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APPROXIMATION THEOREMS FOR COMPACTIFICATIONS

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Abstract. We shall show several approximation theorems for the Hausdorff compactifications of metrizable spaces or locally compact Hausdorff spaces. It is shown that every compactification of the Euclidean n-space $\mathbb{R}^n$ is the supremum of some compactifications homeomorphic to a subspace of $\mathbb{R}^{n+1}$. Moreover, the following are equivalent for any connected locally compact Hausdorff space X:

(i) X has no two-point compactifications,
(ii) every compactification of X is the supremum of some compactifications whose remainder is homeomorphic to the unit closed interval or a singleton,
(iii) every compactification of X is the supremum of some singular compactifications.

We shall also give a necessary and sufficient condition for a compactification to be approximated by metrizable (or Smirnov) compactifications.

1. Introduction. Suppose that X is a non-compact completely regular space and let $\mathcal{K}(X)$ be the class of all Hausdorff compactifications of X. For any two compactifications γX and δX of X, we write $\gamma X \le \delta X$ if there is a continuous map $f : \delta X \to \gamma X$ such that $f|_X = \mathrm{id}_X$. If such a map f can be a homeomorphism (i.e., $\gamma X \le \delta X$ and $\gamma X \ge \delta X$), we say γX is *equivalent* ($\sim$) *to* δX. Identifying γX and δX with $\gamma X \sim \delta X$, we may assume that the quotient $(\mathcal{K}(X)/\sim, \le)$ has a partially ordered structure. Throughout this paper, we identify $\mathcal{K}(X)$ with $\mathcal{K}(X)/\sim$.

In this paper, we shall show the following theorem.

THEOREM 1.1. *Every compactification of Euclidean n-space $\mathbb{R}^n$ is the supremum of compactifications that are subspaces of $\mathbb{R}^{n+1}$.*

There have been many studies about approximating the Stone–Čech compactification βX by simpler compactifications. For example, it is known that βX is the supremum of (a) all singular compactifications having the remainder homeomorphic to a closed interval if X is locally compact non-pseudocompact (Chandler and Faulkner [2]), (b) all singular compactifications if X is locally compact 1-complemented, where a locally compact

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space X is said to be 1-*complemented* (or *connected at infinity*) provided each compact set $L_1 \subset X$ is contained in a compact set L_2 such that $X \setminus L_2$ is connected (cf. [2]), (c) all Smirnov compactifications for any metrizable space X (Woods [8]) and (d) all Higson compactifications for any locally compact separable metrizable space X (Kawamura and Tomoyasu [7]).

On the other hand, it was announced that any compactification of $\mathbb{R}^n$ ($n \geq 2$) is the supremum of compactifications having closed intervals as remainders in [5]. In this paper, we shall give the following.

THEOREM 1.2. *Suppose that X is a non-compact locally compact Hausdorff space. Then X has no two-point compactifications if and only if every compactification of X is the supremum of a collection $K \subset \mathcal{K}(X)$ such that each remainder of $\gamma X \in K$ is homeomorphic to the unit closed interval or a singleton. In this case, we can take for K a collection of singular compactifications.*

THEOREM 1.3. *Suppose that X is a connected non-compact locally compact Hausdorff space. Then the following are equivalent:*

(i) *X has no two-point compactifications,*
(ii) *every compactification of X is the supremum of a collection $K \subset \mathcal{K}(X)$ such that each remainder of $\gamma X \in K$ is homeomorphic to the unit closed interval or a singleton,*
(iii) *every compactification of X is the supremum of singular compactifications.*

It is well known that every compactification of a locally compact separable metrizable space is the supremum of some metrizable compactifications. Since any metrizable compactification is a Smirnov compactification (see Fact 3.3), it also follows that every compactification of a locally compact separable metrizable space is the supremum of some Smirnov compactifications. These results are generalized as follows.

THEOREM 1.4. *Suppose that X is a non-compact metrizable space. Then a compactification δX of X can be realized as the supremum of some metrizable compactifications if and only if X has a metrizable compactification γX such that $\gamma X \leq \delta X$.*

THEOREM 1.5. *Suppose that X is a non-compact metrizable space. Then a compactification δX of X can be realized as the supremum of some Smirnov compactifications if and only if X has a Smirnov compactification $u_d X$ for an admissible metric d on X such that $u_d X \leq \delta X$.*

In Section 4, we shall also give an upper bound κ on the minimal cardinality of a collection K in Theorem 1.2 (see Corollaries 4.3 and 4.5). The paper [6] gave the minimal cardinality to approximate βX by Smirnov compactifications in the case of $X = [0,1)$ or ω.

2. Compactifications and subalgebras of $C^*(X)$. Throughout this paper, X is a non-compact completely regular space. We denote by $C^*(X)$ the unital Banach algebra consisting of all bounded continuous functions from X to $\mathbb{R}$ with the supremum norm $\|f\| = \sup_{x\in X} |f(x)|$.

It is well known that the partially ordered set $(\mathcal{K}(X), \leq)$ is isomorphic to the collection $(\mathcal{A}(X), \subset)$ consisting of all closed unital subalgebras $A \subset C^*(X)$ which generate the topology of X, that is, the original topology of X coincides with the weak topology with respect to A. Indeed, the following function $S : \mathcal{K}(X) \to \mathcal{A}(X)$ (cf. [1]) is an isomorphism:

$$S(\gamma X) = \{f|_X \in C^*(X) \mid f \in C^*(\gamma X)\}.$$

Note that $S(\gamma X)$ is isomorphic to $C^*(\gamma X)$ as a Banach algebra and coincides with all functions in $C^*(X)$ which can be extended over γX. For any $A \in \mathcal{A}(X)$, let $e_A : X \to \mathbf{I}^A$ be the embedding defined by $e_A(x) = (f(x)/\|f\|)_{f\in A}$, where $\mathbf{I}$ denotes the closed interval $[-1,1]$. Then the order homomorphism $T : \mathcal{A}(X) \to \mathcal{K}(X)$ defined by $T(A) = \mathrm{cl}_{\mathbf{I}^A}\, e_A(X)$ is the inverse of S (cf. Theorem 3.7 of [1]).

Since $(\mathcal{A}(X), \subset)$ is a complete upper semilattice, so is $(\mathcal{K}(X), \leq)$, that is, every subset $K \subset \mathcal{K}(X)$ has a supremum $\sup K \in \mathcal{K}(X)$. In particular, $\sup \mathcal{K}(X) = \beta X$ is the Stone–Čech compactification of X. If X is locally compact, then $\mathcal{K}(X)$ has a complete lattice structure and the infimum $\inf \mathcal{K}(X) = \alpha X$ is the Aleksandrov one-point compactification of X.

For a subset D of any unital Banach algebra A, $\overline{D}$ denotes the closure of D in A, and $\langle D\rangle$ is the smallest unital subalgebra of A containing D. The following is a key lemma (cf. Theorem 2 of [5]).

LEMMA 2.1. *Suppose that γX is a compactification of X and $A = S(\gamma X)$. For any $g \in C^*(X)$, the compactification $\gamma_g X = T(\overline{\langle A, g\rangle})$ can be embedded in $\mathbf{I} \times \gamma X$.*

Proof. Put $A_g = \overline{\langle A, g\rangle}$. Note that the map $e : X \to \mathbf{I}^{A\cup\{g\}}$ defined by $e(x) = \mathrm{pr}_{A\cup\{g\}} \circ e_{A_g}$ is an embedding, where $\mathrm{pr}_{A\cup\{g\}} : \mathbf{I}^{A_g} \to \mathbf{I}^{A\cup\{g\}}$ is the natural projection. From the definition of the operator T, $\gamma_g X = \mathrm{cl}_{\mathbf{I}^{A_g}}\, e_{A_g}(X)$. Now, we show that the compactification $\gamma'_g X = \mathrm{cl}_{\mathbf{I}^{A\cup\{g\}}}\, e(X)$ is equivalent to $\gamma_g X$. It is clear that $\gamma_g X \geq \gamma'_g X$ via the quotient map $\mathrm{pr}_{A\cup\{g\}}|_{\gamma_g X} : \gamma_g X \to \gamma'_g X$. On the other hand, since each map $f \in A\cup\{g\}$ has an extension $\|f\|\,\mathrm{pr}_f|_{\gamma'_g X}$ over $\gamma'_g X$, the closed subalgebra $S(\gamma'_g X)$ contains $A \cup \{g\}$. This implies $A_g \subset S(\gamma'_g X)$ and

$$\gamma_g X = T(A_g) \leq T(S(\gamma'_g X)) = \gamma'_g X.$$

Hence, $\gamma_g X$ is equivalent to $\gamma'_g X$. Thus, we have the inclusion

$$(1)\qquad \gamma_g X \sim \gamma'_g X = \mathrm{cl}_{\mathbf{I}^{A\cup\{g\}}}\, e(X) \subset \mathbf{I} \times \mathrm{cl}_{\mathbf{I}^A}\, e_A(X) = \mathbf{I} \times \gamma X. \ \blacksquare$$

In the following, we consider the one-point compactification for a locally compact space X.

Lemma 2.2. *Suppose that X is non-compact locally compact having no two-point compactifications. Let $\alpha X = X \cup \{\infty\}$ be the one-point compactification and $A = S(\alpha X)$ a subalgebra of $C^*(X)$. Then the remainder of the compactification $\alpha_g X = T(\overline{\langle A, g\rangle})$ is homeomorphic to $\mathbf{I}$ or a singleton for each $g \in C^*(X)$.*

Proof. Without loss of generality, we may assume $\|g\| = 1$ because $\langle A, g\rangle = \langle A, g/\|g\|\rangle$. Put $e(X) = \{(g(x), x) \mid x \in X\} \subset \mathbf{I} \times \alpha X$. Note that equation (1) in the proof of Lemma 2.1 implies $\alpha'_g X = \mathrm{cl}_{\mathbf{I}\times\alpha X}\, e(X) \sim \alpha_g X$. Thus, it suffices to show that $\nu X = \alpha'_g X \setminus e(X)$ is homeomorphic to $\mathbf{I}$ or a singleton. Since $e(X)$ is closed in $\mathbf{I}\times X$, we have $\nu X = \alpha'_g X \cap (\mathbf{I}\times\{\infty\})$. Now, we shall show if νX is disconnected then X has a two-point compactification. Indeed, if νX is disconnected, take $t \in \mathbf{I}$ such that $(t, \infty) \notin \nu X$, $\nu X \cap ((t, 1]\times\{\infty\}) \neq \emptyset$ and $\nu X \cap ([-1, t)\times\{\infty\}) \neq \emptyset$. Then X is the disjoint union

$$X = g^{-1}([-1, t)) \cup g^{-1}(t) \cup g^{-1}((t, 1]).$$

As $(t, \infty) \notin \nu X$, $g^{-1}(t)$ is compact. On the other hand, the closures of $V_+ = g^{-1}((t, 1])$ and $V_- = g^{-1}([-1, t))$ are both non-compact. Hence, we have a two-point compactification $\gamma X = X \cup \{\pm\infty\}$ with the topology generated by

$$\{U \mid U \text{ open in } X\} \cup \{\{+\infty\} \cup (V_+ \setminus F), \{-\infty\} \cup (V_- \setminus F) \mid F \text{ compact in } X\}.$$

Thus, νX is a compact connected subset of $\mathbf{I} \times \{\infty\}$ and it is homeomorphic to $\mathbf{I}$ or a singleton. $\blacksquare$

3. Singular compactifications and Smirnov compactifications. We call a continuous map f from X to a compact space L *singular* if $\mathrm{cl}_X\, f^{-1}(U)$ is non-compact for every non-empty open set U in L. Then the *singular compactification of X induced by f*, denoted by $X \cup_f L$, is a topological space $X \cup L$ whose topology is generated by the following collection:

$$\{U \mid U \text{ open in } X\} \cup \{V \cup (f^{-1}(V) \setminus F) \mid V \text{ open in } L,\ F \text{ compact in } X\}.$$

It is known that a compactfication γX is equivalent to some singular compactification if and only if $\gamma X \setminus X$ is a retract of γX (cf. [4]). Then we have the following lemma whose simple proof is left to the reader.

LEMMA 3.1. *In Lemma 2.2, $\alpha_g X$ is a singular compactification for any $g \in C^*(X)$.* ∎

For a metrizable space X and an admissible metric d on X, let $U_d^*(X)$ be the subalgebra of $C^*(X)$ consisting of all bounded uniformly continuous functions with respect to d. The *Smirnov* (or *Samuel*) *compactification* $u_d X$ is defined by $u_d X = T(U_d^*(X))$, which is characterized by the following theorem:

THEOREM 3.2 (Theorem 2.5 of [8]). *Suppose that $X = (X, d)$ is a metric space. Then the following are equivalent:*

(i) $\gamma X \sim u_d X$,
(ii) *for any $A, B \subset X$, $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B \neq \emptyset$ if and only if $d(A, B) = 0$.*

Note that it is well known that every compactification of a locally compact separable metrizable space is the supremum of some metrizable compactifications. The following fact implies that every compactification of a locally compact separable metrizable space is the supremum of some Smirnov compactifications.

FACT 3.3. *Every metrizable compactification is a Smirnov compactification.*

Proof. Let γX be a metrizable compactification. Take an admissible metric ρ on γX. Then $d = \rho|_X$ is an admissible metric on X. It is obvious that $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B \neq \emptyset$ if and only if $d(A, B) = 0$ for any $A, B \subset X$. By Theorem 3.2, we have $u_d X \sim \gamma X$. ∎

The following is well known (see the proof of Theorem 3.5.5 of [3]).

THEOREM 3.4 (Taĭmanov). *Suppose that X is a non-compact completely regular space. Let γX and δX be compactifications of X. Then the following are equivalent:*

(i) $\gamma X \leq \delta X$,
(ii) *if A and B are closed subsets in X with $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \emptyset$ then $\mathrm{cl}_{\delta X} A \cap \mathrm{cl}_{\delta X} B = \emptyset$.*

The following lemma implies that βX is realized as the supremum of some Smirnov compactifications, which was shown by Woods [8].

LEMMA 3.5. *Let $X = (X, d)$ be a metric space. Then $u_{d'} X$ is equivalent to $\gamma X = T(\overline{\langle U_d^*(X), g\rangle})$ for any $g \in C^*(X)$, where d' is the metric on X defined by $d'(x, y) = d(x, y) + |g(x) - g(y)|$.*

Proof. Since g and each member of $U_d^*(X)$ are uniformly continuous with respect to the metric d', it is clear that $U_d^*(X) \subset \overline{\langle U_d^*(X), g\rangle} \subset U_{d'}^*(X)$. This implies $u_d X \leq \gamma X \leq u_{d'} X$. To see $u_{d'} X \leq \gamma X$, by Theorem 3.4 it suffices to check that $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \emptyset$ for any closed subsets A and B in X with

$\operatorname{cl}_{u_{d'}X} A \cap \operatorname{cl}_{u_{d'}X} B = \emptyset$. Assume to the contrary that $\operatorname{cl}_{u_{d'}X} A \cap \operatorname{cl}_{u_{d'}X} B = \emptyset$ and there is $x \in \operatorname{cl}_{\gamma X} A \cap \operatorname{cl}_{\gamma X} B$. Then $\varepsilon = d'(A, B) > 0$ by Theorem 3.2. Let $\pi = \operatorname{pr}_{U_d^*(X)}|_{\gamma X} : \gamma X \to u_d X$ be the quotient map and $N(x)$ the open neighborhood system of x in γX. For every $V \in N(x)$, $\pi(x)$ is in $\operatorname{cl}_{u_d X} \pi(A \cap V) \cap \operatorname{cl}_{u_d X} \pi(A \cap V)$ since $x \in \operatorname{cl}_{\gamma X}(A \cap V) \cap \operatorname{cl}_{\gamma X}(B \cap V)$. Note that $\pi(A \cap V) = A \cap V$ and $\pi(B \cap V) = B \cap V$. Thus, $\pi(x) \in \operatorname{cl}_{u_d X}(A \cap V) \cap \operatorname{cl}_{u_d X}(B \cap V)$ and $d(A \cap V, B \cap V) = 0$ by Theorem 3.2. Hence, we can take $a_V \in A \cap V$ and $b_V \in B \cap V$ such that $d(a_V, b_V) < \varepsilon/2$. Then the nets $(a_V)_{V \in N(x)}$ and $(b_V)_{V \in N(x)}$ converge to x in γX and we have

$$\begin{aligned}\lim_{V \in N(x)} g(a_V) &= \lim_{V \in N(x)} \|g\| \operatorname{pr}_g(a_V) = \|g\| \operatorname{pr}_g(x) \\ &= \lim_{V \in N(x)} \|g\| \operatorname{pr}_g(b_V) = \lim_{V \in N(x)} g(b_V).\end{aligned}$$

Hence, there is $V_0 \in N(x)$ such that $|g(a_{V_0}) - g(b_{V_0})| < \varepsilon/2$ and so

$$d'(a_{V_0}, b_{V_0}) = d(a_{V_0}, b_{V_0}) + |g(a_{V_0}) - g(b_{V_0})| < \varepsilon/2 + \varepsilon/2 = \varepsilon = d'(A, B).$$

This is a contradiction. ■

4. Generators for Banach algebras. Recall that $\mathcal{K}(X)$ is the collection of all compactifications of X.

PROPOSITION 4.1. *Let γX and δX be compactifications of a completely regular space X with $\delta X \geq \gamma X$. Then there exists a collection $K \subset \mathcal{K}(X)$ such that $\sup K = \delta X$ and each member of K can be embedded in $\mathbf{I} \times \gamma X$.*

Proof. Let $A = S(\gamma X)$ and $B = S(\delta X)$ be subalgebras of $C^*(X)$. Take any set $D \subset B$ satisfying $\overline{\langle D \rangle} = B$. For any $g \in D \setminus S(\gamma X)$, $\gamma_g X = T(\overline{\langle A, g \rangle})$ can be embedded in $\mathbf{I} \times \gamma X$ by Lemma 2.1. Let $K = \{\gamma_g X \mid g \in D \setminus S(\gamma X)\}$ be a collection of compactifications. Since T is an isomorphism between complete upper semilattices, we have

$$\begin{aligned}\delta X = T(B) &= T(\sup\{\overline{\langle A, g \rangle} \mid g \in D \setminus S(\gamma X)\}) \\ &= \sup\{T(\overline{\langle A, g \rangle}) \mid g \in D \setminus S(\gamma X)\} \\ &= \sup\{\gamma_g X \mid g \in D \setminus S(\gamma X)\} = \sup K. \ \blacksquare\end{aligned}$$

For a set D, $\operatorname{card} D$ denotes the cardinal of D. From the proof of Proposition 4.1, the following corollaries are derived.

COROLLARY 4.2. *In Proposition 4.1, we can take $K \subset \mathcal{K}(X)$ such that each member of K is $T(\overline{\langle A, g \rangle})$ for some $g \in C^*(X)$, where A is the subalgebra of $C^*(X)$ defined by $A = S(\gamma X)$.* ■

COROLLARY 4.3. *In Proposition 4.1, the minimal cardinality of a collection $K \subset \mathcal{K}(X)$ satisfying $\sup K = \delta X$ has an upper bound*

$$\kappa = \min\{\operatorname{card}(D \setminus S(\gamma X)) \mid D \subset S(\delta X),\ \overline{\langle D \rangle} = S(\delta X)\}. \ \blacksquare$$

For any unital Banach algebra A, $\operatorname{gen}(A)$ denotes the minimal cardinality of a subset of A which generates A, that is,

$$\operatorname{gen}(A) = \min\{\operatorname{card} D \mid D \subset A,\ \overline{\langle D\rangle} = A\}.$$

In Corollary 4.3, it is obvious that $\kappa \leq \operatorname{gen}(S(\delta X)) = \operatorname{gen}(C^*(\delta X))$. Let Y be a completely regular space. We denote by $\operatorname{emb}(Y)$ the minimal dimension of Tikhonov cubes containing Y as a subspace:

$$\operatorname{emb}(Y) = \min\{\operatorname{card} D \mid \exists e : Y \hookrightarrow \mathbf{I}^D \text{ an embedding}\}.$$

Then the following proposition yields $\operatorname{gen}(S(\delta X)) = \operatorname{emb}(\delta X)$.

PROPOSITION 4.4. *If L is a compact Hausdorff space, then* $\operatorname{emb}(L) = \operatorname{gen}(C^*(L))$.

Proof. To see $\operatorname{emb}(L) \geq \operatorname{gen}(C^*(L))$, take an embedding $e : L \hookrightarrow \mathbf{I}^{\operatorname{emb}(L)}$. Let $D = \{\operatorname{pr}_\lambda \circ e \in C^*(L) \mid \lambda \in \operatorname{emb}(L)\}$, where $\operatorname{pr}_\lambda : \mathbf{I}^{\operatorname{emb}(L)} \to \mathbf{I}$ is the projection to the λth coordinate. Since e is an embedding, D separates the points of L. Hence, $\overline{\langle D\rangle} = C^*(L)$ by the Stone–Weierstrass Theorem. This implies $\operatorname{gen}(C^*(L)) \leq \operatorname{card} D = \operatorname{emb}(L)$.

To see $\operatorname{emb}(L) \leq \operatorname{gen}(C^*(L))$, let D be a subset of $C^*(L)$ with $\overline{\langle D\rangle} = C^*(L)$. Then D separates the points of L. Hence, the map

$$e : L \to \mathbf{I}^D, \qquad e(x) = (f(x)/\|f\|)_{f\in D},$$

is an embedding. This implies $\operatorname{emb}(L) \leq \operatorname{card} D = \operatorname{gen}(C^*(L))$. ∎

COROLLARY 4.5. *In Corollary* 4.3, $\kappa \leq \operatorname{emb}(\delta X)$. ∎

We say that a unital Banach algebra A is *countably* (resp. *finitely*) *generated* if there exists a countable (resp. finite) subset $D \subset A$ such that $\overline{\langle D\rangle} = A$. The following is a direct consequence of the previous proposition.

COROLLARY 4.6. *Suppose that L is a compact Hausdorff space. Then L is separable (resp. finite-dimensional separable) metrizable if and only if $C^*(L)$ is countably (resp. finitely) generated.* ∎

THEOREM 4.7. *Suppose that X is a completely regular space. Then X is separable (resp. finite-dimensional separable) metrizable if and only if there exists a compactification γX of X such that $C^*(\gamma X)$ is countably (resp. finitely) generated.*

Proof. To show the "if" part, suppose that $C^*(\gamma X)$ is countably (resp. finitely) generated. By Corollary 4.6, γX is separable (resp. finite-dimensional separable) metrizable, thus so is X.

To show the "only if" part, assume that X is separable (resp. finite-dimensional separable) metrizable. Then X can be embedded in the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ (resp. the n-cube $\mathbf{I}^n$ for some $n \in \mathbb{N}$). Let γX be the closure of X in $\mathbf{I}^{\mathbb{N}}$ (resp. $\mathbf{I}^n$). Then $C^*(\gamma X)$ is countably (resp. finitely) generated by Corollary 4.6. ∎

In particular, if X is locally compact (finite-dimensional) separable metrizable, then so is αX. Thus, we have the following corollary.

COROLLARY 4.8. *Suppose that X is a locally compact Hausdorff space. Then X is separable (resp. finite-dimensional separable) metrizable if and only if $C^*(\alpha X)$ is countably (resp. finitely) generated.* ■

5. The proof of approximation theorems. Now, we shall show the statements in Section 1. The following is a direct consequence of Proposition 4.1.

THEOREM 5.1. *Suppose that X is a non-compact locally compact Hausdorff space. If $\mathbf{I} \times \alpha X$ is homeomorphic to a subspace of a topological space Y, then every compactification of X is the supremum of compactifications homeomorphic to a subspace of Y.* ■

In particular, if $X = \mathbb{R}^n$ then the one-point compactification αX is homeomorphic to the n-dimensional sphere $\mathbb{S}^n$. Since $\mathbf{I} \times \mathbb{S}^n$ can be embedded in $\mathbb{R}^{n+1}$, Theorem 1.1 follows from Theorem 5.1 above.

Proof of Theorem 1.2. It is clear that the "only" if part follows from Lemma 2.2 and Corollary 4.2. On the other hand, it is obvious that no two-point compactification of X can be the supremum of compactifications having the remainder homeomorphic to $\mathbf{I}$ or one-point. This yields the "if" part. The representability by singular compactifications is due to Lemma 3.1. We have completed the proof. ■

Proof of Theorem 1.3. From Theorem 1.2, we have (i)⇔(ii) and (i)⇒(iii). Now we shall show (iii)⇒(i). Assume to the contrary that X has a two-point compactification γX. Since γX is the supremum of some singular compactifications, we have a singular map $f : X \to L$ and the singular compactification $X \cup_f L \leq \gamma X$ whose remainder L is a two-point set. This contradicts the connectedness of X. ■

Proof of Theorems 1.4 and 1.5. Let γX be a metrizable (resp. Smirnov) compactification with $\gamma X \leq \delta X$ and $A = S(\gamma X)$ a subalgebra of $C^*(X)$. It follows from Lemma 2.1 (resp. Lemma 3.5) that $T(\overline{\langle A, g\rangle})$ is also a metrizable (resp. Smirnov) compactification for any $g \in C^*(X)$. Thus, we have the result from Corollary 4.2. ■

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