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ON UNIT BALLS AND ISOPERIMETRICES IN NORMED SPACES

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Abstract. The purpose of this paper is to continue the investigations on the homothety of unit balls and isoperimetrices in higher-dimensional Minkowski spaces for the Holmes–Thompson measure and the Busemann measure. Moreover, we show a strong relation between affine isoperimetric inequalities and Minkowski geometry by proving some new related inequalities.

0. Introduction. One of the most challenging open questions in Minkowski geometry (i.e., the geometry of finite-dimensional real Banach spaces) is whether the unit ball must be an ellipsoid if it is the solution of the isoperimetric problem in a higher-dimensional (Minkowski or) normed space; see [2], [3], and [18]. Before answering this question, one has to define the notion of measure in the given normed space. Among others, there are two well-known definitions of measure: one due to Holmes–Thompson and another one due to Busemann. For the Holmes–Thompson measure, the above question can also be formulated as follows: if a centered convex body B in $\mathbb{R}^d$, $d \geq 3$, and the projection body of its polar are homothetic, must B then be an ellipsoid? The analogous question for the Busemann measure is: if B and the polar of its intersection body are homothetic, must B then be an ellipsoid? In $\mathbb{R}^2$, apart from ellipses, Radon curves (see [12]) have these properties as well. For higher-dimensional normed spaces it is only known that if B is a centered polytope, then the unit ball B and the corresponding isoperimetrix are not homothetic for the Holmes–Thompson measure (see [18]). We prove some results on the homothety of unit balls and isoperimetrices in d-dimensional normed spaces, and derive various related inequalities.

In [2], Busemann and Petty posed ten challenging questions formulated in terms of convex geometry, but important in Minkowski geometry. So far, only one of the questions has been completely answered (see, e.g., [5]). One of the above questions is also among those ten problems raised by Busemann

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and Petty. On the other hand, our investigations relate to affine isoperimetric inequalities (cf. [7]). For example, we prove an estimate related to Petty's conjectured projection inequality; see also [10].

1. Definitions and preliminaries. Recall that a *convex body* $K \subset \mathbb{R}^d$, $d \geq 2$, is a compact, convex set with nonempty interior, and K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$. As usual, S^{d-1} denotes the standard Euclidean unit sphere in $\mathbb{R}^d$. We write λ_i for the *i-dimensional Lebesgue measure* in $\mathbb{R}^d$, $1 \leq i \leq d$; for λ_d we simply write λ. We denote by $u^{\perp}$ the $(d-1)$-dimensional subspace orthogonal to $u \in S^{d-1}$, and l_u is the 1-subspace parallel to u. For a convex body K in $\mathbb{R}^d$, $K^{\circ} = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, x \in K\}$ denotes its *polar body*. We identify $\mathbb{R}^d$ and its *dual space* $(\mathbb{R}^d)^*$ by using the standard basis. In that case, λ_i and its dual measure λ_i^* coincide in $\mathbb{R}^d$. We denote by ϵ_d the volume of the standard Euclidean unit ball in $\mathbb{R}^d$. For K a convex body in $\mathbb{R}^d$ and $u \in S^{d-1}$, the *support function* $h_K(u) = \sup\{\langle u, y\rangle : y \in K\}$ and, for $o \in K$, the *radial function* $\rho_K(u) = \max\{\alpha \geq 0 : \alpha u \in K\}$ satisfy $h_{\alpha K} = \alpha h_K$ and $\rho_{\alpha K} = \alpha \rho_K$, and for all $u \in S^{d-1}$ we have

$$\rho_{K^{\circ}}(u) = \frac{1}{h_K(u)}. \tag{1}$$

The *projection body* ΠK of a convex body K in $\mathbb{R}^d$ is defined by $h_{\Pi K}(u) = \lambda_{d-1}(K|u^{\perp})$ for each $u \in S^{d-1}$, where $K|u^{\perp}$ is the orthogonal projection of K onto $u^{\perp}$ and $\lambda_{d-1}(K|u^{\perp})$ is called the $(d-1)$-*dimensional outer cross-section measure* of K at u. The *intersection body* IK of a convex body $K \subset \mathbb{R}^d$ is defined by $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^{\perp})$ for each $u \in S^{d-1}$. Further, $\lambda_{d-1}(K, u^{\perp})$ and $\lambda_1(K, u)$ denote the *inner cross-section measures of* K, i.e., the maximal measure of a hyperplane section of K normal to u and the *maximal chord length* of K at u, respectively. Note that for any direction a chord of maximal length of a centered convex body passes through the origin. Clearly $\lambda_1(K|l_u)$ denotes the *width* of K at u. All the notions given above can be found in the monographs [3], [16], and [18]; see also [8]. And we refer to [5] for a Fourier-analytic characterization of intersection bodies.

In [9] the following inequalities for cross-section measures were derived (see also [14], [15], and [17] for generalizations).

For a convex body K in $\mathbb{R}^d$, $d \geq 2$, and every direction $u \in S^{d-1}$,

$$\lambda(K) \leq \lambda_{d-1}(K|u^{\perp})\lambda_1(K, u) \leq d\lambda(K), \tag{2}$$

with equality on the left if and only if K is a *compact cylinder* (i.e., the sum of $(d-1)$-dimensional convex body and a line segment) with u as generator direction, and on the right precisely for K an *oblique double cone* with respect to $p - q = u$; this means that each boundary point of K can be connected to the boundary points p or q of K by a boundary segment of K.

Further, for each $u \in S^{d-1}$ a convex body K in $\mathbb{R}^d$, $d \geq 2$, satisfies

$$\lambda(K) \leq \lambda_{d-1}(K, u^{\perp})\lambda_1(K|l_u) \leq d\lambda(K), \tag{3}$$

with equality on the left if and only if K is a *compact cylinder* whose generators are parallel to u and whose basis is normal to u, and on the right exactly for K a *double cone* (possibly *cone*) whose basis is normal to u; see again [9].

We write $(\mathbb{R}^d, \ \|\cdot\|) = \mathbb{M}^d$ for a *d-dimensional real Banach space*, i.e., a *normed* (or *Minkowski*) *space* with *unit ball* B which is a centered convex body. The *unit sphere* of $\mathbb{M}^d$ is the boundary ∂B of the unit ball.

2. Surface areas, volumes, and isoperimetrices in normed spaces. A normed space $\mathbb{M}^d$ possesses a Haar measure μ, and this measure is unique up to multiplying the Lebesgue measure by a constant, i.e., $\mu = \sigma_B \lambda$.

The following notions are well known; see Chapter 5 of [18]. The *d-dimensional Holmes–Thompson volume* of a convex body K in $\mathbb{M}^d$ is defined by

$$\mu_B^{\mathrm{HT}}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\epsilon_d}, \quad \text{i.e.,} \quad \sigma_B = \frac{\lambda(B^\circ)}{\epsilon_d},$$

and the *d-dimensional Busemann volume* of K is defined by

$$\mu_B^{\mathrm{Bus}}(K) = \frac{\epsilon_d}{\lambda(B)}\lambda(K), \quad \text{i.e.,} \quad \sigma_B = \frac{\epsilon_d}{\lambda(B)} \ (\text{and } \mu_B^{\mathrm{Bus}}(B) = \epsilon_d).$$

To define the Minkowskian surface area of a convex body, one similarly has to define σ_B in $\mathbb{M}^{d-1}$. That is, for the Holmes–Thompson measure we have $\sigma_B(u) = \lambda_{d-1}((B \cap u^{\perp})^\circ)/\epsilon_{d-1}$, and for the Busemann measure $\sigma_B(u) = \epsilon_{d-1}/\lambda(B \cap u^{\perp})$ (see [18, pp. 150–151]). The *Minkowskian surface area* of K can also be defined in terms of mixed volumes (see [16] for notation and more about mixed volumes) by

$$\mu_B(\partial K) = dV(K[d-1], I_B), \tag{4}$$

where I_B is the convex body whose support function is $\sigma_B(u)$. For the Holmes–Thompson measure, I_B is defined by $I_B^{\mathrm{HT}} = \mathit{\Pi}(B^\circ)/\epsilon_{d-1}$, and therefore it is a centered zonoid (for zonoids see Proposition 6 below). For the Busemann measure we have $I_B^{\mathrm{Bus}} = \epsilon_{d-1}(IB)^\circ$. Among the homothetic images of I_B, one is specified, called the *isoperimetrix* $\hat{I}_B$ and determined by $\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$. Thus, the *isoperimetrix for the Holmes–Thompson measure* is defined by

$$\hat{I}_B^{\mathrm{HT}} = \frac{\epsilon_d}{\lambda(B^\circ)} I_B^{\mathrm{HT}}, \tag{5}$$

and the *isoperimetrix for the Busemann measure* by

$$\hat{I}_B^{\mathrm{Bus}} = \frac{\lambda(B)}{\epsilon_d} I_B^{\mathrm{Bus}}; \tag{6}$$

see again Chapter 5 of [18].

3. The unit ball and the isoperimetrix. We start by introducing two notions that we need as a useful tool in this paper; for results on these notions we refer to [11]. Namely, if K and L are convex bodies in $\mathbb{M}^d$, the *inner radius* of K with respect to L is defined by $r(K,L) := \max\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \subseteq K + x\}$, and the *outer radius* of K with respect to L is defined by $R(K,L) := \min\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \supseteq K + x\}$.

Notice that when K is a centered convex body, $r(K, \hat{I}_B)$ and $R(K, \hat{I}_B)$ can also be defined in terms of the support functions of K and $\hat{I}_B$. Namely, $r(K, \hat{I}_B)$ is the maximum α such that $\alpha \leq h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$. Similarly, $R(K, \hat{I}_B)$ is the minimum α such that $\alpha \geq h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$.

The following exact bounds have been established for the inner and outer radii of the unit ball for the Holmes–Thompson measure in $\mathbb{M}^d$ (see [10], [11], and [18]):

$$\frac{2\epsilon_{d-1}}{d\epsilon_d} \leq r(B, \hat{I}_B^{\mathrm{HT}}) \leq 1, \qquad R(B, \hat{I}_B^{\mathrm{HT}}) \leq \frac{2\epsilon_{d-1}}{\epsilon_d}.$$

We have $r(B, \hat{I}_B^{\mathrm{HT}}) = 1$ if and only if B is an ellipsoid. The lower bound for $R(B, \hat{I}_B^{\mathrm{HT}})$ is still unknown. One can also see that $R(B, \hat{I}_B^{\mathrm{HT}})/r(B, \hat{I}_B^{\mathrm{HT}}) \leq d$, since the Banach-Mazur distance between two centered convex bodies is at most d, and this bound cannot be reduced; see [6].

For a convex body K, we denote by $w_B(K)$ and $D_B(K)$ the *Minkowskian thickness* (i.e., $w_B(K) = \min_{u \in S^{d-1}} 2w(K,u)/w(B,u)$, where $w(K,u)$ is the Euclidean width of K in the direction u) and the *Minkowskian diameter* (i.e., the maximum of this Minkowskian width function of K), respectively.

PROPOSITION 1. *If B is the unit ball of $\mathbb{M}^d$, then the Minkowskian thickness and diameter of $\hat{I}_B^{\mathrm{HT}}$ have the exact bounds*

$$\frac{\epsilon_d}{\epsilon_{d-1}} \leq w_B(\hat{I}_B^{\mathrm{HT}}) \quad \textit{and} \quad 2 \leq D_B(\hat{I}_B^{\mathrm{HT}}) \leq \frac{d\epsilon_d}{\epsilon_{d-1}}.$$

Proof. One can easily see that

$$r(\hat{I}_B^{\mathrm{HT}}, B) = \frac{1}{R(B, \hat{I}_B^{\mathrm{HT}})} \quad \text{and} \quad R(\hat{I}_B^{\mathrm{HT}}, B) = \frac{1}{r(B, \hat{I}_B^{\mathrm{HT}})}.$$

Hence we have the exact estimates

$$\frac{\epsilon_d}{2\epsilon_{d-1}} \leq r(\hat{I}_B^{\mathrm{HT}}, B), \qquad 1 \leq R(\hat{I}_B^{\mathrm{HT}}, B) \leq \frac{d\epsilon_d}{2\epsilon_{d-1}}.$$

It is easy to establish that if K is a centered convex body in $\mathbb{M}^d$, then $r(K,B) = w_B(K)/2$ and $R(K,B) = D_B(K)/2$. From this we get the exact estimates

$$\frac{\epsilon_d}{\epsilon_{d-1}} \leq w_B(\hat{I}_B^{\mathrm{HT}}), \qquad 2 \leq D_B(\hat{I}_B^{\mathrm{HT}}) \leq \frac{d\epsilon_d}{\epsilon_{d-1}}.$$

Also, $D_B(\hat{I}_B^{\mathrm{HT}}) = 2$ if and only if B is an ellipsoid. ∎

If B and $\hat{I}_B^{\mathrm{HT}}$ are homothetic, then $w_B(\hat{I}_B^{\mathrm{HT}}) = D_B(\hat{I}_B^{\mathrm{HT}})$. Also, if one considers the class of centered convex bodies B in $\mathbb{M}^d$, $d \geq 3$, for which $w_B(\hat{I}_B^{\mathrm{HT}}) \leq 2$, then such bodies B and $\hat{I}_B^{\mathrm{HT}}$ would be homothetic if and only if B is an ellipsoid. The inequality $w_B(\hat{I}_B^{\mathrm{HT}}) \leq 2$ also yields $R(B, \hat{I}_B^{\mathrm{HT}}) \geq 1$. We can expand $w_B(\hat{I}_B^{\mathrm{HT}})$ as follows:

$$\begin{aligned} w_B(\hat{I}_B^{\mathrm{HT}}) &= \min_{u \in S^{d-1}} \frac{2w(\hat{I}_B^{\mathrm{HT}}, u)}{w(B,u)} = \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_B^{\mathrm{HT}}}(u)}{h_B(u)} \\ &= \min_{u \in S^{d-1}} \frac{2\epsilon_d}{\lambda(B^\circ)} h_{I_B^{\mathrm{HT}}}(u)\rho_{B^\circ}(u) = \min_{u \in S^{d-1}} \frac{2\epsilon_d}{\epsilon_{d-1}} \frac{h_{\Pi B^\circ}(u)\rho_{B^\circ}(u)}{\lambda(B^\circ)} \\ &= \min_{u \in S^{d-1}} \frac{\epsilon_d}{\epsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}, \end{aligned}$$

where w denotes the usual width function. Thus, one can raise the following

Problem 2. *If B is a centered convex body in $\mathbb{R}^d$ with $d \geq 3$, does there exist a direction $u \in S^{d-1}$ such that*

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} < \frac{2\epsilon_{d-1}}{\epsilon_d}?$$

In [10] it was proved that there is a direction $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} \geq \frac{2\epsilon_{d-1}}{\epsilon_d},$$

where equality holds for all $u \in S^{d-1}$ if and only if B is an ellipsoid.

From (2) and the considerations above we obtain the following

Theorem 3. *If B is a centered convex body of cylindrical type* (*i.e., a compact cylinder*) *in $\mathbb{R}^d$ with $d \geq 3$, then B and $\hat{I}_B^{\mathrm{HT}}$ of the respective normed space cannot be homothetic.*

If there is a Minkowski space with $w_B(\hat{I}_B^{\mathrm{HT}}) > 2$, then the assumption that B and $\hat{I}_B^{\mathrm{HT}}$ are homothetic would be equivalent to the existence of a constant $c > 2\epsilon_{d-1}/\epsilon_d$ such that

$$\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B \cap l_u)}{\lambda(B)} = c$$

for all $u \in S^{d-1}$.

The inner and outer radii also play important roles for the ratio between the Minkowskian surface area of the unit ball and its Minkowskian volume. Namely, combining them with properties of mixed volumes (such as $V(K[d-1], K_1) \leq V(K[d-1], K_2)$ if $K_1 \subseteq K_2$, and $V(K[d-1], K) = \lambda(K)$) we get

$$r(B, \hat{I}_B^{\mathrm{HT}})V(B[d-1], \hat{I}_B^{\mathrm{HT}}) \leq \lambda(B) \leq R(B, \hat{I}_B^{\mathrm{HT}})V(B[d-1], \hat{I}_B^{\mathrm{HT}}).$$

Hence we have

$$\frac{d}{R(B,\hat{I}_B^{\mathrm{HT}})} \le \frac{\mu_B^{\mathrm{HT}}(\partial B)}{\mu_B^{\mathrm{HT}}(B)} \le \frac{d}{r(B,\hat{I}_B^{\mathrm{HT}})},$$

yielding new approaches to reestablish the bounds

$$\frac{d\epsilon_d}{2\epsilon_{d-1}} \le \frac{\mu_B^{\mathrm{HT}}(\partial B)}{\mu_B^{\mathrm{HT}}(B)} \le \frac{d^2\epsilon_d}{2\epsilon_{d-1}}$$

obtained in [4]. It is known that the right side of this ratio is exact (see [10]), but we cannot claim this for the left side (see [13] for $d = 2$). However, we pose the following

Conjecture 4. *When $d \ge 3$, we have $\mu_B^{\mathrm{HT}}(\partial B)/\mu_B^{\mathrm{HT}}(B) \ge d$ with equality if and only if B is an ellipsoid.*

It is also known that the unit balls not satisfying this conjecture will contradict Petty's conjectured projection inequality (see [7] and [10]). The following statement gives an estimate related to this conjectured inequality.

Proposition 5. *If B a centered convex body in $\mathbb{R}^d$, then*

$$\lambda(\mathit{\Pi} B)\lambda^{1-d}(B) \le \left(\frac{d}{2}\right)^d \epsilon_d^2.$$

Proof. From the isodiametric inequality (see [18]) we get

$$2^d\lambda(\hat{I}_B^{\mathrm{HT}}) \le D_B^d(\hat{I}_B^{\mathrm{HT}})\lambda(B).$$

Using the above estimate on $D_B(\hat{I}_B^{\mathrm{HT}})$, we obtain

$$\lambda(\mathit{\Pi} B^\circ) \le \frac{d}{2}\lambda^d(B^\circ)\lambda(B).$$

With the Blaschke–Santaló inequality (cf. [7]) we obtain

$$\lambda(\mathit{\Pi} B^\circ)\lambda^{1-d}(B^\circ) \le \left(\frac{d}{2}\right)^d \epsilon_d^2. \ \blacksquare$$

Our next proposition refers to unit balls which are *zonoids*, i.e., limits of finite sums of line segments (see [3] and [16]).

Proposition 6. *If B is a centered zonoid in $\mathbb{R}^d$, then*

$$\lambda(\mathit{\Pi} B)\lambda^{1-d}(B) \ge \binom{2d}{d}^{-1}\frac{4^d}{d!}.$$

Proof. The relationship between the volume of a convex body and its minimal width can be given by the sharp inequality $\lambda(\cdot) \ge \alpha(B)w_B^d(\cdot)$, where $\alpha(B)$ is a positive constant satisfying the sharp inequality $\binom{2d}{d}^{-1} \le \alpha(B)/\lambda(B) \le 2^{-d}$. Here equality holds on the left if and only if B is the difference body of a simplex, and on the right if B is a cross-polytope (see [1]).

Using the estimate on $w_B(\hat{I}_B^{\mathrm{HT}})$, we obtain

$$\binom{2d}{d}\lambda(\Pi B^\circ) \geq \lambda^d(B^\circ)\lambda(B).$$

If B is a zonoid, then from the Mahler–Reisner inequality (see [7]) we get

$$\lambda(\Pi B^\circ)\lambda^{1-d}(B^\circ) \geq \binom{2d}{d}^{-1}\frac{4^d}{d!}. \ \blacksquare$$

REMARK. Proposition 6 holds not only for zonoids, but for general (centered) convex bodies K. This follows from the following inequality (see [16, Section 7.4]):

$$\frac{\lambda(\Pi(\Pi K))}{\lambda^{d-1}(\Pi K)} \leq \frac{\lambda(\Pi K)}{\lambda^{d-1}(K)}.$$

Recall that finding the exact bounds on $\lambda(\Pi B)\lambda^{1-d}(B)$ is also one of the most challenging problems in the area of affine isoperimetric inequalities; cf. again [7].

PROPOSITION 7. *If B is the unit ball of $\mathbb{M}^d$, $d \geq 3$, then the Minkowskian thickness and diameter of $\hat{I}_B^{\mathrm{Bus}}$ have the exact bounds*

$$\frac{4\epsilon_{d-1}}{d\epsilon_d} \leq w_B(\hat{I}_B^{\mathrm{Bus}}) \quad \textit{and} \quad 2 \leq D_B(\hat{I}_B^{\mathrm{Bus}}) \leq \frac{4\epsilon_{d-1}}{\epsilon_d}.$$

Proof. For the Busemann measure we have the following exact estimates (see [10] and [11]):

$$\frac{\epsilon_d}{2\epsilon_{d-1}} \leq r(B, \hat{I}_B^{\mathrm{Bus}}) \leq 1, \qquad R(B, \hat{I}_B^{\mathrm{Bus}}) \leq \frac{d\epsilon_d}{2\epsilon_{d-1}}.$$

The sharp lower bound on the outer radius is still an open question. Again, we have $r(\hat{I}_B^{\mathrm{Bus}}, B) = 1/R(B, \hat{I}_B^{\mathrm{Bus}})$ and $R(\hat{I}_B^{\mathrm{Bus}}, B) = 1/r(B, \hat{I}_B^{\mathrm{Bus}})$. Therefore

$$\frac{2\epsilon_{d-1}}{d\epsilon_d} \leq r(\hat{I}_B^{\mathrm{Bus}}, B), \qquad 1 \leq R(\hat{I}_B^{\mathrm{Bus}}, B) \leq \frac{2\epsilon_{d-1}}{\epsilon_d}.$$

From these inequalities we clearly obtain

$$\frac{4\epsilon_{d-1}}{d\epsilon_d} \leq w_B(\hat{I}_B^{\mathrm{Bus}}), \qquad 2 \leq D_B(\hat{I}_B^{\mathrm{Bus}}) \leq \frac{4\epsilon_{d-1}}{\epsilon_d}. \ \blacksquare$$

We can also expand $w_B(\hat{I}_B^{\mathrm{Bus}})$ as

$$\begin{aligned} w_B(\hat{I}_B^{\mathrm{Bus}}) &= \min_{u\in S^{d-1}} \frac{2h_{\hat{I}_B^{\mathrm{Bus}}}(u)}{h_B(u)} = \min_{u\in S^{d-1}} \frac{2\lambda(B)\epsilon_{d-1}h_{(IB)^\circ}(u)}{\epsilon_d h_B(u)} \\ &= \frac{2\epsilon_{d-1}}{\epsilon_d} \min_{u\in S^{d-1}} \frac{\lambda(B)}{\rho_{IB}(u)h_B(u)} = \frac{2\epsilon_{d-1}}{\epsilon_d} \min_{u\in S^{d-1}} \frac{2\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}. \end{aligned}$$

Then $w_B(\hat{I}_B^{\mathrm{Bus}}) \leq 2$ is equivalent to asking whether

$$\max_{u \in S^{d-1}} \frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \geq \frac{2\epsilon_{d-1}}{\epsilon_d}.$$

In [11] it was proved that, when $d \geq 3$, there is a direction $u \in S^{d-1}$ such that

$$\frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\epsilon_{d-1}}{\epsilon_d},$$

with equality for all $u \in S^{d-1}$ if and only if B is an ellipsoid.

From (3), we obtain the following

Theorem 8. *If B is a centered convex body of the type of a double cone in $\mathbb{R}^d$ with $d \geq 3$, then B and $\hat{I}_B^{\mathrm{Bus}}$ of the corresponding normed space cannot be homothetic.*

If there is a normed space with $w_B(\hat{I}_B^{\mathrm{Bus}}) > 2$, then the assumption that B and $\hat{I}_B^{\mathrm{Bus}}$ are homothetic is equivalent to asking wether there is a constant $c < 2\epsilon_{d-1}/\epsilon_d$ such that

$$\frac{\lambda_{d-1}(B \cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} = c$$

for all $u \in S^{d-1}$.

Similar to the Holmes–Thompson measure, for the ratio of the Busemann surface area of the unit ball to its volume (which is ϵ_d) we have

$$r(B, \hat{I}_B^{\mathrm{Bus}})V(B[d-1], \hat{I}_B^{\mathrm{Bus}}) \leq \lambda(B) \leq R(B, \hat{I}_B^{\mathrm{Bus}})V(B[d-1], \hat{I}_B^{\mathrm{Bus}}).$$

Thus,

$$\frac{d}{R(B, \hat{I}_B^{\mathrm{Bus}})} \leq \frac{\mu_B^{\mathrm{Bus}}(\partial B)}{\epsilon_d} \leq \frac{d}{r(B, \hat{I}_B^{\mathrm{Bus}})},$$

and therefore bounds on $r(B, \hat{I}_B^{\mathrm{Bus}})$ and $R(B, \hat{I}_B^{\mathrm{Bus}})$ yield again a new approach to

$$2\epsilon_{d-1} \leq \mu_B^{\mathrm{Bus}}(\partial B) \leq 2d\epsilon_{d-1}.$$

It is known that the right side is exact (see [2] and p. 242 in [18]). This cannot be claimed for the left side. Thus, we formulate the following conjecture.

Conjecture 9. *For $d \geq 3$, $\mu_B^{\mathrm{Bus}}(\partial B) \geq d\epsilon_d$.*

Our final proposition is also related to the Busemann measure in normed spaces.

Proposition 10. *If B is a centered convex body in $\mathbb{R}^d$, then*

$$\binom{2d}{d}^{-1}\left(\frac{4}{d}\right)^d \leq \lambda^{d-1}(B)\lambda((IB)^\circ) \leq 2^d.$$

Proof. As in the proof of Proposition 6 we get, using the relationship between the volume of $\hat{I}_B^{\mathrm{Bus}}$ and its minimal width and the lower estimate on $w_B(\hat{I}_B^{\mathrm{Bus}})$, the inequality

$$\lambda(\hat{I}_B^{\mathrm{Bus}}) \geq \binom{2d}{d}^{-1} \lambda(B) \left(\frac{4\epsilon_{d-1}}{d\epsilon_d} \right)^d.$$

Hence the left inequality in the proposition follows. Using the isodiametric inequality (cf. [18]) and the upper estimate on $D_B(\hat{I}_B^{\mathrm{Bus}})$, we also obtain the right inequality. ∎

Finding the exact bounds on $\lambda^{d-1}(B)\lambda((IB)^\circ)$ is another interesting open problem in the area of affine isoperimetric inequalities (cf. [7]).

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