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NECESSARY CONDITION FOR KOSTYUCHENKO TYPE SYSTEMS TO BE A BASIS IN LEBESGUE SPACES

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Abstract. A necessary condition for Kostyuchenko type systems and system of powers to be a basis in L_p $(1 \le p < +\infty)$ spaces is obtained. In particular, we find a necessary condition for a Kostyuchenko system to be a basis in L_p $(1 \le p < +\infty)$.

1. Introduction. The system $S_\alpha^+ \equiv \{e^{i\alpha nt} \sin nt\}_{n\in\mathbb{N}}$ (where $\alpha \in \mathbb{C}$ is, in general, a complex number), usually called a *Kostyuchenko system*, arises in connection with the spectral problem

$$-y''(t) + 2\alpha\lambda y'(t) + (\alpha^2+1)\lambda^2 y(t) = 0, \qquad t \in (0,\pi),$$
$$y(0) = y(\pi) = 0.$$

Beginning from the paper [Dz] many papers have been dedicated to the investigation of basis properties of the system S_α^+ in $L_p(0,\pi), 1 \le p < +\infty$ ([Le], [LyT], [Ly1], [Ly2], [Shk1], [Shk2], [B1], [K]). The following results have been obtained in this direction: for $\alpha \in \mathbb{C} \setminus \{(-\infty,-1) \cup (1,+\infty)\}$ a criterion for the completeness and minimality of the system S_α^+ in $L_2(0,\pi)$ has been obtained ([Ly1], [Ly2]); in particular, the system is complete and minimal in $L_2(0,\pi)$ for all $\alpha \in i\mathbb{R}$. For $\alpha \in (-1,1)$ a criterion for this system to be a Riesz basis in $L_2(0,\pi)$ has been found (see [B1]). It is shown in [K] that when $\operatorname{Im}\alpha \ne 0$ the system S_α^+ is not uniformly minimal in $L_2(0,\pi)$ and therefore is not a basis in $L_2(0,\pi)$.

2. Necessary condition for Kostyuchenko type systems to be a basis in L_p spaces. In this note we consider the most general system

$$\{\varphi^n(t) \sin nt\}_{n\in\mathbb{N}} \tag{2.1}$$

where $\varphi : [a,b] \to \mathbb{C}$ is a measurable, a.e. finite function and a, b are some reals. A necessary condition for this system to be a basis in $L_p = L_p(a,b)$, $1 \le p < +\infty$, will be obtained. In particular, it follows from this result that S_α^+ is not a basis in $L_p, 1 \le p < +\infty$, when $\operatorname{Im}\alpha \ne 0$.

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Theorem 2.1. *If the system* (2.1) *is a basis in* L_p $(1 \le p < +\infty)$, *then* $|\varphi(t)| = \text{const}$ *a.e. on* $[a, b]$.

To prove this theorem we need the following lemma.

Lemma 2.2. *Let* $E \subset [a, b]$ *be a Lebesgue measurable subset of* $[a, b]$. *If there exists a sequence* $\{n_k\}$ *of natural numbers and a number* p $(1 \le p < +\infty)$ *such that*

$$\int_E |\sin n_k t|^p \, dt \to 0 \quad \text{as } k \to \infty, \tag{2.2}$$

then $\operatorname{mes} E = 0$.

Proof. Assume the contrary: there exists p_0 with $1 \le p_0 < +\infty$, a sequence $\{n_k\}$ of natural numbers and a set $E \subset [a, b]$ of positive measure for which (2.2) holds. Then there exists a subsequence $\{\sin m_k t\}$ of $\{\sin n_k t\}$ which converges to zero a.e. on E, i.e. there is a set $E' \subset E$ such that $\sin m_k t \to 0$ on E' as $k \to \infty$ and $\operatorname{mes} E' = \operatorname{mes} E > 0$. This contradicts the Cantor–Lebesgue theorem. ■

The next proposition is proved in the same way.

Lemma 2.3. *Let* $E \subset [a, b]$ *be a Lebesgue measurable subset of* $[a, b]$. *If there exists a sequence* $\{n_k\}$ *of natural numbers and a number* p $(1 \le p < +\infty)$ *such that*

$$\int_E |\cos n_k t|^p dt \to 0 \quad \text{as } k \to \infty,$$

then $\operatorname{mes} E = 0$.

Proof of Theorem 2.1. Since φ is a measurable function on $[a, b]$, by Denjoy's theorem [N, p. 247], it is approximately continuous a.e. on $[a, b]$. Denote by A' the set of points of approximate continuity and put $A = A' \setminus \{a, b\}$.

Every function $f \in L_p$ has an expansion (in L_p norm)

$$f(t) = \sum_{n=1}^{\infty} a_n \varphi^n(t) \sin nt. \tag{2.3}$$

Consider the power series $\sum_{n=1}^{\infty} a_n z^n$. Let R_0 be its radius of convergence. We assert that $R_0 \ge R$, where $R = \sup_A |\varphi(t)|$. Suppose $R_0 < R$. Then there exists $t_0 \in A$ such that $|\varphi(t_0)| > R_0$. Since φ is approximately continuous at t_0, there is a set $E_0 \subset [a, b]$ and a number $\delta_0 > 0$ such that $\operatorname{mes} E(t_0, \delta_0) > 0$ and

$$|\varphi(t)| > R' > R_0 \quad \text{for } t \in E(t_0, \delta_0), \tag{2.4}$$

where $E(t_0, \delta_0) = E_0 \cap [t_0 - \delta_0, t_0 + \delta_0]$ and R' is some fixed number.

There exists a sequence $\{n_k\}$ of natural numbers for which

$$R_0 = \lim_{k\to\infty} \frac{1}{|a_{n_k}|^{1/n_k}}.$$

Taking into account (2.4), we have

$$|\varphi(t)| > \frac{1}{|a_{n_k}|^{1/n_k}},$$

i.e.

$$|a_{n_k}\varphi^{n_k}(t)| > 1, \qquad t \in E(t_0, \delta_0), \tag{2.5}$$

for sufficiently large k.

It is obvious that $\|a_n\varphi^n(t)\sin nt\|_{L_p(a,b)} \to 0$ as $n \to \infty$. Then we also have $\|a_n\varphi^n(t)\sin nt\|_{L_p(E(t_0,\delta_0))} \to 0$, as $n \to \infty$. Therefore, according to (2.5), $\|\sin n_k t\|_{L_p(E(t_0,\delta_0))} \to 0$ as $k \to \infty$. This contradicts Lemma 2.2. So $R_0 \ge R$.

We write the system (2.1) in the form

$$\varphi^n(t)\sin nt = \frac{1}{2i}(\varphi(t)e^{it})^n - \frac{1}{2i}(\varphi(t)e^{-it})^n.$$

Thus

$$f(t) = \frac{1}{2i}\sum_{n=1}^{\infty} a_n((\varphi(t)e^{it})^n - (\varphi(t)e^{-it})^n).$$

To prove the theorem it suffices to show that

$$R = \sup_A |\varphi(t)| = \inf_A |\varphi(t)|.$$

Assume the contrary: $R = \sup_A |\varphi(t)| > \inf_A |\varphi(t)|$. Then there exist $t_1 \in A$ and $R'' > 0$ such that $|\varphi(t_1)| < R'' < R_0$ (since $R \le R_0$). Since φ is approximately continuous at t_1, there exists a set $E_1 \subset [a,b]$ and $\delta_1 > 0$ such that $\operatorname{mes} E(t_1,\delta_1) > 0$ and $|\varphi(t)| < R'' < R_0$ for $t \in E(t_1,\delta_1) = E_1 \cap [t_1-\delta_1, t_1+\delta_1]$. Thus

$$|\varphi(t)e^{\pm it}| < R'' < R_0 \qquad \text{for } t \in E(t_1,\delta_1).$$

Therefore the series $\sum_{n=1}^{\infty} a_n(\varphi(t)e^{it})^n$ and $\sum_{n=1}^{\infty} a_n(\varphi(t)e^{-it})^n$ are uniformly convergent on $E(t_1,\delta_1)$ and so the function

$$\begin{aligned} F(t) &= \frac{1}{2i}\sum_{n=1}^{\infty} a_n((\varphi(t)e^{it})^n - (\varphi(t)e^{-it})^n) \\ &= \frac{1}{2i}\sum_{n=1}^{\infty} a_n(\varphi(t)e^{it})^n - \frac{1}{2i}\sum_{n=1}^{\infty} a_n(\varphi(t)e^{-it})^n \end{aligned}$$

is continuous at t_1 along the set $E(t_1,\delta_1)$. Since uniform convergence implies convergence in L_p, $f(t) = F(t)$ a.e. on $E(t_1,\delta_1)$. So we find that the restriction of every function $f \in L_p$ is equivalent to a function that is continuous at t_1 along $E(t_1, \delta_1)$ (note that the choice of $E(t_1, \delta_1)$ does not depend on $f \in L_p$). This is a contradiction, since the restriction of the function

$$f_0(t) = \begin{cases} 0 & \text{for } t \in [a, t_1], \\ 1 & \text{for } t \in [t_1, b] \end{cases}$$

is not equivalent to a function that is continuous at t_1 along $E(t_1, \delta_1)$. ∎

Using Lemma 2.3 instead of Lemma 2.2, it is easy to see that a similar proposition is true for the system

$$\{\varphi^n(t) \cos nt\}_{n \in \mathbb{N} \cup \{0\}}. \tag{2.6}$$

THEOREM 2.4. *If the system* (2.6) *is a basis in* L_p $(1 \le p < +\infty)$*, then* $|\varphi(t)| = \text{const}$ *a.e. on* $[a, b]$.

3. Necessary condition for Kostyuchenko systems to be a basis. In particular, for the systems S_α^+ and $C_\alpha^+ \equiv \{e^{i\alpha nt} \cos nt\}_{n \in \mathbb{N} \cup \{0\}}$ we obtain the following

COROLLARY 3.1. *If* $\operatorname{Im} \alpha \neq 0$*, then the systems* S_α^+ *and* C_α^+ *are not bases in* L_p*,* $1 \le p < +\infty$*.*

Proof. Indeed, if $\operatorname{Im} \alpha \neq 0$, then it is obvious that $|e^{i\alpha t}| \not\equiv \text{const}$ and the corollary follows directly from Theorems 2.1 and 2.4. ∎

4. Necessary condition for a system of powers to be a basis. The proof of Theorem 2.1 with minor changes allows us to prove the following

THEOREM 4.1. *If the system* $\{\varphi^n(t)\}_{n \in \mathbb{N}}$ *(or* $\{\varphi^n(t)\}_{n \in \mathbb{N} \cup \{0\}}$*) is a basis in* L_p $(1 \le p < +\infty)$*, then* $|\varphi(t)| = const$ *a.e. on* $[a, b]$.

A similar result for double systems of powers was obtained in [B2]. But the result there is proved under strong restrictions on φ and is not applicable to L_p spaces when $p \neq 2$; a similar result for double systems of powers is also obtained in [Sh] under the condition that φ is continuous.

Note that Theorem 4.1 generalizes a result from [AG, p. 52].

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