canonical isomorphism $M_2(G,X) = M_2(G)$ if and only if X is isomorphic to a Hilbert space.*

Proof. We consider several cases.

CASE 1: G is not a torsion group. Then G contains a copy of $\mathbb{Z}$. Suppose that $M_2(G,X) = M_2(G)$. By Theorem 2.6, we have $M_2(\mathbb{Z},X) = M_2(\mathbb{Z})$. By Proposition 3.6, we deduce that X is isomorphic to a Hilbert space.

CASE 2: G is a torsion group, but contains elements of arbitrarily large order. We may therefore assume that there is a sequence $G_1, G_2, \ldots$ of cyclic subgroups of G of orders $n_1, n_2, \ldots$ with $n_j \to \infty$ as $j \to \infty$.

We will construct contractive Fourier multipliers C_{a_n} on the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with large $\|C_{a_n} \otimes \mathrm{Id}_X\|_{\ell^2_n(X) \to \ell^2_n(X)}$. We use a similar method to the one in the proof of Proposition 3.6. By Proposition 3.3, there exists a bounded Fourier multiplier $M_\psi \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ such that $M_\psi \otimes \mathrm{Id}_X$ is not bounded on $L^2(\mathbb{T},X)$. By Theorem 2.5, there exists a net $(b_i)_{i \in I}$ of continuous functions such that

$$\|C_{b_i}\|_{L^2(\mathbb{T}) \to L^2(\mathbb{T})} \leq \|M_\psi\|_{L^2(\mathbb{T}) \to L^2(\mathbb{T})} \quad \text{and} \quad C_{b_i} \xrightarrow[i]{so} M_\psi.$$

Let $C > 1$. It is not difficult to deduce that there exists a continuous function $b \colon \mathbb{T} \to \mathbb{C}$ such that

$$\|C_b\|_{L^2(\mathbb{T}) \to L^2(\mathbb{T})} \leq 1 \quad \text{and} \quad \|C_b \otimes \mathrm{Id}_X\|_{L^2(\mathbb{T},X) \to L^2(\mathbb{T},X)} \geq 2C.$$

Now, we use the identification $L^2(\mathbb{T}) = L^2([0,1])$. We consider b as a 1-periodic function $b \colon \mathbb{R} \to \mathbb{C}$. Then, we define by (3.4) the sequence $(a_n)_{n \geq 1}$ of complex sequences indexed by $\{0, \ldots, n\}$. For each $n \geq 1$, C_{a_n} is a convolution operator on ℓ^2_n. For $n \geq 1$, we introduce the conditional expectation $\mathbb{E}_n \colon L^2([0,1]) \to L^2([0,1])$ with respect to the σ-algebra generated by the intervals $[k/n, (k+1)/n[$, $k \in \{0, \ldots, n\}$. For $n \geq 1$ and $f \in L^2([0,1])$, we have

$$\mathbb{E}_n f = n \sum_{k=0}^{n-1} \Big(\int_{k/n}^{(k+1)/n} f(t)\, dt \Big) 1_{[k/n,(k+1)/n[}. \tag{3.5}$$

Now, we define the linear map $J_n \colon \ell^2_n \to \mathbb{E}_n(L^2([0,1]))$ by

$$J_n(u) = n^{1/p} \sum_{k=0}^{n-1} u_k 1_{[k/n,(k+1)/n[} \quad \text{if } u \in \ell^2_n.$$

It is easy to check that J_n is an isometry of ℓ^2_n onto $\mathbb{E}_n(L^2([0,1]))$. For any $u \in \ell^n_p$, by a computation similar to the one in the proof of [A, Theorem 3.5], we show that

$$\mathbb{E}_n C_b J_n(u) = J_n C_{a_n}(u).$$

Thus, it is not difficult to deduce that there exists $N \geq 1$ such that for any $n \geq N$ we have

$$\|C_{a_n}\|_{\ell^2_n \to \ell^2_n} \leq 1 \quad \text{and} \quad \|C_{a_n} \otimes \mathrm{Id}_X\|_{\ell^2_n(X) \to \ell^2_n(X)} \geq C.$$

Now, recall that $n_j \to \infty$ as $j \to \infty$. Hence, there exists an integer $j \geq 1$ and a convolution operator $C_a \colon L^2(G_{n_j}) \to L^2(G_{n_j})$ such that

$$\|C_a\|_{L^2(G_{n_j}) \to L^2(G_{n_j})} \leq 1 \quad \text{and} \quad \|C_a \otimes \mathrm{Id}_X\|_{L^2(G_{n_j},X) \to L^2(G_{n_j},X)} \geq C.$$

We conclude by using Theorem 2.6 and the closed graph theorem.

Case 3: G is a group of bounded order. In this case, the remark following Theorem 2.10 allows us to claim that G contains a subgroup isomorphic to $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$ where q is a prime. We conclude by invoking Theorem 2.6 and Proposition 3.7. ∎

We now recall a particular case of [DJ, Theorem 1]. We give an independent proof of this result.

Proposition 3.9. *Let G be an infinite compact abelian group and X be a Banach space. There is a canonical isomorphism $M_2(G,X) = M_2(G)$ if and only if X is isomorphic to a Hilbert space.*

Proof. Let G be an infinite compact group. Suppose that $M_2(G,X) = M_2(G)$.

Case 1: The discrete group $\widehat{G}$ is not a torsion group. Then $\widehat{G}$ contains a copy of $\mathbb{Z}$. Note that we have $G/\mathbb{Z}^\perp = \widehat{\mathbb{Z}} = \mathbb{T}$ isomorphically. By Proposition 2.8, we deduce that $M_2(\mathbb{T},X) = M_2(\mathbb{T})$. By Proposition 3.3, X is isomorphic to a Hilbert space.

Case 2: $\widehat{G}$ is a torsion group, but contains elements of arbitrarily large order. We may therefore assume that there is a sequence $G_1, G_2, \ldots$ of cyclic subgroups of $\widehat{G}$ of orders $n_1, n_2, \ldots$ with $n_j \to \infty$ as $j \to \infty$. Note that for any $j \geq 1$, we have group isomorphisms

$$G/G_j^\perp = \widehat{G_j} = \mathbb{Z}/n_j\mathbb{Z}.$$

Using Proposition 2.8, we conclude as in Case 2 of the proof of Proposition 3.8.

Case 3: $\widehat{G}$ is a group of bounded order. In this case, the remark following Theorem 2.10 shows that $\widehat{G}$ contains a subgroup isomorphic to $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$ where q is a prime. Observe that we have group isomorphisms

$$G/\big(\textstyle\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}\big)^\perp = \widehat{\textstyle\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}} = \textstyle\prod_1^\infty \mathbb{Z}/q\mathbb{Z}.$$

Using the fact that $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$ is a subgroup of $\prod_1^\infty \mathbb{Z}/q\mathbb{Z}$, the result follows by applying Proposition 2.8, Theorem 2.6 and Proposition 3.7. ∎

The next theorem is the principal result of this section.

Theorem 3.10. *Let G be an infinite locally compact abelian group and X a Banach space. We have a canonical isomorphism $M_2(G,X) = M_2(G)$ if and only if X is isomorphic to a Hilbert space.*

Proof. By Theorem 2.9, G is isomorphic to a product $\mathbb{R}^n \times G_0$ where G_0 is a locally compact abelian group containing a compact subgroup K such that G_0/K is discrete. Suppose $n \geq 1$. If $M_2(G,X) = M_2(G)$, Theorem 2.6 yields a canonical isomorphism $M_2(\mathbb{R},X) = M_2(\mathbb{R})$. Hence, by Proposition 3.6, X is isomorphic to a Hilbert space. If the group K is infinite, we apply a similar reasoning by using Proposition 3.9 instead of Proposition 3.6. If $n = 0$ and K is finite then it is not difficult to see that G is discrete. In this case we use Proposition 3.8. ■

4. Bounded but not completely bounded Fourier multipliers. In this section, we prove that if $1 < p < \infty$, $p \neq 2$, then there exists a bounded Fourier multiplier on $L^p(G)$ which is not completely bounded, where G is an infinite locally compact abelian group. The cases of $\mathbb{R}$, $\mathbb{Z}$ and infinite compact abelian groups are already known. We start by extending these results to the discrete group $\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z}$, where $q \geq 2$ is an integer. In the proof, we will use the notation introduced before Proposition 3.7.

PROPOSITION 4.1. *Suppose $1 < p < \infty$, $p \neq 2$. Let $q \geq 2$ be an integer. There exists a bounded Fourier multiplier on $L^p(\bigoplus_1^\infty \mathbb{Z}/q\mathbb{Z})$ which is not completely bounded.*

Proof. By Theorem 2.6 and the closed graph theorem, it suffices to prove that there exist contractive Fourier multipliers on the group $\Omega_q^n \times \Omega_q^n = (\mathbb{Z}/q\mathbb{Z} \times \cdots \times \mathbb{Z}/q\mathbb{Z}) \times (\mathbb{Z}/q\mathbb{Z} \times \cdots \times \mathbb{Z}/q\mathbb{Z})$ with arbitrarily large completely bounded norms in n. By Theorem 2.3 and Lemma 2.4, we have

$$\Big\| \sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \Big\|_{\mathrm{Rad}(\mathrm{Rad}(S^p))} \approx \Big\| \sum_{i,j=1}^{n} \varepsilon_{i,q} \otimes \varepsilon_{j,q} \otimes x_{ij} \Big\|_{L^p(\Omega_q^n \times \Omega_q^n, S^p)}$$

where $n \in \mathbb{N}$ and $x_{ij} \in S^p$. We let $\mathcal{R}_{2,q}^p$ denote the closed span of the $\varepsilon_{i,q} \otimes \varepsilon_{j,q}$'s in $L^p(\Omega_q^n \times \Omega_q^n)$ where $1 \leq i,j \leq n$. For any family $\tau = (t_{ij})_{i,j\geq 1}$ of complex numbers we consider the linear map

$$T_\tau : \mathcal{R}_{2,q}^p \to L^p(\Omega_q^n \times \Omega_q^n), \qquad \varepsilon_{i,q} \otimes \varepsilon_{j,q} \mapsto t_{ij}\varepsilon_{i,q} \otimes \varepsilon_{j,q}.$$

Note that for any $n \in \mathbb{N}$ and $\alpha_{ij} \in \mathbb{C}$, we have

$$\begin{aligned}\Big\| \sum_{i,j=1}^{n} \alpha_{ij}\varepsilon_{i,q} \otimes \varepsilon_{j,q} \Big\|_{L^p(\Omega_q^n \times \Omega_q^n)} &\approx \Big\| \sum_{i,j=1}^{n} \alpha_{ij}\varepsilon_i \otimes \varepsilon_j \Big\|_{L^p(\Omega_0 \times \Omega_0)} \\ &\approx \Big(\sum_{i,j=1}^{n} |\alpha_{ij}|^2 \Big)^{1/2}\end{aligned}$$

(see [P1, Lemma 2.1] or [DF, p. 455]). Then, for any $\alpha_{ij} \in \mathbb{C}$,

$$\Big\|T_\tau\Big(\sum_{i,j=1}^{n} \alpha_{ij}\varepsilon_{i,q} \otimes \varepsilon_{j,q}\Big)\Big\|_{L^p(\Omega_q^n \times \Omega_q^n)} = \Big\|\sum_{i,j=1}^{n} t_{ij}\alpha_{ij}\varepsilon_{i,q} \otimes \varepsilon_{j,q}\Big\|_{L^p(\Omega_q^n \times \Omega_q^n)}$$

$$\approx \Big(\sum_{i,j=1}^{n} |t_{ij}\alpha_{ij}|^2\Big)^{1/2} \lesssim \sup_{1\leq i,j\leq n} |t_{ij}| \Big\|\sum_{i,j=1}^{n} \alpha_{ij} \otimes \varepsilon_{i,q} \otimes \varepsilon_{j,q}\Big\|_{L^p(\Omega_q^n \times \Omega_q^n)}.$$

Consequently, $\|T_\tau\|_{\mathcal{R}^p_{2,q} \to L^p(\Omega_q^n \times \Omega_q^n)} \lesssim \|\tau\|_\infty$. Since S^p does not have property (α) there exist $t_{ij} \in \mathbb{C}$ with $|t_{ij}| = 1$ and large $\|T_\tau \otimes \mathrm{Id}_{S^p}\|$. Now, using the canonical bounded projection from $L^p(\prod_1^\infty \mathbb{Z}/q\mathbb{Z})$ onto the closure of $\mathrm{Span}\{\varepsilon_{i,q} \mid i \geq 1\}$ in $L^p(\prod_1^\infty \mathbb{Z}/q\mathbb{Z})$, we see that there exists a bounded projection from $L^p(\prod_1^\infty \mathbb{Z}/q\mathbb{Z} \times \prod_1^\infty \mathbb{Z}/q\mathbb{Z})$ onto $\mathcal{R}^p_{2,q}$. Applying the inclusion map $L^p(\Omega_q^n \times \Omega_q^n) \to L^p(\prod_1^\infty \mathbb{Z}/q\mathbb{Z} \times \prod_1^\infty \mathbb{Z}/q\mathbb{Z})$ we obtain a bounded projection from $L^p(\Omega_q^n \times \Omega_q^n)$ onto $\mathcal{R}^p_{2,q}$ with norm bounded independently of n. Finally, by composing with this projection, we obtain contractive Fourier multipliers on the group $\Omega_q^n \times \Omega_q^n$ with arbitrary completely bounded norms in n. ■

Now, we can prove the second main result of this paper.

THEOREM 4.2. *Suppose $1 < p < \infty$, $p \neq 2$. Let G be an infinite locally compact abelian group. There exists a bounded Fourier multiplier on $L^p(G)$ which is not completely bounded.*

Proof. The proof is similar to the ones of Proposition 3.8 and Theorem 3.10. The case of a discrete group of torsion needs some minor modifications. We prove it by a reasoning similar to the one used in the proof of Proposition 3.8 using the conditional expectation defined by (3.5) as an operator $\mathbb{E}_n \colon L^p([0,1]) \to L^p([0,1])$ and using the isometric map $J_n \colon \ell^p_n \to \mathbb{E}_n(L^p([0,1]))$ defined by

$$J_n(u) = n^{1/p} \sum_{k=0}^{n-1} u_k 1_{[k/n,(k+1)/n[}, \qquad u \in \ell^p_n. \ ■$$

REMARK 4.3. Using the fact that the space S^p does not have property (α) if $1 < p < \infty$, $p \neq 2$, and the method used at the beginning of the proof of Proposition 3.3, one can give a proof of Theorem 4.2 for the case $G = \mathbb{T}$. The more general case where G is an infinite compact abelian group can also be obtained with the method of the proof of Proposition 3.9.

REMARK 4.4. Recall the following classical result of S. Kwapień [K2]. Suppose $1 < p < \infty$. A Banach space X is isomorphic to an SQL^p-space, i.e. a subspace of a quotient of an L^p-space, if and only if for any measure space Ω and any bounded operator $T \colon L^p(\Omega) \to L^p(\Omega)$, the operator $T \otimes \mathrm{Id}_X \colon L^p(\Omega, X) \to L^p(\Omega, X)$ is bounded. The results of Section 3 and of this section lead to the following general open question. Let X be a Banach space and G be an infinite locally compact abelian group. Suppose we have a canonical isomorphism $M_p(G,X)=M_p(G)$; does there exist an isomorphism from the Banach space X onto an SQL^p-space?

5. Unconditionality and Schur multipliers. Suppose $1<p<\infty$. In this section, we use the notation $S^p_{\mathbb{Z}}=S^p(\ell^2_{\mathbb{Z}})$ and $S^p_{\mathbb{Z}\times\mathbb{N}}=S^p(\ell^2_{\mathbb{Z}\times\mathbb{N}})$. Recall that a *Schur multiplier* on S^p is a linear map $M_A\colon S^p\to S^p$ defined by a scalar matrix A such that $M_A(B)=[a_{ij}b_{ij}]$ belongs to S^p for any $B\in S^p$. We have a similar notion for $S^p_{\mathbb{Z}}$. Below, $(\varepsilon_{ij})_{i,j\geq 1}$ denotes a doubly indexed family of independent Rademacher variables.

The paper [LE] contains the following result:

THEOREM 5.1. *Let E be an operator space. Then E is completely isomorphic to the operator Hilbert space $\mathrm{OH}(I)$ for some index set I if and only if*

$$\Big\|\sum_{i,j=1}^{n} e_{ij}\otimes x_{ij}\Big\|_{S^2(E)}\approx\Big\|\sum_{i,j=1}^{n}\varepsilon_{ij}\otimes x_{ij}\Big\|_{\mathrm{Rad}(E)},\qquad n\in\mathbb{N},\ x_{ij}\in E.$$

First, we show a link between a property of the Banach space $S^2(E)$ and a property of the operator space E.

PROPOSITION 5.2. *Let E be an operator space. The following assertions are equivalent:*

- *The Banach space $S^2(E)$ is isomorphic to a Hilbert space.*
- *The operator space E is completely isomorphic to the operator Hilbert space $\mathrm{OH}(I)$ for some index set I.*

Proof. Suppose that $S^2(E)$ is isomorphic to a Hilbert space. By Lemma 2.2 we have

$$\Big\|\sum_{i,j=1}^{n} e_{ij}\otimes x_{ij}\Big\|_{S^2(E)}=\Big\|\sum_{i,j=1}^{n}\varepsilon_i\otimes\varepsilon_j\otimes e_{ij}\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(S^2(E)))},\quad n\in\mathbb{N},\ x_{ij}\in E.$$

Moreover, the Banach space $\mathrm{Rad}(S^2(E))$ is also isomorphic to a Hilbert space. Hence, for any $n\in\mathbb{N}$ and $x_{ij}\in E$,

$$\Big\|\sum_{i,j=1}^{n} e_{ij}\otimes x_{ij}\Big\|_{S^2(E)}\approx\Big(\sum_{i,j=1}^{n}\|e_{ij}\otimes x_{ij}\|^2_{S^2(E)}\Big)^{1/2}=\Big(\sum_{i,j=1}^{n}\|x_{ij}\|^2_E\Big)^{1/2}.$$

The space E is a closed subspace of $S^2(E)$. Hence it is isomorphic to a Hilbert space. Then we conclude that

$$\Big\|\sum_{i,j=1}^{n} e_{ij}\otimes x_{ij}\Big\|_{S^2(E)}\approx\Big\|\sum_{i,j=1}^{n}\varepsilon_{ij}\otimes x_{ij}\Big\|_{\mathrm{Rad}(E)},\qquad n\in\mathbb{N},x_{ij}\in E.$$

By Theorem 5.1, we deduce that E is completely isomorphic to $\mathrm{OH}(I)$ for some index set I. The reverse implication is obvious. ■

We need the next theorem [NR, Remark 3.1].

THEOREM 5.3. *Let E be an operator space and $\varphi\colon \mathbb{Z} \to \mathbb{C}$ be a function. Consider the infinite matrix $A = [\varphi_{i-j}]_{i,j\in\mathbb{Z}}$. If the map $M_A \otimes \mathrm{Id}_E$ is bounded on $S^2_{\mathbb{Z}}(E)$ then $M_\varphi \otimes \mathrm{Id}_E$ is bounded on $L^2(\mathbb{T}, E)$ and*

$$\|M_\varphi \otimes \mathrm{Id}_E\|_{L^2(\mathbb{T},E)\to L^2(\mathbb{T},E)} \leq \|M_A \otimes \mathrm{Id}_E\|_{S^2_{\mathbb{Z}}(E)}.$$

The following result shows that if the matrix units form an 'unconditional system' of $S^2(E)$ then the operator space E is completely isomorphic to an operator Hilbert space.

THEOREM 5.4. *Let E be an operator space. The following assertions are equivalent:*

- *The Banach space $S^2(E)$ has property (α).*
- *There exists a positive constant C such that*

$$\Big\|\sum_{i,j=1}^{n} t_{ij} e_{ij} \otimes x_{ij}\Big\|_{S^2(E)} \leq C \sup_{1\leq i,j\leq n} |t_{ij}| \Big\|\sum_{i,j=1}^{n} e_{ij} \otimes x_{ij}\Big\|_{S^2(E)} \tag{5.1}$$

  for any $n \in \mathbb{N}$, $t_{ij} \in \mathbb{C}$ and $x_{ij} \in E$.
- *The operator space E is completely isomorphic to the operator Hilbert space $\mathrm{OH}(I)$ for some index set I.*

Proof. Suppose that $S^2(E)$ has property (α). For any $n \in \mathbb{N}$, $y_{ij} \in S^2(E)$ and $t_{ij} \in \mathbb{C}$ we have

$$\Big\|\sum_{i,j=1}^{n} t_{ij} \varepsilon_i \otimes \varepsilon_j \otimes y_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(S^2(E)))} \lesssim \sup_{1\leq i,j\leq n} |t_{ij}| \Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes y_{ij}\Big\|.$$

For any $1 \leq i, j \leq n$, choose $x_{ij} \in E$. Using $y_{ij} = e_{ij} \otimes x_{ij}$, we obtain

$$\Big\|\sum_{i,j=1}^{n} t_{ij} \varepsilon_i \otimes \varepsilon_j \otimes e_{ij} \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(S^2(E)))} \lesssim \sup_{1\leq i,j\leq n} |t_{ij}| \Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes e_{ij} \otimes x_{ij}\Big\|.$$

By Lemma 2.2, we conclude that

$$\Big\|\sum_{i,j=1}^{n} t_{ij} e_{ij} \otimes x_{ij}\Big\|_{S^2(E)} \lesssim \sup_{1\leq i,j\leq n} |t_{ij}| \Big\|\sum_{i,j=1}^{n} e_{ij} \otimes x_{ij}\Big\|_{S^2(E)}.$$

Now suppose that inequality (5.1) is true. Using the completely isometric isomorphisms

$$S^2_{\mathbb{Z}}(S^2(E)) = S^2_{\mathbb{Z}\times\mathbb{N}}(E) = S^2(E),$$

it is easy to see that

$$\Big\| \sum_{i,j=-n}^{n} t_{ij} e_{ij} \otimes x_{ij} \Big\|_{S^2_{\mathbb{Z}}(S^2(E))} \leq C \sup_{-n\leq i,j\leq n} |t_{ij}| \Big\| \sum_{i,j=-n}^{n} e_{ij} \otimes x_{ij} \Big\|_{S^2_{\mathbb{Z}}(S^2(E))} \tag{5.2}$$

for any integer n and any $t_{ij} \in \mathbb{C}$ and $x_{ij} \in S^2(E)$. Let $\varphi : \mathbb{Z} \to \mathbb{C}$ be a function with finite support. By (5.2), the map $M_A \otimes \mathrm{Id}_{S^2(E)}$ on $S^2_{\mathbb{Z}}(S^2(E))$ associated with the matrix $A = [\varphi_{i-j}]_{i,j\in\mathbb{Z}}$ is bounded with

$$\|M_A \otimes \mathrm{Id}_{S^2(E)}\|_{S^2_{\mathbb{Z}}(S^2(E))\to S^2_{\mathbb{Z}}(S^2(E))} \leq C.$$

Then, by Theorem 5.3, the map $M_\varphi \otimes \mathrm{Id}_{S^2(E)}$ is bounded on $L^2(\mathbb{T}, S^2(E))$ and

$$\|M_\varphi \otimes \mathrm{Id}_{S^2(E)}\|_{L^2(\mathbb{T},S^2(E))\to L^2(\mathbb{T},S^2(E))} \leq \|M_A \otimes \mathrm{Id}_{S^2_{\mathbb{Z}}(E)}\|_{S^2_{\mathbb{Z}}(S^2(E))\to S^2_{\mathbb{Z}}(S^2(E))} \leq C.$$

For any sequence (x_k) of elements of E, we deduce that

$$\Big\| \sum_{k=-\infty}^{\infty} \varphi(k) e^{2\pi i k\,\cdot} \otimes x_k \Big\|_{L^2(S^2(E))} \leq C \sup_{k\in\mathbb{Z}} |\varphi(k)| \Big\| \sum_{k=-\infty}^{\infty} e^{2\pi i k\,\cdot} \otimes x_k \Big\|_{L^2(S^2(E))}.$$

By Theorem 1.1, the Banach space $S^2(E)$ is isomorphic to a Hilbert space. Finally, by Theorem 5.2, the operator space E is completely isomorphic to $\mathrm{OH}(I)$ for some index set I.

The remaining implication is trivial. ■

Remark 5.5. The results of Section 4 raise the question of proving an analogous result for Schur multipliers. Indeed, in this context, G. Pisier conjectured that there exists a Schur multiplier which is bounded on S^p but not completely bounded if $1 < p < \infty$, $p \neq 2$ (see [P2, Conjecture 8.1.12]).

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