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ON φ-INNER AMENABLE BANACH ALGEBRAS

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Abstract. Generalizing the concept of inner amenability for Lau algebras, we define and study the notion of φ-inner amenability of any Banach algebra A, where φ is a homomorphism from A onto $\mathbb{C}$. Several characterizations of φ-inner amenable Banach algebras are given.

1. Introduction. In his famous work, Lau [6] introduced a wide class of Banach algebras, called F-algebras, and studied the notion of left amenability for these algebras. By definition, an *F-algebra* A is a Banach algebra which is the predual of a W^*-algebra M such that the identity ϵ of M is a multiplicative linear functional on A. Although M need not be unique [6], we shall identify M with the continuous dual A^* of A if no confusion can arise. Later on, F-algebras were termed *Lau algebras* by Pier [14]. Such an algebra A was called *left amenable* if there exists a positive linear functional m of norm 1 on the W^*-algebra A^* such that $m(f \cdot a) = m(f)$ for all $f \in A^*$ and $a \in P_1(A) = \{a \in A : \epsilon(a) = \|a\| = 1\}$. Left amenability of F-algebras has been characterized in different ways by Lau [6].

Lau algebras have been studied under various aspects in [6, 7], [9], and [11–13]. In [12], Nasr-Isfahani introduced the concept of inner amenability for Lau algebras. A Lau algebra A was said to be *inner amenable* if there exists a *topological inner invariant mean* on the W^*-algebra A^*, that is, a positive linear functional m of norm 1 on A^* such that $m(f\cdot a) = m(a\cdot f)$ for all $f \in A^*$ and all $a \in P_1(A) = \{a \in A : \epsilon(a) = \|a\| = 1\}$ (or equivalently, for all $a \in A$). Commutative Lau algebras, like the Fourier algebra $A(G)$ of a locally compact group G, are examples of inner amenable algebras. Also the group algebra $L^1(G)$ of any locally compact group G is inner amenable. In [12], the author obtained several characterizations of inner amenability of Lau algebras, for instance, inner amenability was shown to be equivalent to a fixed point property. The idea behind this definition was the notion of inner amenability for discrete semigroups studied by Ling [10]. A discrete semigroup S is called *inner amenable* if there is an element m of $P_1(\ell^\infty(S)^*)$

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such that $m(ft) = m(tf)$ for all $f \in \ell^\infty(S)$ and $t \in S$, where $ft(s) = f(ts) = sf(t)$ for $s, t \in S$. As pointed out in [12], a discrete semigroup S is inner amenable if and only if $\ell^1(S)$ is inner amenable.

In an interesting recent work [4] (continued in [5]), the authors have studied the notion of φ-amenability for an arbitrary Banach algebra A, where φ is a homomorphism from A onto $\mathbb{C}$, generalizing left amenability for Lau algebras of [6]. A is called (left) *φ-amenable* if there exists a bounded linear functional m on A^* satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A^*$. They characterized φ-amenability in different ways. One may define that A is *two-sided φ-amenable* if there exists $m \in A^{**}$ with $m(\varphi) = 1$ and $m(f \cdot a) = m(a \cdot f) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A^*$.

In this paper, as in the case of φ-amenability in [4], we are going to define and study the concept of φ-inner amenability for any Banach algebra. Let A be an arbitrary Banach algebra and φ a homomorphism from A onto $\mathbb{C}$. Let $A_\varphi = \{a \in A : \varphi(a) = 1\}$. We call A *φ-inner amenable* if there exists a bounded linear functional m on A^* satisfying $m(\varphi) = 1$ and $m(f \cdot a) = m(a \cdot f)$ for all $f \in A^*$ and for all $a \in A_\varphi$ (hence for all $a \in A$, since if $\varphi(a) = 0$ and $b \in A_\varphi$ is arbitrary, then $b - a \in A_\varphi$, thus $m(f \cdot (b-a)) = m((b-a) \cdot f)$, that is, $m(f \cdot a) = m(a \cdot f)$ because $m(f \cdot b) = m(b \cdot f)$). Such a linear functional m will sometimes be referred to as a *φ-inner mean*, and we denote by φ-IM(A^*) the set of all φ-inner means on A^*. In case φ is identically zero, it is clear that there is no non-trivial 0-inner amenable Banach algebra. So we always assume that φ is non-zero.

Commutative Banach algebras, two-sided φ-amenable Banach algebras and Banach algebras with a bounded approximate identity are examples of φ-inner amenable algebras (for the latter see Corollary 2.2). As we shall see, the concept of φ-inner amenability is more general than the notion of inner amenability for Lau algebras (Remark 2.4). We give several characterizations of φ-inner amenable Banach algebras. In accomplishing these, the methods employed in [12] and [4] prove extremely useful. Below we outline the content of this paper.

In Section 2, among other things, it is shown that φ-inner amenability of a Banach algebra A is equivalent to; the existence of a bounded net (ν_α) in A_φ such that $\|\nu_\alpha a - a\nu_\alpha\| \to 0$ for all $a \in A_\varphi$, and the existence of a φ-inner invariant mean (see Section 2 for the definition) on $C_{\mathrm{au}}(A_\varphi)$, the set of all additively uniformly continuous functions on A_φ (Theorem 2.1). The aim of Section 3 is to show that the φ-inner amenability of a Banach algebra A with a bounded right approximate identity is equivalent to the existence of a certain element $\Lambda \in B(X^{**})$ (the Banach space of all bounded operators on X^{**}) such that for all $a \in A_\varphi$, $\Lambda\Lambda_a = \Lambda_a\Lambda$, for every left Banach A-module X (Theorem 3.3).

2. Characterization of φ-inner amenability. Unless otherwise stated, throughout this paper A denotes an arbitrary Banach algebra, $0 \neq \varphi \in \Delta(A)$, the set of all homomorphisms from A onto $\mathbb{C}$, and $A_\varphi = \{a \in A : \varphi(a) = 1\}$. The set A_φ, endowed with the induced norm topology of A and the product of A, is a topological semigroup. Let $C_{\mathrm{b}}(A_\varphi)$ denote the Banach space of all bounded and continuous functions on A_φ with the supremum norm, and define the left and right translation operators l_a and r_a on $C_{\mathrm{b}}(A_\varphi)$ by $l_a\phi(b) = \phi(ab) = r_b\phi(a)$ for all $a, b \in A_\varphi$ and $\phi \in C_{\mathrm{b}}(A_\varphi)$.

As in [12], a function $\phi \in C_{\mathrm{b}}(A_\varphi)$ is called *additively uniformly continuous* on A_φ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|\phi(a) - \phi(b)| < \varepsilon$ whenever $a, b \in A_\varphi$ with $\|a - b\| < \delta$. Let $C_{\mathrm{au}}(A_\varphi)$ denote the set of all additively uniformly continuous functions on A_φ. Then $C_{\mathrm{au}}(A_\varphi)$ is a norm closed, translation invariant subspace of $C_{\mathrm{b}}(A_\varphi)$ containing the constants and the restrictions to A_φ of elements of A^*. An element m of $C_{\mathrm{au}}(A_\varphi)^*$ is called a *φ-inner invariant mean* if $\langle m, \varphi|_{A_\varphi}\rangle = 1$ and $\langle m, l_a\phi\rangle = \langle m, r_a\phi\rangle$ for all $a \in A_\varphi$ and $\phi \in C_{\mathrm{au}}(A_\varphi)$, where $\varphi|_{A_\varphi}$ denotes the restriction of φ to A_φ.

Recall that the second dual A^{**} of A is a Banach algebra with respect to the first and second Arens products denoted by $\odot$ and $\diamond$, respectively, defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $f \cdot a$, $a \cdot f$, $m \cdot f$ and $f \cdot m$ of A^* and the elements $m \odot n$ and $m \diamond n$ of A^{**} are defined by

$$\langle m \odot n, f\rangle = \langle m, n \cdot f\rangle, \qquad \langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle, \qquad \langle f \cdot a, b\rangle = \langle f, ab\rangle,$$
$$\langle m \diamond n, f\rangle = \langle n, f \cdot m\rangle, \qquad \langle f \cdot m, a\rangle = \langle m, a \cdot f\rangle, \qquad \langle a \cdot f, b\rangle = \langle f, ba\rangle.$$

Obviously, $a \odot m = a \diamond m$ and $m \odot a = m \diamond a$ for all $a \in A$ and $m \in A^{**}$. A Banach algebra A is called *Arens regular* if $m \odot n = m \diamond n$ for all $m, n \in A^{**}$. Now we state and prove the main result of this section.

THEOREM 2.1. *For a Banach algebra A and $\varphi \in \Delta(A)$ the following statements are equivalent:*

- (i) *A is φ-inner amenable.*
- (ii) *There is a bounded net (ν_α) in A_φ such that for all $a \in A_\varphi$, $\nu_\alpha a - a\nu_\alpha \to 0$ in the weak topology of A.*
- (iii) *There is a bounded net (ν_α) in A_φ such that for all $a \in A_\varphi$, $\|\nu_\alpha a - a\nu_\alpha\| \to 0$.*
- (iv) *There is a φ-inner invariant mean on $C_{\mathrm{au}}(A_\varphi)$.*

Proof. (i)$\Rightarrow$(ii). Assume that A is φ-inner amenable. Then there exists $m \in A^{**}$ such that $m(\varphi) = 1$ and $\langle m, f \cdot a\rangle = \langle m, a \cdot f\rangle$ for all $a \in A_\varphi$ and $f \in A^*$. Choose a net (ν_α) in A with the property that $\nu_\alpha \to m$ in the weak* topology on A^{**} and $\|\nu_\alpha\| \le \|m\|$ for all α. Since $\varphi(\nu_\alpha) \to m(\varphi) = 1$, after passing to a subnet and replacing ν_α by $(1/\varphi(\nu_\alpha))\nu_\alpha$, we can assume that $\varphi(\nu_\alpha) = 1$ and $\|\nu_\alpha\| \le \|m\| + 1$ for all α. For all $a \in A_\varphi$ and $f \in A^*$, we have $\langle m \odot a, f\rangle = \langle a \odot m, f\rangle$, thus $\langle w^*\text{-}\lim_\alpha \nu_\alpha \odot a, f\rangle = \langle a \odot w^*\text{-}\lim_\alpha \nu_\alpha, f\rangle$, that is, $\lim_\alpha f(\nu_\alpha a) = \lim_\alpha f(a\nu_\alpha)$ or equivalently $\lim_\alpha f(\nu_\alpha a - a\nu_\alpha) = 0$. The latter means that $\nu_\alpha a - a\nu_\alpha \to 0$ in the weak topology of A.

(ii)⇒(iii). Let Y be the vector space $\prod\{A : b \in A_\varphi\}$ and let $T : A \to Y$ be the linear map defined by $T(a)(b) = ba - ab$ for all $a \in A$ and $b \in A_\varphi$. By assumption, the weak closure of $T(A_\varphi)$ contains 0. Since Y is a locally convex space with the product of the norm topologies and A_φ is convex, the closure of $T(A_\varphi)$ in this topology contains 0. That is, (iii) holds.

(iii)⇒(iv). Let (ν_α) be as in (iii). If we define $m_\alpha \in C_{\mathrm{au}}(A_\varphi)^*$ by $\langle m_\alpha, \phi\rangle = \phi(\nu_\alpha)$ for all $\phi \in C_{\mathrm{au}}(A_\varphi)$, then any weak* cluster point of (m_α) in $C_{\mathrm{au}}(A_\varphi)^*$ is a φ-inner invariant mean.

(iv)⇒(i). Let m be a φ-inner invariant mean on $C_{\mathrm{au}}(A_\varphi)$, and define $M \in A^{**}$ by $\langle M, f\rangle = \langle m, f|_{A_\varphi}\rangle$ for $f \in A^*$. Then M is a φ-inner mean on A^*. ∎

The next corollary gives us a variety of φ-inner amenable Banach algebras.

Corollary 2.2. *Let A be a Banach algebra with a bounded approximate identity. Then A is φ-inner amenable for all $\varphi \in \Delta(A)$.*

Proof. Let $\varphi \in \Delta(A)$. Let $\{e_\alpha\}$ be a bounded approximate identity of A. Then $\varphi(e_\alpha) \to 1$. Hence, without loss of generality, we may assume that $\varphi(e_\alpha) \neq 0$ for all α. Let $\nu_\alpha = e_\alpha/\varphi(e_\alpha)$. Then the net $\{\nu_\alpha\}$ satisfies condition (iii) of Theorem 2.1, and hence A is φ-inner amenable. ∎

Example 2.3. (1) Let G be a locally compact group and let $L^1(G)$ denote the group algebra of G. It is well-known that $L^1(G)$ has a bounded approximate identity. Hence $L^1(G)$ is φ-inner amenable for all $\varphi \in \Delta(L^1(G))$.

(2) As pointed out in the introduction, every commutative Banach algebra A is φ-inner amenable for all $\varphi \in \Delta(A)$. In fact $A_\varphi \subseteq \varphi\text{-IM}(A^*)$. In particular, if G is a locally compact group and $A(G)$ is the Fourier algebra of G [3], then $\Delta(A(G))$ consists of all point evaluations $\varphi_t(f) = f(t)$, $f \in A(G)$, $t \in G$, and so $A(G)$ is φ_t-inner amenable for all $t \in G$.

The following remark asserts that the concept of φ-inner amenability generalizes that of inner amenability of Lau algebras in [12].

Remark 2.4. Let A be a Lau algebra with ϵ being the identity of A^*. Then it is readily seen that A is ϵ-inner amenable if and only if A is inner amenable. In fact, that inner amenability implies ϵ-inner amenability follows easily from definitions. For the converse, assume that A is ϵ-inner amenable, hence there exists an ϵ-inner mean m on A^*. Thus $a \odot m = m \odot a$ and $a \odot m^* = m^* \odot a$ for all $a \in P_1(A) = \{a \in A : \epsilon(a) = \|a\| = 1\}$ (note in particular that the elements of $P_1(A)$ are positive). So we may assume that m is self-adjoint. Write $m = m^+ - m^-$, the orthogonal decomposition of m. If $a \in P_1(A)$, then $a \odot m = a \odot m^+ - a \odot m^-$ and $m \odot a = m^+ \odot a - m^- \odot a$.

Let $a \in P_1(A)$. Since $m^+ \odot a$, $m^- \odot a$, $a \odot m^+$ and $a \odot m^-$ are all positive and

$$\|a \odot m^+\| + \|a \odot m^-\| = \|a \odot m\| = \|m \odot a\| = \|m^+ \odot a\| + \|m^- \odot a\|$$

it follows that $a \odot m^+ = m^+ \odot a$ and $a \odot m^- = m^- \odot a$ [15, Theorem 1.14.3]. Therefore if $m^+ \neq 0$ (say) and $n = m^+/m^+(\epsilon)$, then n is the desired topological inner invariant mean.

For a Banach algebra A and $\varphi \in \Delta(A)$ let $\tilde{\varphi}$ denote the unique extension of φ to A^{**}. Clearly, any $\tilde{\varphi}$-inner mean on A^{***} restricted to A^* is a φ-inner mean on A^*. Thus we have the following proposition.

Proposition 2.5. *Let A be an Arens regular Banach algebra. Then A is φ-inner amenable if and only if A^{**} is $\tilde{\varphi}$-inner amenable.*

Proof. Assume that A is φ-inner amenable. Then there exists $m \in A^{**}$ such that $\langle m, \varphi\rangle = 1$ and $\langle m, f.a\rangle = \langle m, a.f\rangle$ for all $a \in A_\varphi$ and $f \in A^*$. For given $n \in A^{**}_{\tilde{\varphi}}$ and $u \in A^{***}$, choose nets $(a_\alpha)_\alpha$ in A and $(f_\beta)_\beta$ in A^* such that $a_\alpha \to n$ and $f_\beta \to u$ with respect to the corresponding w^*-topologies. Now $\varphi(a_\alpha) = \langle a_\alpha, \varphi\rangle \to \langle n, \varphi\rangle = \tilde{\varphi}(n) = 1$, hence after passing to a subnet and replacing a_α by $(1/\varphi(a_\alpha))a_\alpha$, one may assume that $\varphi(a_\alpha) = 1$. Consider m as an element $\hat{m}$ of A^{****}. Then clearly $\hat{m}(\tilde{\varphi}) = 1$ and

$$\begin{aligned}
\langle \hat{m}, u.n\rangle &= \langle u.n, m\rangle = \langle u, n \odot m\rangle = \lim_\beta \langle f_\beta, n \odot m\rangle = \lim_\beta \langle n, m \cdot f_\beta\rangle \\
&= \lim_\beta \lim_\alpha \langle a_\alpha, m \cdot f_\beta\rangle = \lim_\beta \lim_\alpha \langle m, f_\beta \cdot a_\alpha\rangle = \lim_\beta \lim_\alpha \langle m, a_\alpha \cdot f_\beta\rangle \\
&= \lim_\beta \lim_\alpha \langle m \cdot a_\alpha, f_\beta\rangle = \lim_\beta \langle m \odot n, f_\beta\rangle = \lim_\beta \langle f_\beta, m \odot n\rangle \\
&= \langle u, m \odot n\rangle = \langle n \cdot u, m\rangle = \langle \hat{m}, n \cdot u\rangle.
\end{aligned}$$

Hence A^{**} is $\tilde{\varphi}$-inner amenable. ∎

Recall that an element E of A^{**} is called a *mixed identity* if $a \odot E = E \odot a = a$ for all $a \in A$. It is easily seen that an element E of A^{**} is a mixed identity if and only if it is a weak* cluster point of a bounded approximate identity in A, [1]. A Lau algebra A is called *strictly inner amenable* (see [2] and also [8]) if there exists a topological inner invariant mean on A^* which is not a mixed identity of A^{**}. For $\varphi \in \Delta(A)$, let us call an element E of A^{**} a *φ-mixed identity* if $a \odot E = E \odot a = a$ for all $a \in A_\varphi$. Therefore any φ-mixed (or equivalently mixed) identity M of A^{**} such that $M(\varphi) = 1$ is in φ-IM(A^*).

We say that A is *strictly φ-inner amenable* if there exists a φ-inner mean on A^* which is not a φ-mixed identity. When $\varphi = 1$ and $A = L^1(G)$, the group algebra of a locally compact group G, the notion of strict φ-inner amenability was studied by Effros [2] and also by Lau and Paterson [8].

As an application of the above proposition we have the next corollary.

Corollary 2.6. *If A is Arens regular and A^{**} is not strictly $\tilde{\varphi}$-inner amenable, then A is not strictly φ-inner amenable.*

Proof. Let $M \in \varphi\text{-IM}(A^*)$. Then by the proof of the above proposition, $M \in \tilde{\varphi}\text{-IM}(A^{***})$. Since A^{**} is not strictly $\tilde{\varphi}$-inner amenable, M is a φ-mixed identity of A^{****}. In particular, $a \odot M = M \odot a = a$ for all $a \in A_\varphi$, that is, M is a φ-mixed identity of A^{**} and A is not strictly φ-inner amenable. ∎

Remark 2.7. We remark that every strictly ϵ-inner amenable Lau algebra A is strictly inner amenable, where ϵ is the identity of A^*. Indeed, if A is strictly ϵ-inner amenable, then there exists an ϵ-inner mean m on A^* which is not an ϵ-mixed identity, that is, there exists $b \in A$ with $\epsilon(b) = 1$ such that $m \odot b = b \odot m \neq b$. Suppose that $m^+(\epsilon) \neq 0$. By Remark 2.4, $n = m^+/m^+(\epsilon)$ is a topological inner invariant mean on A^*. Now two cases may occur:

First, $m^-(\epsilon) = 0$. In this case, since m^- is positive we have $\|m^-\| = m^-(\epsilon) = 0$. Hence $m^- = 0$ and therefore $n = m^+ = m$, and m is the desired topological inner invariant mean which is not a mixed identity.

Second, $m^-(\epsilon) \neq 0$. Then the same method as in Remark 2.4 shows that $n' := m^-/m^-(\epsilon)$ is also a topological inner invariant mean on A^*. We are going to show that at least one of the means n or n' is not a mixed identity of A^{**}. To this end, it is enough to show that $n \odot b = b \odot n \neq b$ or $n' \odot b = b \odot n' \neq b$. But this is clear, since otherwise $m \odot b = b \odot m = b$, which is a contradiction.

For every commutative Banach algebra A of dimension more than 1, if $\varphi \in \Delta(A)$ and $a \in A_\varphi$ with $a^2 \neq a$, then a is a φ-inner mean on A^* which is not a φ-mixed identity, hence A is strictly φ-inner amenable.

Now we wish to raise the following question:

Question. Can (strictly) 1-inner amenability be characterized in terms of a property of the von Neumann algebra A^* where A is in a certain class of Lau algebras?

To end this section, we prove the next heredity property.

Theorem 2.8. *Let A and B be Banach algebras and suppose that $h : A \to B$ is a continuous homomorphism with dense range. If $\varphi \in \Delta(B)$ and A is $\varphi \circ h$-inner amenable, then B is φ-inner amenable.*

Proof. Let $m \in A^{**}$ satisfy $\langle m, \varphi \circ h\rangle = 1$ and $\langle m, f \cdot a\rangle = \langle m, a \cdot f\rangle$ for all $f \in A^*$ and $a \in A_{\varphi\circ h}$. Define $n \in B^{**}$ by $\langle n, g\rangle = \langle m, g \circ h\rangle$, where $g \in B^*$. Then $\langle n, \varphi\rangle = 1$. Since $h(A)$ is dense in B, for $b \in B_\varphi$ there is a net (a_α) in A such that $h(a_\alpha) \to b$. Therefore $\varphi(h(a_\alpha)) \to \varphi(b) = 1$. After passing to a subnet and replacing $h(a_\alpha)$ by $(1/\varphi(h(a_\alpha)))h(a_\alpha)$ we can assume that $\varphi(h(a_\alpha)) = 1$, that is, $h(a_\alpha) \in B_\varphi$. Now for $\langle n, g \cdot b\rangle = \langle n, b \cdot g\rangle$ to hold for all

$b \in B_\varphi$ and $g \in B^*$, it suffices to verify this equality for $b \in B_\varphi$ of the form $b = h(a)$, $a \in A$. Let a and b be as above. Since $b \in B_\varphi$, we have $a \in A_{\varphi\circ h}$. Now for all $a' \in A$,

$$\langle (g \cdot h(a)) \circ h, a' \rangle = \langle g, h(a)h(a') \rangle = \langle g \circ h, aa' \rangle = \langle (g \circ h) \cdot a, a' \rangle,$$

hence $(g \cdot h(a)) \circ h = (g \circ h) \cdot a$. Similarly, $(h(a) \cdot g) \circ h = a \cdot (g \circ h)$. Hence for all $g \in B^*$,

$$\begin{aligned} \langle n, g \cdot b \rangle &= \langle n, g \cdot h(a) \rangle = \langle m, (g \cdot h(a)) \circ h \rangle = \langle m, (g \circ h) \cdot a \rangle \\ &= \langle m, a \cdot (g \circ h) \rangle = \langle m, (h(a) \cdot g) \circ h \rangle = \langle n, h(a) \cdot g \rangle = \langle n, b \cdot g \rangle, \end{aligned}$$

and the result follows. ■

3. Bounded right approximate identities and φ-inner amenability. In this section we study the concept of φ-inner amenability for Banach algebras with a bounded right approximate identity. To this end, first we fix some notation and definitions.

Let A be a Banach algebra and let X be a left Banach A-module, i.e. a Banach space X equipped with a bounded bilinear map from $A \times X$ into X, denoted by $(a, x) \mapsto a \cdot x$, such that $a \cdot (b \cdot x) = (ab) \cdot x$ for all $a, b \in A$ and $x \in X$. For all $a \in A$, $x \in X$, $x^* \in X^*$ and $x^{**} \in X^{**}$ define

$$\langle a \cdot x^{**}, x^* \rangle = \langle x^{**}, x^* \cdot a \rangle, \qquad \langle x^* \cdot a, x \rangle = \langle x^*, a \cdot x \rangle.$$

Let $B(X^{**})$ denote the Banach space of all bounded operators on X^{**}. By *weak* operator topology* on $B(X^{**})$ we shall mean the locally convex topology of $B(X^{**})$ determined by the family

$$\{T \mapsto |\langle Tx^{**}, x^* \rangle| : x^{**} \in X^{**},\ x^* \in X^*\}$$

of seminorms on $B(X^{**})$. We denote by $B_\varphi(A, X^{**})$ the closure of the set $\{\Lambda_a : a \in A_\varphi\}$ in the weak* operator topology, where $\Lambda_a \in B(X^{**})$ is defined by $\Lambda_a(x^{**}) = a \cdot x^{**}$ for all $x^{**} \in X^{**}$.

It is well-known that $(X^{**} \otimes X^*)^*$ is isometrically isomorphic to $B(X^{**})$ with the isomorphism $\phi : (X^{**} \otimes X^*)^* \to B(X^{**})$ defined by $\phi(F) = \phi_F$, where $\phi_F(x^{**})(x^*) = F(x^{**} \otimes x^*)$ for all $x^{**} \in X^{**}$ and $x^* \in X^*$. So the weak* operator topology of $B(X^{**})$ coincides with the weak* topology of $(X^{**} \otimes X^*)^*$ (see [1]).

Note that for each $a \in A_\varphi$, $\Lambda_a \in B(X^{**})$, and since ϕ is an isomorphism there exists a unique element $F_a \in (X^{**} \otimes X^*)^*$ such that $\phi(F_a) = \Lambda_a$. Therefore for all $x^{**} \in X^{**}$ and $x^* \in X^*$, $\phi(F_a)(x^{**})(x^*) = \Lambda_a(x^{**})(x^*)$, that is, $F_a(x^{**} \otimes x^*) = \langle a \cdot x^{**}, x^* \rangle = \langle x^{**}, x^* \cdot a \rangle$.

Lemma 3.1. *If $H = \{F_a : a \in A_\varphi\} \subset (X^{**} \otimes X^*)^*$. Then $\phi(\overline{H}^{w^*}) = B_\varphi(A, X^{**})$, where $\overline{H}^{w^*}$ denotes the weak* closure of H in $(X^{**} \otimes X^*)^*$.*

Proof. Indeed, $\phi(H) = \{\Lambda_a : a \in A_\varphi\}$. Let $D = \{\Lambda_a : a \in A_\varphi\}$, and let $F \in \overline{H}^{w^*}$. Then there is a net (F_{a_α}) in H such that $F_{a_\alpha} \to F$ in the weak* topology of $(X^{**} \otimes X^*)^*$. Since the weak* operator topology of $B(X^{**})$ coincides with the weak* topology of $(X^{**} \otimes X^*)^*$, $\phi_{F_{a_\alpha}} \to \phi_F$ in the weak* operator topology on $B(X^{**})$, thus $\Lambda_{a_\alpha} \to \phi_F$ in the weak* operator topology on $B(X^{**})$. Therefore ϕ_F belongs to the weak* operator closure of D, which is equal to $B_\varphi(A, X^{**})$. Hence $\phi_F \in B_\varphi(A, X^{**})$ and so $\phi(\overline{H}^{w^*}) \subseteq B_\varphi(A, X^{**})$.

Conversely, let $\Lambda \in B_\varphi(A, X^{**})$. Then there is a net $a_\alpha \in A_\varphi$ such that $\phi_{F_{a_\alpha}} = \Lambda_{a_\alpha} \to \Lambda$ in the weak* operator topology. Since ϕ is onto, there exists $F \in (X^{**} \otimes X^*)^*$ such that $\Lambda = \phi(F)$. Hence $\phi_{F_{a_\alpha}} \to \phi_F$ in the weak* operator topology, and so $F_{a_\alpha} \to F$ in the weak* topology. That is, $F \in \overline{H}^{w^*}$ and $\Lambda = \phi(F) \in \phi(\overline{H}^{w^*})$. ∎

PROPOSITION 3.2. *If the Banach algebra A is φ-inner amenable, then for each left Banach A-module X there exists $\Lambda \in B_\varphi(A, X^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in A_\varphi$.*

Proof. By Theorem 2.1, there exists a bounded net $a_\alpha \in A_\varphi$ such that $\|a_\alpha a - a a_\alpha\| \to 0$ for all $a \in A_\varphi$. Furthermore, if p denotes the projective tensor norm on $X^{**} \otimes X^*$, then for each α,

$$\begin{aligned}
\|F_{a_\alpha}\| &= \sup\{\|F_{a_\alpha}(x^{**} \otimes x^*)\| : p(x^{**} \otimes x^*) = \|x^{**}\| \cdot \|x^*\| = 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad x^{**} \in X^{**},\, x^* \in X^*\} \\
&= \sup\{\|\langle x^{**}, x^* \cdot a_\alpha\rangle\| : \|x^{**}\| \cdot \|x^*\| = 1,\, x^{**} \in X^{**},\, x^* \in X^*\} \\
&\le \sup\{\|x^{**}\|\, \|x^*\|\, \|a_\alpha\| : \|x^{**}\| \cdot \|x^*\| = 1\} = \|a_\alpha\|.
\end{aligned}$$

But (a_α) is bounded, hence the net (F_{a_α}) is bounded. Therefore (F_{a_α}) has a cluster point, say F. Assume that $F_{a_\delta} \to F$ in the weak* topology on $(X^{**} \otimes X^*)^*$, where (a_δ) is a subnet of (a_α). Put $\Lambda = \phi(F)$. Then clearly $\Lambda_{a_\delta} \to \Lambda$ in the weak* operator topology. Thus for each $a \in A_\varphi$, $\Lambda_{a_\delta}\Lambda_a \to \Lambda\Lambda_a$ and $\Lambda_a\Lambda_{a_\delta} \to \Lambda_a\Lambda$ in the weak* operator topology. Moreover $\|\Lambda_{a_\delta}\Lambda_a - \Lambda_a\Lambda_{a_\delta}\| \le K\|a_\delta a - a a_\delta\| \to 0$, where K is a constant satisfying

$$\|b \cdot x\| \le K\|b\| \cdot \|x\|$$

for all $b \in A$ and $x \in X$. Consequently, $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in A_\varphi$. ∎

We are now in a position to give a characterization of φ-inner amenability of a Banach algebra A with a bounded right approximate identity.

THEOREM 3.3. *Suppose that the Banach algebra A has a bounded right approximate identity and let $\varphi \in \Delta(A)$. Then the following are equivalent:*

(i) *A is φ-inner amenable.*
(ii) *There exists $\Lambda \in B_\varphi(A, A^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in A_\varphi$.*

(iii) *For each left Banach A-module X, there exists $\Lambda \in B_\varphi(A, X^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in A_\varphi$.*

Proof. (i)⇒(iii) follows from Proposition 3.2. (iii)⇒(ii) is trivial. Now suppose that (ii) holds, and choose an element Λ of $B_\varphi(A, A^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in A_\varphi$. We prove that (i) holds. By Lemma 3.1, $\phi(\overline{H}^{w^*}) = B_\varphi(A, A^{**})$, thus for $\Lambda \in B_\varphi(A, A^{**})$ there exists $F \in \overline{H}^{w^*}$ such that $\phi(F) = \Lambda$. On the other hand, there is a net (a_α) in A_φ such that $F_{a_\alpha} \to F$ in the weak* topology on $(A^{**} \otimes A^*)^*$, therefore $\phi_{F_{a_\alpha}} \to \phi(F)$ in the weak* operator topology on $B(A^{**})$, that is, $\Lambda_{a_\alpha} \to \Lambda$ in the weak* operator topology.

Define $M \in A^{**}$ by $\langle M, f\rangle = \langle F, E \otimes f\rangle$ for all $f \in A^*$, where $E \in A^{**}$ is a weak* cluster point of a bounded right approximate identity of A. Hence E is a right identity of A^{**}. Now

$$\begin{aligned}\langle M, \varphi\rangle &= \langle F, E \otimes \varphi\rangle = \langle w^*\text{-}\lim_\alpha F_{a_\alpha}, E \otimes \varphi\rangle = \lim_\alpha \langle F_{a_\alpha}, E \otimes \varphi\rangle \\ &= \lim_\alpha \langle E, \varphi \cdot a_\alpha\rangle = \lim_\alpha \langle a_\alpha \diamond E, \varphi\rangle = \lim_\alpha \langle a_\alpha, \varphi\rangle = 1.\end{aligned}$$

It remains to show that $M \odot a = a \odot M$ for all $a \in A_\varphi$. To this end, observe that for $a \in A_\varphi$ and $f \in A^*$ one has

$$\begin{aligned}\langle M \odot a, f\rangle &= \langle M, a \cdot f\rangle = \langle F, E \otimes (a \cdot f)\rangle = \langle w^*\text{-}\lim_\alpha F_{a_\alpha}, E \otimes (a \cdot f)\rangle \\ &= \lim_\alpha \langle F_{a_\alpha}, E \otimes (a \cdot f)\rangle = \lim_\alpha \langle E, (a \cdot f) \cdot a_\alpha\rangle = \lim_\alpha \langle a_\alpha \diamond E, a \cdot f\rangle \\ &= \lim_\alpha \langle a_\alpha, a \cdot f\rangle = \lim_\alpha \langle a_\alpha, (a \cdot E) \cdot f\rangle = \lim_\alpha \langle a_\alpha \odot (a \cdot E), f\rangle \\ &= \lim_\alpha \langle (a_\alpha a) \diamond E, f\rangle = \lim_\alpha \langle E, f \cdot (a_\alpha a)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle a \odot M, f\rangle &= \langle a, M \cdot f\rangle = \langle M, f \cdot a\rangle = \langle F, E \otimes (f \cdot a)\rangle \\ &= \langle w^*\text{-}\lim_\alpha F_{a_\alpha}, E \otimes (f \cdot a)\rangle = \lim_\alpha \langle F_{a_\alpha}, E \otimes (f \cdot a)\rangle \\ &= \lim_\alpha \langle E, (f \cdot a) \cdot a_\alpha\rangle = \lim_\alpha \langle E, f \cdot (a a_\alpha)\rangle\end{aligned}$$

It is enough to observe that the right hand sides of the above equalities coincide, that is,

$$(\star) \qquad \lim_\alpha \langle E, f \cdot (a_\alpha a)\rangle = \lim_\alpha \langle E, f \cdot (a a_\alpha)\rangle$$

Fix $a \in A_\varphi$ and $f \in A^*$. We have $\Lambda_a\Lambda(E) = \Lambda\Lambda_a(E)$, hence $\langle a \cdot (\Lambda E), f\rangle = \langle \Lambda(a \cdot E), f\rangle$, and $\langle a \diamond (\Lambda E), f\rangle = \langle \Lambda(a \cdot E), f\rangle$. Therefore $\langle \Lambda E, f \cdot a\rangle = \langle \Lambda(a \cdot E), f\rangle$.

Since $\Lambda_{a_\alpha} \to \Lambda$ in the weak* operator topology, $\Lambda_{a_\alpha}(E)(f) \to \Lambda(E)(f)$ for all $f \in A^*$. Thus $\lim_\alpha \langle \Lambda_{a_\alpha}(E), f \cdot a)\rangle = \lim_\alpha \langle \Lambda_{a_\alpha}(a \cdot E), f\rangle$. It follows that $\lim_\alpha \langle a_\alpha \cdot E, f \cdot a\rangle = \lim_\alpha \langle a_\alpha \cdot (a \cdot E), f\rangle$. Hence $\lim_\alpha \langle f, a a_\alpha E\rangle = \lim_\alpha \langle f, a_\alpha a E\rangle$, and therefore $\lim_\alpha \langle f \cdot (a a_\alpha), E\rangle = \lim_\alpha \langle f \cdot (a_\alpha a), E\rangle$. It follows that $(\star)$ holds. Consequently, $a \odot M = M \odot a$ for all $a \in A_\varphi$, and A is φ-inner amenable. ∎

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