COLLOQUIUM MATHEMATICUM

VOL. 127 2012 NO. 1

ON SELF-INJECTIVE ALGEBRAS OF FINITE REPRESENTATION TYPE

BY

MARTA BŁASZKIEWICZ and ANDRZEJ SKOWROŃSKI (Toruń)

Abstract. We describe the structure of finite-dimensional self-injective algebras of finite representation type over a field whose stable Auslander–Reiten quiver has a sectional module not lying on a short chain.

Introduction. Throughout the paper, by an *algebra* we mean a basic indecomposable finite-dimensional associative K-algebra with an identity over a (fixed) field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules, and by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$. We denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander–Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We will not distinguish between an indecomposable module in $\operatorname{mod} A$ and the vertex of Γ_A corresponding to it. An algebra A is called *self-injective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective modules in $\operatorname{mod} A$ are injective. In the representation theory of self-injective algebras an important role is played by the self-injective algebras A which admit Galois coverings of the form $\widehat{B} \to \widehat{B}/G = A$, where $\widehat{B}$ is the repetitive category of an algebra B and G is an admissible group of automorphisms of $\widehat{B}$ (see [22], [29]).

We are concerned with the problem of describing the Morita equivalence classes of self-injective algebras of finite representation type, that is, the self-injective algebras A for which $\operatorname{mod} A$ admits only finitely many indecomposable modules up to isomorphism. For K algebraically closed, the problem was solved in the early 1980's by Riedtmann (see [4], [16], [17], [18]) via the combinatorial classification of the Auslander–Reiten quivers of self-injective algebras of finite representation type over K. Equivalently, Riedtmann's classification can be presented as follows (see [22, Section 3]): a non-simple self-injective algebra A over an algebraically closed field K is of finite representation type if and only if A is a socle deformation of an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Dynkin type $\mathbb{A}_n$ $(n \geq 1)$,

2010 *Mathematics Subject Classification*: Primary 16D50, 16G10, 16G60.

Key words and phrases: self-injective algebra, tilted algebra, repetitive algebra, Auslander–Reiten quiver, short chain.

$\mathbb{D}_n$ ($n \geq 4$), $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$. It was conjectured in [29, Problem 2.4] that a non-simple self-injective algebra A over an arbitrary field K is of finite representation type if and only if A is a socle deformation of an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Dynkin type $\mathbb{A}_n$ ($n \geq 1$), $\mathbb{B}_n$ ($n \geq 2$), $\mathbb{C}_n$ ($n \geq 3$), $\mathbb{D}_n$ ($n \geq 4$), $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ or $\mathbb{G}_2$. This is currently an exciting open problem. An important known result towards solution of this problem is the Riedtmann–Todorov description of the stable Auslander–Reiten quivers of self-injective algebras of finite representation type over an arbitrary field (see [16], [31], [30, Section IV.15]). We also refer to [28] for related results on stable equivalences of self-injective algebras of finite representation type.

The main aim of the paper is to show that a non-simple self-injective algebra A of a finite representation type whose stable Auslander–Reiten quiver admits a section with good behaviour in the module category $\operatorname{mod} A$ is isomorphic to an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Dynkin type and G is an infinite cyclic group of automorphisms of $\widehat{B}$.

For basic background on the representation theory applied in this paper we refer to [1] and [30].

1. The main result and related background. Let B be an algebra and $1_B = e_1 + \cdots + e_n$ a decomposition of the identity of B into a sum of pairwise orthogonal primitive idempotents. We associate to B a self-injective locally bounded K-category $\widehat{B}$, called the *repetitive category* of B (see [11], [20]). The objects of $\widehat{B}$ are $e_{m,i}$, $m \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$, and the morphism spaces are defined as follows:

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i, & r = m, \\ D(e_i B e_j), & r = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $e_j B e_i = \operatorname{Hom}_B(e_i B, e_j B)$, $D(e_i B e_j) = e_j D(B) e_i$ and

$$\bigoplus_{(r,i) \in \mathbb{Z} \times \{1,\ldots,n\}} \widehat{B}(e_{m,i}, e_{r,j}) = e_j B \oplus D(B e_j)$$

for any $r \in \mathbb{Z}$ and $j \in \{1, \ldots, n\}$. We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i} \quad \text{for all } (m,i) \in \mathbb{Z} \times \{1, \ldots, n\}.$$

An automorphism φ of the K-category $\widehat{B}$ is said to be:

- *positive* if for each pair $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$ we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1, \ldots, n\}$;

- *rigid* if for each pair $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$;
- *strictly positive* if it is positive but not rigid.

Then the automorphisms $\nu_{\widehat{B}}^r$, $r \geq 1$, are strictly positive automorphisms of $\widehat{B}$.

A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if G acts freely on the set of objects of $\widehat{B}$ and has finitely many orbits. Then we may consider the orbit category $\widehat{B}/G$ of $\widehat{B}$ with respect to G whose objects are the G-orbits of objects in $\widehat{B}$, and the morphism spaces are given by

$$(\widehat{B}/G)(a,b) = \Big\{ f_{y,x} \in \prod_{(x,y)\in a\times b} \widehat{B}(x,y) \,\Big|\, g f_{y,x} = f_{gy,gx},\ \forall_{g\in G,\,(x,y)\in a\times b} \Big\}$$

for all objects a, b of $\widehat{B}/G$. Since $\widehat{B}/G$ has finitely many objects and the morphism spaces in $\widehat{B}/G$ are finite-dimensional, we have the associated finite-dimensional, self-injective K-algebra $\bigoplus(\widehat{B}/G)$ which is the direct sum of all morphism spaces in $\widehat{B}/G$, called the *orbit algebra* of $\widehat{B}$ with respect to G. We will identify $\widehat{B}/G$ with $\bigoplus(\widehat{B}/G)$. For example, for each positive integer r, the infinite cyclic group $(\nu_{\widehat{B}}^r)$ generated by the rth power $\nu_{\widehat{B}}^r$ of $\nu_{\widehat{B}}$ is an admissible group of automorphisms of $\widehat{B}$, and we have the associated self-injective orbit algebra

$$T(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r) = \left\{ \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ f_2 & b_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & f_3 & b_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_{r-1} & b_{r-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \dots, b_{r-1} \in B,\ f_1, \dots, f_{r-1} \in D(B) \right\},$$

called the *r-fold trivial extension algebra of B*. In particular, $T(B)^{(1)} \cong T(B) = B \ltimes D(B)$ is the trivial extension of B by the injective cogenerator $D(B)$.

Let H be a hereditary algebra and Q_H its valued quiver. Following [3], [9], a module T in $\operatorname{mod} H$ is called a *tilting module* if $\operatorname{Ext}_H^1(T,T) = 0$ and T is a direct sum of n pairwise non-isomorphic, indecomposable modules, where n is the rank of the Grothendieck group $K_0(H)$ of H (equivalently, the number of vertices of Q_H). Then the endomorphism algebra $B = \operatorname{End}_H(T)$ is called a *tilted algebra* of H. Further, the images $\operatorname{Hom}_H(T, I)$ of indecomposable injective modules I in $\operatorname{mod} H$ via the functor $\operatorname{Hom}_H(T,-)\colon \operatorname{mod} H \to \operatorname{mod} B$ form a section Δ_T of a connected component $\mathcal{C}_T$ of Γ_B, called the *connecting component* of Γ_B determined by T, which connects the torsion-free part $\mathcal{Y}(T) = \{Y \in \operatorname{mod} B \mid \operatorname{Tor}_1^B(Y,T) = 0\}$ and the torsion part $\mathcal{X}(T) = \{X \in \operatorname{mod} B \mid X \otimes_B T = 0\}$ (see [9]). Moreover, by a criterion of Liu–Skowroński (see [14], [21]), an algebra B is a tilted algebra of a hereditary algebra H if and only if the Auslander–Reiten quiver Γ_B of B admits a connected component $\mathcal{C}$ having a faithful section Δ such that $\operatorname{Hom}_B(U, \tau_B V) = 0$ for all modules U, V from Δ.

Assume now that H is a hereditary algebra of finite representation type, or equivalently, Q_H is a Dynkin quiver (see [5], [6], [7]). Then for any tilting module T in $\operatorname{mod} H$, the associated tilted algebra $B = \operatorname{End}_H(T)$, called a *tilted algebra of Dynkin type*, is of finite representation type, and $\Gamma_B = \mathcal{C}_T$. Further, it follows from [10], [11] that the repetitive category $\widehat{B}$ of a tilted algebra B of Dynkin type is locally representation-finite in the sense of [8]. In particular, by a theorem of Gabriel [8, Theorem 3.6] the orbit algebra $A = \widehat{B}/G$ of $\widehat{B}$, with respect to an admissible infinite cyclic group G of automorphisms of $\widehat{B}$, is a self-injective algebra of finite representation type, and the stable Auslander–Reiten quiver Γ_A^s of A is the orbit quiver $\mathbb{Z}\Delta/G$, where $\Delta = Q_H$.

Let A be a non-simple self-injective algebra of finite representation type. Then by the Riedtmann–Todorov theorem (see [16], [31]) the stable Auslander–Reiten quiver Γ_A^s of A is isomorphic to the orbit quiver $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver and G is an infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$. Therefore, we may associate to any self-injective algebra A of finite representation type a Dynkin graph $\Delta(A)$, called the *Dynkin type* of A, such that $\Gamma_A^s = \mathbb{Z}\Delta/G$ for a quiver Δ having $\Delta(A)$ as underlying graph. We also note that $\mathbb{Z}\Delta = \mathbb{Z}\Delta'$ for any quivers Δ and Δ' having $\Delta(A)$ as underlying graph. A module M in $\operatorname{mod} A$ is said to be *sectional* if M is a direct sum of pairwise non-isomorphic indecomposable non-projective modules forming a connected full-valued subquiver Δ of Γ_A^s with $\Delta(A)$ as underlying graph. Finally, a sectional module M in $\operatorname{mod} A$ is said to be *pure* if no direct summand of M is the radical of a projective module in $\operatorname{mod} A$.

Let A be an algebra. Following [2], [15], a sequence $N \to M \to \tau_A N$ of non-zero homomorphisms in $\operatorname{mod} A$ with N indecomposable is called a *short chain*, and M is the *middle* of this chain. We mention that, if M is a module in $\operatorname{mod} A$ which is not the middle of a short chain, then every indecomposable direct summand Z of M is uniquely determined (up to isomorphism) by the simple composition factors (see [15, Corollary 2.2]). It has been recently proved in [12, Theorem] that an algebra B is a tilted algebra if and only if $\operatorname{mod} B$ contains a sincere module M which is not the middle of a short chain.

Recall that M is called *sincere* if every simple module in $\operatorname{mod} B$ occurs as a composition factor of M. We also refer to [13] for a description of finite-dimensional modules over algebras which are not the middle of a short chain of modules, using injective and tilting modules over hereditary algebras.

The aim of this paper is to prove the following theorem.

THEOREM 1.1. *Let A be a non-simple finite-dimensional basic indecomposable self-injective algebra of finite representation type over a field K. The following statements are equivalent:*

(i) $\operatorname{mod} A$ *admits a pure sectional module M which is not the middle of a short chain.*
(ii) *A is isomorphic to a self-injective orbit algebra $\widehat{B}/(\rho\nu_{\widehat{B}}^2)$, where B is a tilted algebra of the form $B = \operatorname{End}_H(T)$ with H a hereditary algebra of Dynkin type and T is a tilting module in* $\operatorname{mod} H$ *without indecomposable projective direct summands, and ρ is a positive automorphism of $\widehat{B}$.*

We note that the module category $\operatorname{mod} H$ of a hereditary algebra H of Dynkin type admits a tilting module T without indecomposable projective direct summands if and only if H is not a Nakayama algebra, or equivalentely, the quiver Q_H of H is not an equioriented quiver

$$\bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet \to \bullet$$

of type $\mathbb{A}_n$ $(n \geq 1)$.

2. Self-injective algebras of Dynkin type. Let B be a triangular algebra (the quiver Q_B has no oriented cycles) and $e_1, \ldots, e_n$ be pairwise orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$. We identify B with the full subcategory B_0 of the repetitive category $\widehat{B}$ given by the objects $e_{0,i}$, $1 \leq i \leq n$. For a sink i of Q_B, the *reflection* $S_i^+ B$ of B at i is the full subcategory of $\widehat{B}$ given by the objects

$$e_{0,j}, \quad 1 \leq j \leq n, \quad j \neq i, \quad \text{and} \quad e_{1,i} = \nu_{\widehat{B}}(e_{0,i}).$$

Then the quiver $Q_{S_i^+ B}$ of $S_i^+ B$ is the reflection $\sigma_i^+ Q_B$ of Q_B at i (see [11]). Observe that $\widehat{B} = \widehat{S_i^+ B}$. By a *reflection sequence of sinks* of Q_B we mean a sequence $i_1, \ldots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \ldots \sigma_{i_1}^+ Q_B$ for all s in $\{1, \ldots, t\}$. Moreover, for a sink i of Q_B, we denote by $T_i^+ B$ the full subcategory of $\widehat{B}$ given by the objects

$$e_{0,j}, \quad 1 \leq j \leq n, \quad \text{and} \quad e_{1,i} = \nu_{\widehat{B}}(e_{0,i}).$$

Observe that $T_i^+ B$ is the one-point extension $B[I_B(i)]$ of B by the indecomposable injective B-module $I_B(i)$ at the vertex i. By a *finite-dimensional*

$\widehat{B}$*-module* we mean a contravariant K-linear functor M from $\widehat{B}$ to the category of K-vector spaces such that $\sum_{x\in \mathrm{ob}\,\widehat{B}} \dim_K M(x)$ is finite. We denote by $\mathrm{mod}\,\widehat{B}$ the category of all finite-dimensional $\widehat{B}$-modules. Finally, for a module M in $\mathrm{mod}\widehat{B}$, we denote by $\mathrm{supp}(M)$ the full subcategory of $\widehat{B}$ formed by all objects x with $M(x) \neq 0$, and call it the *support of* M.

The following consequence of results proved in [10], [11] describes the supports of finite-dimensional indecomposable modules over the repetitive categories $\widehat{B}$ of tilted algebras B of Dynkin type.

THEOREM 2.1. *Let B be a tilted algebra of Dynkin type and n the rank of $K_0(B)$. Then there exists a reflection sequence $i_1,\dots,i_n$ of sinks of Q_B such that the following statements hold:*

(i) $S^+_{i_n}\dots S^+_{i_1} = \nu_{\widehat{B}}(B)$.

(ii) *For every indecomposable non-projective module M in* $\mathrm{mod}\,\widehat{B}$, $\mathrm{supp}(M)$ *is contained in one of the full subcategories of $\widehat{B}$ given by*
$$\nu^m_{\widehat{B}}(S^+_{i_r}\dots S^+_{i_1}B), \qquad r\in\{1,\dots,n\},\ m\in\mathbb{Z}.$$

(iii) *For every indecomposable projective module P in* $\mathrm{mod}\,\widehat{B}$, $\mathrm{supp}(P)$ *is contained in one of the full subcategories of $\widehat{B}$ given by*
$$\nu^m_{\widehat{B}}(T^+_{i_r}S^+_{i_{r-1}}\dots S^+_{i_1}B), \qquad r\in\{1,\dots,n\},\ m\in\mathbb{Z}.$$

The aim of this section is to prove the following theorem playing a prominent role in the proof of Theorem 1.1.

THEOREM 2.2. *Let B be a tilted algebra* $\mathrm{End}_H(T)$ *of Dynkin type, Δ_T the canonical section of Γ_B given by the images* $\mathrm{Hom}_H(T,I)$ *of indecomposable injective H-modules I via the functor* $\mathrm{Hom}_H(T,-)\colon \mathrm{mod}\,H \to \mathrm{mod}\,B$, *and M_T the direct sum of indecomposable B-modules lying on Δ_T. Moreover, let φ be a strictly positive automorphism of $\widehat{B}$, $A = \widehat{B}/(\varphi)$, and $F^\varphi_\lambda\colon$* $\mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,A$ *the associated push-down functor. The following statements are equivalent:*

(i) $F^\varphi_\lambda(M_T)$ *is not the middle of a short chain in* $\mathrm{mod}\,A$.

(ii) $\varphi = \rho\nu^2_{\widehat{B}}$ *for a positive automorphism ρ of $\widehat{B}$.*

Proof. It follows from Theorem 2.1 that $\widehat{B}$ is a locally representation-finite locally bounded category [8], that is, for any indecomposable module N in $\mathrm{mod}\,\widehat{B}$ the number of objects x in $\widehat{B}$ with $N(x)\neq 0$ is finite. Then, applying [8, Theorem 3.6], the push-down functor $F^\varphi_\lambda\colon \mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,A$ is a Galois covering of module categories preserving almost split sequences. In particular, for any indecomposable modules X and Y in $\mathrm{mod}\,\widehat{B}$, $F^\varphi_\lambda(X)$ and $F^\varphi_\lambda(Y)$ are indecomposable modules in $\mathrm{mod}\,A$, and F^φ_λ induces K-linear isomorphisms

$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}(X, \varphi^r Y) \xrightarrow{\sim} \operatorname{Hom}_A(F_\lambda^\varphi(X), F_\lambda^\varphi(Y)),$$

$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}(\varphi^r X, Y) \xrightarrow{\sim} \operatorname{Hom}_A(F_\lambda^\varphi(X), F_\lambda^\varphi(Y)).$$

Here, $\varphi^r X$ and $\varphi^r Y$ denote the shifts of X and Y by the automorphism of $\operatorname{mod} \widehat{B}$ induced by φ^r.

Assume that $F_\lambda^\varphi(M_T)$ is the middle of a short chain in $\operatorname{mod} A$. Then there is an indecomposable non-projective module N in $\operatorname{mod} A$, indecomposable direct summands U and V of $F_\lambda^\varphi(M_T)$, and non-zero homomorphisms $N \to U$ and $V \to \tau_A N$. Therefore, there exist indecomposable direct summands X and Y of M_T, an indecomposable non-projective module Z in $\operatorname{mod} \widehat{B}$, and non-zero homomorphisms $Y \to \tau_{\widehat{B}} Z$ and $Z \to \varphi^r X$ in $\operatorname{mod} \widehat{B}$ with $r \geq 1$ such that $F_\lambda^\varphi(X) = F_\lambda^\varphi(\varphi^r X) = U$, $F_\lambda^\varphi(Y) = V$, and $F_\lambda^\varphi(Z) = N$. Observe that for modules L, L' in $\operatorname{mod} \widehat{B}$, $\operatorname{Hom}_{\widehat{B}}(L, L') \neq 0$ implies that $\operatorname{supp}(L)$ and $\operatorname{supp}(L')$ have a common object. Since $\operatorname{supp}(M_T) = B = B_0$ and Y is a direct summand of M_T, we conclude that $\operatorname{supp}(Y)$ is contained in B. Similarly, $\varphi^r X$ is a direct summand of $\varphi^r M_T$ and $\operatorname{supp}(\varphi^r M_T) = \varphi^r B$, and so $\operatorname{supp}(\varphi^r X)$ is contained in $\varphi^r B$. Applying now Theorem 2.1, we infer that $\operatorname{supp}(\tau_{\widehat{B}} Z)$ is contained in B or one of the full subcategories $S_{i_p}^+ \dots S_{i_1}^+ B$ for some $p \in \{1, \dots, n-1\}$ and the corresponding reflection sequence $i_1, \dots, i_n$ of sinks of Q_B. Note that $B = \nu_{\widehat{B}}^{-1}(\nu_{\widehat{B}}(B)) = \nu_{\widehat{B}}^{-1}(S_{i_n}^+ \dots S_{i_1}^+ B)$. Then it follows that $\operatorname{supp}(Z)$ is contained in $S_{i_p}^+ \dots S_{i_1}^+ B$ or in $S_{i_n}^+ \dots S_{i_1}^+ B = \nu_{\widehat{B}}(B)$ (if $p = n-1$). Hence $\operatorname{Hom}_{\widehat{B}}(Z, \varphi^r X) \neq 0$ forces that $\operatorname{supp}(\varphi^r X)$ is contained in a full subcategory of $\widehat{B}$ of one of the forms $S_{i_r}^+ \dots S_{i_1}^+ B$ for $r \in \{1, \dots, n\}$, or $\nu_{\widehat{B}}(S_{i_q}^+ \dots S_{i_1}^+ B)$ for $q \in \{1, \dots, n-1\}$. This shows that $\operatorname{supp}(\varphi^r X) = \varphi^r(\operatorname{supp}(X))$ is contained in the full subcategory $T_{i_n}^+ \dots T_{i_1}^+ B$ of $\widehat{B}$ given by the objects of B and $\nu_{\widehat{B}}(B)$. Summing up, we have proved that if $\varphi = \rho\nu_{\widehat{B}}^2$ for a positive automorphism ρ of $\widehat{B}$, then $F_\lambda^\varphi(M_T)$ is not the middle of a short chain in $\operatorname{mod} A$. Therefore, (ii) implies (i).

Assume now that φ is not of the form $\rho\nu_{\widehat{B}}^2$ for a positive automorphism ρ of $\widehat{B}$. Then φB is a full subcategory of $T_{i_n}^+ \dots T_{i_1}^+ B$ of $\widehat{B}$ given by the objects of B and $\nu_{\widehat{B}}(B)$. Take an indecomposable direct summand X of M_T. Then φX is an indecomposable direct summand φM_T, and so $\operatorname{supp}(\varphi X)$ is a full subcategory of $\operatorname{supp}(\varphi M_T) = \varphi(\operatorname{supp} M_T) = \varphi B$. Thus $\operatorname{supp}(\varphi X)$ is a full subcategory of $T_{i_n}^+ \dots T_{i_1}^+ B$. We have two cases to consider.

Assume first that $\operatorname{supp}(\varphi X)$ contains an object j which is not in B. Then $j = \nu_{\widehat{B}}(i)$ for some object i of B. Take the indecomposable projective-injective $\widehat{B}$-module $P_{\widehat{B}}(j)$ at j. Clearly, we have $\mathrm{Hom}_{\widehat{B}}(P_{\widehat{B}}(j), \varphi X) \neq 0$. In fact, since X is not a projective-injective $\widehat{B}$-module, φX is not a projective-injective $\widehat{B}$-module, and hence $\mathrm{Hom}_{\widehat{B}}(P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j), \varphi X) \neq 0$. Clearly then $\mathrm{Hom}_{\widehat{B}}(P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j), \varphi M_T) \neq 0$. Observe also that we have in $\mathrm{mod}\, B$ a canonical almost split sequence

$$0 \to \mathrm{rad}\, P_{\widehat{B}}(j) \to (\mathrm{rad}\, P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j)) \oplus P_{\widehat{B}}(j) \to P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j) \to 0,$$

and then $\mathrm{rad}\, P_{\widehat{B}}(j) = \tau_{\widehat{B}}(P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j))$. Since $j = \nu_{\widehat{B}}(i)$ for some vertex i of Q_B, we conclude that $\mathrm{soc}\, P_{\widehat{B}}(j)$ is the simple $\widehat{B}$-module $S_{\widehat{B}}(i)$ at i, and consequently $\mathrm{Hom}_{\widehat{B}}(M_T, \mathrm{rad}\, P_{\widehat{B}}(j)) \neq 0$. This shows that $F_\lambda^\varphi(M_T) = F_\lambda^\varphi(\varphi M_T)$ is the middle of a short chain

$$F_\lambda^\varphi(P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j)) \to F_\lambda^\varphi(M_T) \to \tau_A F_\lambda^\varphi(P_{\widehat{B}}(j)/\mathrm{soc}\, P_{\widehat{B}}(j))$$

since $\tau_A F_\lambda^\varphi(L) \cong F_\lambda^\varphi(\tau_{\widehat{B}} L)$ for any indecomposable non-projective module L in $\mathrm{mod}\, \widehat{B}$.

Assume now that $\mathrm{supp}(\varphi X)$ is contained in B. Since φ is a strictly positive automorphism of $\widehat{B}$, the support $\mathrm{supp}(\tau_{\widehat{B}}\varphi X)$ of $\tau_{\widehat{B}}\varphi X$ is also contained in B. Clearly, φX is an indecomposable $\widehat{B}$-module which is a successor of an indecomposable direct summand of M_T, because X is an indecomposable direct summand of M_T. Moreover, every indecomposable module in $\mathrm{mod}\, B$ is cogenerated or generated by M_T. Hence $\mathrm{Hom}_{\widehat{B}}(M_T, \tau_{\widehat{B}}\varphi X) = \mathrm{Hom}_B(M_T, \tau_{\widehat{B}}\varphi X) \neq 0$. This shows that $F_\lambda^\varphi(M_T)$ is the middle of a short chain in $\mathrm{mod}\, A$ of the form

$$F_\lambda^\varphi(X) \to F_\lambda^\varphi(M_T) \to \tau_A F_\lambda^\varphi(X)$$

because $F_\lambda^\varphi(X)$ is an indecomposable direct summand of $F_\lambda^\varphi(M_T)$ and $F_\lambda^\varphi(\tau_{\widehat{B}}\varphi X) \cong \tau_A F_\lambda^\varphi(\varphi X) \cong \tau_A F_\lambda^\varphi(X)$. Therefore, (i) implies (ii). ■

3. Self-injective algebras with deforming ideals. In this section we present criteria for self-injective algebras to be orbit algebras of the repetitive categories of algebras with respect to infinite cyclic automorphism groups, playing a fundamental role in the proof of the main theorem.

Let A be a self-injective algebra. For a subset X of A, we may consider the left annihilator $l_A(X) = \{a \in A \mid ax = 0\}$ of X in A and the right annihilator $r_A(X) = \{a \in A \mid xa = 0\}$ of X in A. Then by a theorem due to Nakayama (see [30, Theorem IV.6.10]) the annihilator operation l_A induces a Galois correspondence from the lattice of right ideals of A to the lattice of left ideals of A, and r_A is the inverse Galois correspondence to l_A. Let I be an ideal of A, $B = A/I$, and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $1_A = e_1 + \cdots + e_r$ with $e_1, \ldots, e_r$ pairwise orthogonal primitive idempotents of A, $e = e_1 + \cdots + e_n$ for some $n \leq r$, and $\{e_i \mid 1 \leq i \leq n\}$ is the set of all idempotents in $\{e_i \mid 1 \leq i \leq r\}$ which are not in I. Then such an idempotent e is uniquely determined by I up to an inner automorphism of A, and is called a *residual identity* of $B = A/I$. Observe also that $B \cong eAe/eIe$.

We have the following lemma from [27, Lemma 5.1].

LEMMA 3.1. *Let A be a self-injective algebra, I an ideal of A, and e an idempotent of A such that $l_A(I) = Ie$ or $r_A(I) = eI$. Then e is a residual identity of A/I.*

We also recall the following proposition proved in [23, Proposition 2.3].

PROPOSITION 3.2. *Let A be a self-injective algebra, I an ideal of A, $B = A/I$, e a residual identity of B, and assume that $IeI = 0$. The following conditions are equivalent:*

(i) *Ie is an injective cogenerator in* $\operatorname{mod} B$.
(ii) *eI is an injective cogenerator in* $\operatorname{mod} B^{\mathrm{op}}$.
(iii) $l_A(I) = Ie$.
(iv) $r_A(I) = eI$.

Moreover, under these equivalent conditions, we have $\operatorname{soc} A \subseteq I$ *and* $l_{eAe}(I) = eIe = r_{eAe}(I)$.

The following theorem proved in [25, Theorem 3.8] (sufficiency part) and [27, Theorem 5.3] (necessity part) will be fundamental for our considerations.

THEOREM 3.3. *Let A be a self-injective algebra. The following conditions are equivalent:*

(i) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra and φ is a positive automorphism of $\widehat{B}$.*
(ii) *There is an ideal I of A and an idempotent e of A such that*
   (1) $r_A(I) = eI$;
   (2) *the canonical algebra epimorphism $eAe \to eAe/eIe$ is a retraction.*

Moreover, in this case, B is isomorphic to A/I.

Let A be an algebra, I an ideal of A, and e a residual identity of A/I. Following [23], I is said to be a *deforming ideal* of A if the following conditions are satisfied:

(D1) $l_{eAe}(I) = eIe = r_{eAe}(I)$;
(D2) the valued quiver $Q_{A/I}$ of A/I is acyclic.

Assume I is a deforming ideal of A. Then we have a canonical isomorphism of algebras $eAe/eIe \to A/I$ and I can be considered as an (eAe/eIe)-(eAe/eIe)-bimodule. Denote by $A[I]$ the direct sum of K-vector spaces

$(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (c, y) = (bc, by + xc + xy)$$

for $b, c \in eAe/eIe$ and $x, y \in I$. Then $A[I]$ is a K-algebra with the identity $(e + eIe, 1_A - e)$, and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$. Observe that $e = (e + eIe, 0)$ is a residual identity of $A[I]/I = eAe/eIe \xrightarrow{\sim} A/I$, $eA[I]e = (eAe/eIe) \oplus eIe$ and the canonical algebra epimorphism $eA[I]e \to eA[I]e/eIe$ is a retraction.

The following properties of the algebra $A[I]$ were established in [23, Theorem 4.1] and [24, Theorem 3].

THEOREM 3.4. *Let A be a self-injective algebra and I a deforming ideal of A. The following statements hold.*

(i) *$A[I]$ is a self-injective algebra with the same Nakayama permutation as A and I is a deforming ideal of $A[I]$.*
(ii) *A and $A[I]$ are socle equivalent.*
(iii) *A and $A[I]$ are stably equivalent.*

We note that if A is a self-injective algebra, I an ideal of A, $B = A/I$, e an idempotent of A such that $r_A(I) = eI$, and the valued quiver Q_B of B is acyclic, then by Lemma 3.1 and Proposition 3.2, I is a deforming ideal of A and e is a residual identity of B.

The following theorem proved in [25, Theorem 4.1] shows the importance of the algebras $A[I]$.

THEOREM 3.5. *Let A be a self-injective algebra, I an ideal of A, $B = A/I$ and e an idempotent of A. Assume that $r_A(I) = eI$ and Q_B is acyclic. Then $A[I]$ is isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

We point out that there are self-injective algebras A with deforming ideals I such that the algebras A and $A[I]$ are not isomorphic (see [25, Example 4.2]).

The following result proved in [26, Proposition 3.2] describes a situation when the algebras A and $A[I]$ are isomorphic.

THEOREM 3.6. *Let A be a self-injective algebra with a deforming ideal I, $B = A/I$, e be a residual identity of B and ν the Nakayama permutation of A. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$, for any primitive summand e_i of e. Then the algebras A and $A[I]$ are isomorphic. In particular, A is isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

4. Proof of Theorem 1.1. Let A be a non-simple, finite-dimensional, basic, indecomposable, self-injective K-algebra over a field K.

Assume $\operatorname{mod} A$ admits a pure sectional module M which is not the middle of a short chain. We will show first that A is socle equivalent to the self-injective orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a tilted algebra of the form $B = \operatorname{End}_H(T)$ for a hereditary algebra H of Dynkin type and a tilting module T in $\operatorname{mod} H$ without indecomposable projective direct summands, and φ is a positive automorphism of $\widehat{B}$. Let Δ be the full-valued subquiver of the stable Auslander–Reiten quiver Γ_A^{s} of given by the indecomposable direct summands of M. We recall that then $\Gamma_A^{\mathrm{s}} \cong \mathbb{Z}\Delta/G$ for an infinite cyclic group G of automorphisms of the translation quiver $\mathbb{Z}\Delta$, and Δ is a Dynkin quiver whose underlying graph is the Dynkin type $\Delta(A)$ of A. Let $I = r_A(M)$ and $B = A/I$. Then M is a faithful, hence sincere, right B-module which is not the middle of a short chain in $\operatorname{mod} B$, because M is not the middle of a short chain in $\operatorname{mod} A$ (see [15, Proposition 2.3]). So B is a tilted algebra, by the main result of [12]. Further, $H = \operatorname{End}_A(M) = \operatorname{End}_B(M)$ is the hereditary algebra, by [13, Corollary 1.2]. Clearly, H is then a hereditary algebra of Dynkin type with $Q_H = \Delta^{\mathrm{op}}$. Observe also that M is a faithful B-module with $\operatorname{Hom}_B(M, \tau_B M) = 0$, and hence $\operatorname{pd}_B(M) \le 1$ and $\operatorname{Ext}_B^1(M, M) \cong D\overline{\operatorname{Hom}}_B(M, \tau_B(M)) = 0$ (see [1, Lemma VIII.5.1 and Theorem IV.2.13]). Therefore, M is a partial tilting B-module. Since the rank of $K_0(B)$ coincides with the number of indecomposable direct summands of M, we conclude that M is a tilting B-module. Hence, by the Brenner–Butler theorem [1, Theorem VI.3.8], M is a tilting module in $\operatorname{mod} H^{\mathrm{op}}$, $T = D(M)$ is a tilting module in $\operatorname{mod} H$, $B \cong \operatorname{End}_H(T)$, and M is isomorphic to the right B-module $\operatorname{Hom}_H(T, D(H))$. In particular, we conclude that the indecomposable direct summands of M form the canonical section $\Delta_T = \Delta$ of the connecting component $\mathcal{C}_T = \Gamma_B$. Moreover, since M is a pure sectional module in $\operatorname{mod} A$, we find that no indecomposable injective B-module is a direct summand of M, or equivalently, the indecomposable direct summands of $\tau_B^{-1} M$ form another section $\tau_B^{-1}\Delta_T$ of $\mathcal{C}_T = \Gamma_B$. Finally, we note that T is a splitting tilting module in $\operatorname{mod} H$, since H is a hereditary algebra [1, Corollary VI.5.7]. Then, invoking the description of the indecomposable injective modules in $\operatorname{mod} B$, given in [1, Proposition VI.5.8], and $M \cong \operatorname{Hom}_H(T, D(H))$, we conclude that T has no indecomposable projective direct summand.

Let $e_1, \dots, e_r$ be a set of pairwise orthogonal, primitive idempotents of A such that $1_A = e_1 + \dots + e_r$ and that $e = e_1 + \dots + e_n$, for some $n \le r$, is a residual identity of B. We claim that I is a deforming ideal of A satisfying $IeI = 0$. Observe that the valued quiver Q_B of $B = A/I$ is acyclic, because B is a tilted algebra. Therefore, by Proposition 3.2, it remains to show that $r_A(I) = eI$.

Denote by J the trace ideal of M in A, that is, the ideal of A generated by the images of all homomorphisms from M to A in $\operatorname{mod} A$, and by J'

the trace ideal of the left A-module $D(M)$ in A. Observe that I is the left annihilator of $D(M)$ in A.

LEMMA 4.1. *We have* $J \cup J' \subseteq I$.

Proof. First we show that $J \subseteq I$. By definition, there exists an epimorphism $\varphi\colon M^r \to J$ for some integer $r \geq 1$. Suppose that there exists a homomorphism $f\colon A \to M$ in $\operatorname{mod} A$ with $f(J) \neq 0$. Since M has no projective-injective indecomposable direct summands, the homomorphism f factors through $A/\operatorname{soc} A$. Hence we have in $\operatorname{mod} A$ a sequence of homomorphisms

$$M^r \xrightarrow{\varphi} J \xrightarrow{\omega} A \xrightarrow{\pi} A/\operatorname{soc} A \xrightarrow{g} M$$

with $g\pi\omega\varphi \neq 0$, where $\omega\colon J \to A$ is the canonical inclusion homomorphism, $\pi\colon A \to A/\operatorname{soc} A$ is the canonical epimorphism, and $f = g\pi$. Observe that $g\pi\omega\varphi$ factors through a module from $\operatorname{add}(\tau_A^{-1}M)$, and consequently $\operatorname{Hom}_A(\tau_A^{-1}M, M) \neq 0$. This is a contradiction because M is not the middle of a short chain in $\operatorname{mod} A$. Hence we conclude

$$J \subseteq \bigcap_{f\colon A_A \to M} \operatorname{Ker} f = I.$$

Suppose now that there is a homomorphism $f'\colon A \to D(M)$ in $\operatorname{mod} A^{\mathrm{op}}$ such that $f'(J') \neq 0$. Then f' factors through $A/\operatorname{soc} A$, because $D(M)$ has no projective-injective indecomposable direct summands. Moreover, we have in $\operatorname{mod} A^{\mathrm{op}}$ an epimorphism $\varphi'\colon D(M)^s \to J'$ for some integer $s \geq 1$. Hence we obtain in $\operatorname{mod} A^{\mathrm{op}}$ a sequence of homomorphisms

$$D(M)^s \xrightarrow{\varphi'} J' \xrightarrow{\omega'} A \xrightarrow{\pi} A/\operatorname{soc} A \xrightarrow{g'} D(M)$$

with $g'\pi\omega'\varphi' \neq 0$, where $\omega'\colon J' \to A$ is the canonical inclusion homomorphism and $f' = g'\pi$. Observe also that $g'\pi\omega'\varphi'$ factors through a module from $\operatorname{add}(\tau_{A^{\mathrm{op}}}^{-1}D(M))$, and consequently $\operatorname{Hom}_{A^{\mathrm{op}}}(\tau_{A^{\mathrm{op}}}^{-1}D(M), D(M)) \neq 0$. Since $\tau_{A^{\mathrm{op}}}^{-1}D(M) = \operatorname{Tr}M = D(\tau_A M)$, we conclude that $\operatorname{Hom}_A(M, \tau_A M) \neq 0$. This is again a contradiction, because M is not the middle of a short chain in $\operatorname{mod} A$. Therefore we obtain

$$J' \subseteq \bigcap_{f'\colon {}_A A \to D(M)} \operatorname{Ker} f' = I. \ \blacksquare$$

LEMMA 4.2. *We have* $l_A(I) = J$, $r_A(I) = J'$ *and* $I = r_A(J) = l_A(J')$.

Proof. We prove the lemma only for J, the proof for J' being dual. Since J is a right B-module, we have $JI = 0$, and hence $I \subseteq r_A(J)$. In order to show the converse inclusion, take a monomorphism $u\colon M \to A_A^t$ for some integer $t \geq 1$, and let $u_i\colon M \to A$ be the composite of u with the projection of A_A^t on the ith component. Then there is a monomorphism $v\colon M \to \bigoplus_{i=1}^t \operatorname{Im} u_i$ induced by u. Moreover, by definition of J, $\bigoplus_{i=1}^t \operatorname{Im} u_i$ is contained in $\bigoplus_{i=1}^t J$.

This leads to the inclusions

$$r_A(J) = r_A\Big(\bigoplus_{i=1}^{t} J\Big) \subseteq r_A(M) = I.$$

Hence $I = r_A(J)$. Finally, applying a theorem by Nakayama (see [30, Theorem IV.6.10]), we obtain $J = l_A r_A(J) = l_A(I)$. ∎

LEMMA 4.3. *We have $eIe = eJe = eJ'e$. In particular, $(eIe)^2 = 0$.*

Proof. Since e is a residual identity of $B = A/I$, we have $B \cong eAe/eIe$. Thus M is a faithful right eAe/eIe-module and the direct sum of indecomposable modules forming a section of $\Gamma_{eAe/eIe}$. Further, it follows from Lemma 4.1 that $eJe = eJ$ is an ideal of eAe with $eJe \subseteq eIe$. Consider the algebra $B' = eAe/eJe$. Then M is a sincere right B'-module which is not the middle of a short chain in $\operatorname{mod} B'$, because B' is a factor algebra of B and M is not the middle of a short chain in $\operatorname{mod} B$ [15, Proposition 2.3]. Applying [15, Corollary 3.2] we conclude that M is a faithful B'-module. This implies that $eIe/eJe = r_{B'}(M) = 0$, and hence $eIe = eJe$. In a similar way we show that $eIe = eJ'e$. Finally, it follows from Lemma 4.2 that $(eIe)^2 = (eJe)(eIe) = eJIe = 0$. ∎

We shall also use the following general lemma on almost split sequences over triangular matrix algebras (see [19, (2.5)], [23, Lemma 5.6]).

LEMMA 4.4. *Let R and S be algebras and N be an (S,R)-bimodule. Let $\Lambda = \left(\begin{smallmatrix} S & N \\ 0 & R \end{smallmatrix}\right)$ be the matrix algebra defined by the bimodule ${}_SN_R$. Then an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{mod} R$ is an almost split sequence in $\operatorname{mod} \Lambda$ if and only if $\operatorname{Hom}_R(N, X) = 0$.*

LEMMA 4.5. *Let f be a primitive idempotent in I such that $fJ \neq fAe$. Then $K = fAeAf + fJ + fAeAfAe + eAf + eIe$ is an ideal of $F = (e+f)A(e+f)$, and $N = fAe/fKe$ is a B-module such that $\operatorname{Hom}_B(N, M) = 0$ and $\operatorname{Hom}_B(M, N) \neq 0$.*

Proof. It follows from Lemma 4.3 that $fAeIe \subseteq fJ$. Then the fact that K is an ideal of F is a direct consequence of $f \in I$. Observe also that $fKe = fJ + fAeAfAe$, $fKf \subseteq \operatorname{rad}(fAf)$, $eKe = eIe$ and $eKf = eAf$. We have $N \neq 0$. Indeed, if $fAe = fKe$ then, since $eAfAe \subseteq \operatorname{rad}(eAe)$, we obtain $fAe = fJ + fAe(\operatorname{rad}(eAe))$, and so $fAe = fJ$ (Nakayama lemma, [30, Lemma I.3.3]), which contradicts our assumption. Further, $B = eAe/eIe$ and $(fAe)(eIe) = fAeJ \subseteq fJ \subseteq fKe$, and hence N is a B-module. Moreover, N is also a left module over $S = fAf/fKf$ and $\Lambda = F/K$ is isomorphic to the triangular matrix algebra $\left(\begin{smallmatrix} S & N \\ 0 & R \end{smallmatrix}\right)$. Invoking now our assumption that M is a pure sectional module in $\operatorname{mod} A$, we conclude that, for any indecomposable direct summand X of M, we have in $\operatorname{mod} B$ an almost split sequence $0 \to X \to Y \to Z \to 0$ which is also an almost split sequence in

$\operatorname{mod} A$, and so an almost split sequence in $\operatorname{mod} \Lambda$. Applying Lemma 4.4, we obtain $\operatorname{Hom}_B(N, M) = 0$. On the other hand, since every indecomposable module in $\operatorname{mod} B$ is either generated or cogenerated by M, we conclude that $\operatorname{Hom}_B(M, N) \neq 0$. ∎

Proposition 4.6. *We have $Ie = J$ and $eI = J'$.*

Proof. This follows exactly as [23, Proposition 5.9] by applying Lemmas 4.1, 4.2, 4.3, 4.5. ∎

The following direct consequence of Lemma 4.2 and Proposition 4.6 completes the proof that I is a deforming ideal of A with $IeI = 0$.

Corollary 4.7. *We have $r_A(I) = eI$ and $l_A(I) = Ie$.*

Applying Theorems 3.4 and 3.5 we conclude that:

(1) A is socle equivalent to $A[I]$;
(2) A is stably equivalent to $A[I]$;
(3) $A[I]$ is isomorphic to a self-injective orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.

Since A and $A[I]$ are socle equivalent, the quotient algebras $A/\operatorname{soc} A$ and $A[I]/\operatorname{soc} A[I]$ are isomorphic, and consequently there is a canonical isomorphism $\Phi\colon \operatorname{mod}(A/\operatorname{soc} A) \to \operatorname{mod}(A[I]/\operatorname{soc} A[I])$ of their module categories. Observe also that the indecomposable modules in $\operatorname{mod}(A/\operatorname{soc} A)$ (respectively, $\operatorname{mod}(A[I]/\operatorname{soc} A[I])$) are precisely the indecomposable non-projective modules in $\operatorname{mod} A$ (respectively, $\operatorname{mod} A[I]$). Further, for any non-projective indecomposable modules L, N in $\operatorname{mod} A$ and non-projective indecomposable modules U, V in $\operatorname{mod} A[I]$ we have the equalities of homomorphism spaces $\operatorname{Hom}_A(L,N) = \operatorname{Hom}_{A/\operatorname{soc} A}(L,N)$ and $\operatorname{Hom}_{A[I]}(U,V) = \operatorname{Hom}_{A[I]/\operatorname{soc} A[I]}(U,V)$. We also note that the Auslander–Reiten quiver $\Gamma_{A/\operatorname{soc} A}$ of $A/\operatorname{soc} A$ (respectively, $\Gamma_{A[I]/\operatorname{soc} A[I]}$ of $A[I]/\operatorname{soc} A[I]$) is obtained from Γ_A (respectively, $\Gamma_{A[I]}$) by removing all indecomposable projective modules P, making their radicals $\operatorname{rad} P$ injective modules and the socle factors $P/\operatorname{soc} P$ projective modules, and keeping the indecomposable non-projective modules as well their Auslander–Reiten translations unchanged. Finally, the functor Φ induces a canonical isomorphism of the stable Auslander–Reiten quivers $\Gamma_A^s \xrightarrow{\sim} \Gamma_{A[I]}^s$. Summing up, we conclude that the image $\Phi(M)$ of the pure sectional module in $\operatorname{mod} A$ is a pure sectional module M in $\operatorname{mod} A[I]$ and is not the middle of a short chain. Applying Theorem 2.2, we conclude that $\varphi\nu_{\widehat{B}} = \rho\nu_{\widehat{B}}^2$ for some positive automorphism ρ of $\widehat{B}$. Since, by Theorem 3.4, the Nakayama permutations of A and $A[I]$ are the same, an isomorphism $A[I] \cong \widehat{B}/(\rho\nu_{\widehat{B}}^2)$ forces that $e_i \neq e_{\nu(i)}$ for any primitive direct summand e_i of the common residual identity e of $A/I \cong A[I]/I$. Applying now Theorem 3.6, we conclude that the algebras A and $A[I]$ are isomorphic. Therefore, A is isomorphic to the orbit algebra $\widehat{B}/(\rho\nu_{\widehat{B}}^2)$. This proves the implication (i)⇒(ii) of Theorem 1.1. The converse implication (ii)⇒(i) follows from Theorem 2.2.

Acknowledgements. The research of both authors has been supported by the research grant No. 2011/02/A/ST1/00216 of the Polish National Science Center.

REFERENCES

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge, 2006.

[2] M. Auslander and I. Reiten, *Modules determined by their composition factors*, Illinois J. Math. 29 (1985), 280–301.

[3] K. Bongartz, *Tilted algebras*, in: Representations of Algebras, Lecture Notes in Math. 903, Springer, Berlin, 1981, 26–38.

[4] O. Bretscher, C. Läser and C. Riedtmann, *Selfinjective and simply connected algebras*, Manuscripta Math. 361 (1981), 253–307.

[5] V. Dlab and C. M. Ringel, *On algebras of finite representation type*, J. Algebra 33 (1975), 306–394.

[6] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. 6 (1976), no. 173.

[7] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1972), 71–103.

[8] P. Gabriel, *The universal cover of a representation-finite algebra*, in: Representations of Algebras, Lecture Notes in Math. 903, Springer, Berlin, 1981, 68–105.

[9] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.

[10] M. Hoshino, *Trivial extensions of tilted algebras*, Comm. Algebra 10 (1982), 1965–1999.

[11] D. Hughes and J. Waschbüsch, *Trivial extensions of tilted algebras*, Proc. London Math. Soc. 46 (1983), 347–364.

[12] A. Jaworska, P. Malicki, and A. Skowroński, *Tilted algebras and short chains of modules*, Math. Z., DOI 10.1007/s00209-012-0993-0.

[13] A. Jaworska, P. Malicki, and A. Skowroński, *Classification of modules not lying on short chains*, arXiv:1112.3464v1 [math.RT].

[14] S. Liu, *Tilted algebras and generalized standard Auslander–Reiten components*, Arch. Math. (Basel) 61 (1993), 12–19.

[15] I. Reiten, S. O. Smalø and A. Skowroński, *Short chains and short cycles of modules*, Proc. Amer. Math. Soc. 117 (1993), 343–354.

[16] C. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen and zurück*, Comment. Math. Helv. 55 (1980), 199–224.

[17] C. Riedtmann, *Representation-finite selfinjective algebras of class A_n*, in: Representation Theory II, Lecture Notes in Math. 832, Springer, Berlin, 1980, 449–520.

[18] C. Riedtmann, *Representation-finite selfinjective algebras of class D_n*, Compos. Math. 49 (1983), 231–282.

[19] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin, 1984.

[20] D. Simson and A. Skowroński, *Extensions of artinian rings by hereditary injective modules*, in: Representations of Algebras, Lecture Notes in Math. 903, Springer, Berlin, 1981, 315–330.

[21] A. Skowroński, *Generalized standard Auslander–Reiten components without oriented cycles*, Osaka J. Math. 30 (1993), 515–527.

[22] A. Skowroński, *Selfinjective algebras: Finite and tame type*, in: Trends in Representation Theory of Algebras and Related Topics, Contemp. Math. 406, Amer. Math. Soc., Providence, RI, 2006, 169–238.

[23] A. Skowroński and K. Yamagata, *Socle deformations of self-injective algebras*, Proc. London Math. Soc. 72 (1996), 545–566.

[24] A. Skowroński and K. Yamagata, *Stable equivalence of selfinjective algebras of tilted type*, Arch. Math. (Basel) 70 (1998), 341–350.

[25] A. Skowroński and K. Yamagata, *Galois coverings of selfinjective algebras by repetitive algebras*, Trans. Amer. Math. Soc. 351 (1999), 715–734.

[26] A. Skowroński and K. Yamagata, *On invariability of self-injective algebras of tilted type under stable equivalences*, Proc. Amer. Math. Soc. 132 (2004), 659–667.

[27] A. Skowroński and K. Yamagata, *Positive Galois coverings of self-injective algebras*, Adv. Math. 194 (2005), 398–436.

[28] A. Skowroński and K. Yamagata, *Stable equivalence of selfinjective Artin algebras of Dynkin type*, Algebr. Represent. Theory 9 (2006), 33–45.

[29] A. Skowroński and K. Yamagata, *Selfinjective algebras of quasitilted type*, in: Trends in Representation Theory of Algebras and Related Topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, 639–708.

[30] A. Skowroński and K. Yamagata, *Frobenius Algebras I. Basic Representation Theory*, EMS Textbooks Math., Eur. Math. Soc., Zürich, 2011.

[31] G. Todorov, *Almost split sequences for* TrD*-periodic modules*, in: Representation Theory II, Lecture Notes in Math. 832, Springer, Berlin, 1980, 600–631.

Marta Błaszkiewicz, Andrzej Skowroński
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: marta22@mat.uni.torun.pl
skowron@mat.uni.torun.pl

Received 16 April 2012;
revised 8 May 2012 (5668)