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SYMMETRIC BESSEL MULTIPLIERS

BY

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Abstract. We study the L^p-boundedness of linear and bilinear multipliers for the symmetric Bessel transform.

1. Introduction. Bessel functions occur in the analysis of radial problems. The simplest case is the analysis of structures on $\mathbb{R}^n$ which are invariant under the action of the orthogonal group $O(n)$. In the present paper we are concerned with radiality on matrix spaces $M_{p,q} = M_{p,q}(\mathbb{F})$ over one of the skew-fields $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, as invariance under the left action of the unitary group $U_p = U_p(\mathbb{F})$,

$$U_p \times M_{p,q} \to M_{p,q}, \quad (u, x) \mapsto ux.$$

Note that in [FT], the authors gave the basic elements of radial analysis in $M_{p,q}$. The mapping $U_p x \mapsto \sqrt{x^\star x}$ establishes a homeomorphism between the space of U_p-orbits in $M_{p,q}$ and the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite hermitian $q \times q$-matrices over $\mathbb{F}$. Radial functions on $M_{p,q}$ can thus be considered as functions on the cone Π_q and the Fourier transform of a radial function can be expressed in terms of a generalized Hankel transform involving Bessel functions of a matrix argument. These functions occur in the theory of multi-variable hypergeometric functions of Dunkl type. Let $G = U(p, q)$ denote the indefinite unitary group of index (p, q) over $\mathbb{F}$. Its maximal compact subgroup is naturally isomorphic to $U_p \times U_q$. We may identify $M_{p,q}$ with the tangent space of the Riemannian symmetric space G/K at the coset eK. This action induces an action of $U_p \times U_q$ on $M_{p,q}$ via

$$((u, v), x) \mapsto uxv^{-1}, \quad u \in U_p,\ v \in U_q.$$

The associated orbit space is canonically parameterized by the possible singular spectra of matrices from $M_{p,q}$ and is homeomorphic to

$$\Xi_q = \{\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \cdots \geq \xi_q \geq 0\},$$

which is a Weyl chamber of type B_q.

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Let

$$\mathcal{M}_q := \{pd/2 : p = q, q+1, \dots\} \cup]\rho - 1, \infty[, \qquad d = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4,$$

where $\rho = d(q - 1/2) + 1$ is a real parameter. M. Rösler [R3] has shown that for $\mu \in \mathcal{M}_q$ the set Ξ_q is a locally compact Hausdorff space endowed with a convolution structure $\circ_\mu : M^b(\Xi_q) \times M^b(\Xi_q) \to M^b(\Xi_q)$ such that $(\Xi_q, \circ_\mu)$ is a hypergroup. The characters of Ξ_q were identified with multi-variable Bessel functions of Dunkl type which are associated with root system of type B_q,

$$\xi \mapsto J_k^{B_q}(\xi, i\eta), \qquad \eta \in \Xi_q.$$

These functions satisfy the positive product formula

$$J_k^B(\xi, z) J_k^B(\eta, z) = \int_{\Xi_q} J_k^B(\zeta, z)\, d(\delta_\xi \circ_\mu \delta_\eta)(\zeta), \qquad \xi, \eta \in \Xi_q,\ z \in \mathbb{C}^q.$$

This allows us to introduce the symmetric Bessel translation and symmetric Bessel convolution on Ξ_q by

$$(\tau_\eta f)(\xi) = \int_{\Xi_q} f(\zeta)\, d(\delta_\xi \circ_\mu \delta_\eta)(\zeta), \qquad f \in \mathcal{C}_c(\Xi_q),$$

and

$$(f \circ_\mu g)(\xi) = \int_{\Xi_q} (\tau_\xi f)(\eta) g(\eta)\, d\tilde{\omega}_\mu(\eta), \qquad f \in \mathcal{C}_c(\Xi_q).$$

By analogy with the ordinary Fourier analysis, one can define the symmetric Bessel transform on Ξ_q by

$$\hat{f}(\eta) = \int_{\Xi_q} f(\xi) J_k^B(\xi, i\eta)\, d\tilde{\omega}_\mu(\xi)$$

where $\tilde{\omega}_\mu$ is a Haar measure on Ξ_q.

Let $m : \mathbb{R}^q \to \mathbb{C}$ be a bounded function and define the linear multiplier operator T_m associated with m by $T_m(f) = \mathcal{F}^{-1}(m\mathcal{F}f)$, where $\mathcal{F}$ denotes the ordinary Fourier transform on $\mathbb{R}^q$. The multiplier theorem of Hörmander [Ho] gives a sufficient condition on m guaranteeing the boundedness of T_m on $L^p(\mathbb{R}^q)$ for $1 < p < \infty$. It states that is enough for m to be a bounded C^ℓ-function satisfying

$$\Big(\int_{R/2 \le |\xi| \le R} |\partial_\xi^s m(\xi)|^2 d\xi \Big)^{1/2} \le C R^{q/2 - |s|} \quad \text{for all } R > 0,$$

where ℓ is the least integer greater than $q/2$ and $s = (s_1, \dots, s_q)$, $|s| = s_1 + \dots + s_q \le \ell$.

Anker [A] proves a result analogous to the Hörmander–Mikhlin multiplier theorem on a general Riemannian symmetric space G/K of noncompact type. Next, Gosselin and Stempak [GS] develop Hörmander's original technique to establish an analogous multiplier theorem with respect to the Fourier–Bessel transform.

The aim of this work is to prove the Hörmander multiplier theorem for the symmetric Bessel transform by using Hörmander's technique. This is done in our Theorem 3.1.

The second part of this paper is devoted to the study of L^p-boundedness of bilinear multiplier operators for the symmetric Bessel transform. By means of Littlewood–Paley theory we establish the analogue of Coifman and Meyer's result for a smooth multiplier. Analogous results were obtained in [AGS] for the Dunkl transform in the one-dimensional case.

This paper is organized as follows. In Section 2, we collect the important results of [R3] about the hypergroup $(\Xi_q, \circ_\mu)$. Next, we introduce Bessel functions associated with root systems, and we identify the characters of the hypergroup Ξ_q with Bessel functions of Dunkl type associated with a root system of type B_q. We define a translation operator τ_η, $\eta \in \Xi_q$, which satisfies, for $f \in C_c(\Xi_q)$ (the space of continuous functions on Ξ_q with compact support),

$$\int_{\Xi_q} (\tau_\eta f)(\xi)\, d\tilde{\omega}_\mu(\xi) = \int_{\Xi_q} f(\xi)\, d\tilde{\omega}_\mu(\xi), \qquad \eta \in \Xi_q.$$

Next we define the convolution of two functions on Ξ_q. We give the properties of the translation operator and convolution on Ξ_q. This provides a handy tool for extending some results from the classical Fourier transform to the symmetric Bessel transform. In Section 3, we prove the Hörmander multiplier theorem in greater generality for the symmetric Bessel transform by using Hörmander's techniques. Section 4 is devoted to the study of bilinear multiplier operators for the symmetric Bessel transform.

In what follows, C represents a suitable positive constant which is not necessarily the same at each occurrence. Furthermore, we denote by

- $\mathcal{D}(\mathbb{R}^q)$ the space of C^∞-functions on $\mathbb{R}^q$ with compact support;
- $\mathcal{S}(\mathbb{R}^q)$ (rep. $\mathcal{S}(\Xi_q)$) the space of Schwartz functions on $\mathbb{R}^q$ (resp. Ξ_q);
- $\|\cdot\|_{p,\mu}$ the usual norm of $L^p(\tilde{\omega}_\mu)$.

2. The symmetric Bessel hypergroup $(\Xi_q, \circ_\mu)$

2.1. Preliminaries. In this subsection we collect some basic notation and facts about matrix Bessel hypergroups associated with rational Dunkl operators of type B_q (see [FK], [BH], [J], [R3]).

Let V denote a simple Euclidean Jordan algebra of rank q and of dimension constant d corresponding to the symmetric cone Ω. Then the hypergeometric function ${}_0\mathrm{F}_1^\alpha(\mu;\,\cdot\,)$ essentially coincides, for $\alpha = 2/d$, with the Bessel function $\mathcal{J}_\mu$ associated with Ω in the sense of [FK]. Indeed, the latter is defined by

$$\mathcal{J}_\mu(x) = \sum_{\lambda\geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(x), \qquad x \in V,$$

where $(\mu)_\lambda$ is the generalized Pochhammer symbol, $Z_\lambda = c_\lambda \Phi_\lambda$ with constants $c_\lambda > 0$ are the normalized spherical polynomials, and $\mu \in \mathbb{C}$ is an index with $(\mu)_\lambda \neq 0$ for a partition $\lambda \geq 0$.

In this paper we shall work with Bessel functions of two variables,

$$\mathcal{J}_\mu(x, y) = \sum_{\lambda\geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} \frac{Z_\lambda(x) Z_\lambda(y)}{Z_\lambda(e)}, \qquad x, y \in V,$$

where e is the unit of V. For $x, y \in V$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q)$ and $\eta = (\eta_1, , \ldots, \eta_q)$ respectively, we thus have

$$\mathcal{J}_\mu(x, y) = {}_0\mathrm{F}_1^{2/d}(\mu; i\xi, i\eta).$$

We consider in this work the set H_q of Hermitian $q \times q$ matrices over $\mathbb{F}$ and regard it as a Euclidean vector space with scalar product $(x \mid y) = \operatorname{Re}\operatorname{tr}(xy)$, where tr denotes the trace on $M_q(\mathbb{F})$. The dimension of H_q over $\mathbb{R}$ is $n = q + \frac{d}{2}q(q-1)$ where $d = \dim_{\mathbb{R}} \mathbb{F}$. With the above scalar product and the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, the space H_q becomes a Euclidean Jordan algebra with unit $I = I_q$. The set $\Omega = \Omega_q$ of positive definite matrices from H_q is a symmetric cone.

The following properties summarize the important results shown in [R3]:

PROPERTIES.

(i) For each $\mu \in \mathcal{M}_q$, the set Ξ_q carries a commutative hypergroup structure with convolution

$$(\delta_\xi \circ_\mu \delta_\eta)(f) := \int_{U_q} (f \circ \pi)(\xi *_\mu u\eta u^{-1})\, du, \qquad f \in C(\Xi_q).$$

The neutral element of the hypergroup $\Xi_{q,\mu} := (\Xi_q, \circ_\mu)$ is $0 \in \Xi_q$ and the involution is given by the identity mapping.

(ii) A Haar measure on Ξ_q is given by

$$\tilde{\omega}_\mu = d_\mu h_\mu(\xi)\, d\xi, \quad \text{with} \quad h_\mu(\xi) = \prod_{i=1}^q \xi_i^{2\gamma+1} \prod_{i<j} (\xi_i^2 - \xi_j^2)^d$$

and a constant $d_\mu > 0$.

(iii) The characters of the hypergroup Ξ_q are all defined by

$$\psi_\xi(\eta) := \int_{U_q} \varphi_\xi(u\eta u^{-1})\, du, \qquad \xi \in \Xi_q,$$

where $\varphi_\xi(r) = \mathcal{J}_\mu(\frac{1}{4} r\xi^2 r)$, and $\psi_\xi \in C_b(\Xi_q)$. We easily verify that $\psi_\xi(\eta) = \psi_\eta(\xi)$ for all $\xi, \eta \in \Xi_q$.

The dual space of the hypergroup $\Xi_q = (\Xi_q, \circ_\mu)$ is given by

$$\widehat{\Xi}_{q,\mu} = \{\psi_\xi : \xi \in \Xi_q\}.$$

(iv) The hypergroup Ξ_q is self-dual via the homeomorphism $\Xi_q \to \widehat{\Xi}_q$, $\xi \mapsto \psi_\xi$. Under this identification, the Plancherel measure $\tilde{\pi}_\mu$ of Ξ_q coincides with the Haar measure $\tilde{\omega}_\mu$.

(v) For $\xi, \eta \in \Xi_q$, we have the integral representation

$$\psi_\xi(\eta) = \int\limits_{U_q} \mathcal{J}_\mu\left(\frac{1}{4}\xi u \eta^2 u^{-1} \xi\right) du = \mathcal{J}_\mu(\xi^2/2, \eta^2/2).$$

2.2. Dunkl theory and Dunkl Bessel functions. Let G be a finite reflection group on $\mathbb{R}^q$ equipped with the usual scalar product $\langle\cdot,\cdot\rangle$, and R be a reduced root system of G. From now on we assume that R is normalized in the sense that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in R$; this simplifies formulas, but is no loss of generality. We extend the action of G to $\mathbb{C}^q$ and $\langle\cdot,\cdot\rangle$ to a bilinear form on $\mathbb{C}^q \times \mathbb{C}^q$. The Dunkl operators associated with R can be considered as perturbations of the usual partial derivatives by reflection parts. These reflections parts are coupled by means of parameters, which are given in terms of multiplicity functions:

A function $k : R \to \mathbb{C}$ which is invariant under G is called a *multiplicity function* on R. For a finite reflection group G and a fixed multiplicity function k on its root system, the associated (rational) *Dunkl operators* are defined by

$$(T_n f)(x) = \frac{\partial f}{\partial x_n}(x) + \frac{1}{2}\sum_{\alpha\in R} k(\alpha)\alpha_n \frac{f(x) - f(\sigma_\alpha x)}{\langle\alpha, x\rangle}, \qquad 1 \le n \le q,\ x \in \mathbb{R}^q;$$

here σ_α denotes the reflection in the hyperplane perpendicular to α and the action of G is extended to functions on $\mathbb{C}^q$ via $g.f(x) := f(g^{-1}x)$ (see [D] and [DO] for more properties of T_n, $1 \le n \le q$). Moreover, for each fixed $\omega \in \mathbb{C}^q$, the joint eigenvalue problem

$$T_n f = \omega_n f, \qquad f(0) = 1, \qquad 1 \le n \le q,\ \xi \in \mathbb{C}^q,$$

has a unique holomorphic solution $f(z) = E_k(z,\omega)$ called the *Dunkl kernel*. It is symmetric in its arguments and satisfies $E_k(\lambda z, \omega) = E(z, \lambda\omega)$ for all $\lambda \in \mathbb{C}$ as well as $E_k(gz,\omega) = E(z, g\omega)$ for all $g \in G$. The *generalized Bessel function*

$$J_k(z,\omega) := \frac{1}{|G|}\sum_{g\in G} E_k(z, g\omega) \tag{2.1}$$

is G-invariant in both arguments. Moreover, $g(z) = J_k(z,\omega)$ is the unique holomorphic solution of the Bessel system

$$p(T)f = p(\omega)f, \qquad g(0) = 1, \qquad p \in \mathcal{P}^G,$$

where $T = (T_1, \dots, T_q)$ and $\mathcal{P}^G$ denotes the subalgebra of G-invariant polynomials in $\mathcal{P}$ (see [O]). The Dunkl kernel E_k gives rise to an integral transform on $\mathbb{R}^q$ called the *Dunkl transform*. Let ω_k denote the weight function

$$\omega_k(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{2k(\alpha)}$$

on $\mathbb{R}^q$.

Let us denote by J_k^B the Dunkl type Bessel function associated with the root system $R = B_q$, given by

$$B_q = \{\pm e_j : 1 \le j \le q\} \cup \{\pm e_i \pm e_j : 1 \le i < j \le q\},$$

where $(e_i)_{1 \le i \le q}$ is the canonical basis of $\mathbb{R}^q$ and $k = (k_1, k_2)$ a multiplicity function, and by $[\cdot, \cdot]_k^B$ the associated Dunkl pairing (see [D], [R2]). For $z = (z_1, \dots, z_q) \in \mathbb{C}^q$ we put $z^2 = (z_1^2, \dots, z_q^2)$.

The key result of [R3] identifies J_k^B with a generalized ${}_0\mathrm{F}_1$ hypergeometric function of two arguments: For $z, \omega \in \mathbb{C}^q$, we have

$$J_k^B(z, \omega) = {}_0\mathrm{F}_1^{\alpha}(\mu; z^2/2, \omega^2/2) \quad \text{with} \quad \alpha = \frac{1}{k_2},\ \mu = k_1 + (m-1)k_2 + 1/2.$$

As a consequence, Bessel functions associated with a symmetric cone can be identified with Dunkl Bessel functions of type B_q with specific multiplicities.

Let Ω be an irreducible symmetric cone in a Euclidean Jordan algebra of rank q. Then for $r, s \in \overline{\Omega}$ with eigenvalues $\xi = (\xi_1, \dots, \xi_q)$ and $\eta = (\eta_1, \dots, \eta_q)$ respectively, we have

$$\mathcal{J}_\mu(r^2/2, s^2/2) = J_k^B(\xi, i\eta)$$

where $k = k(\mu, d) = (\mu - (d/2)(q-1) - 1/2, d/2)$.

A consequence of the above identification can be formulated in two ways:

1) The characters of the hypergroup $\Xi_{q,\mu}$, $\mu \in \mathcal{M}_q$, are given by

$$\psi_\eta(\xi) = J_k^B(\xi, i\eta), \qquad \eta \in \Xi_q,$$

with multiplicity $k = k(\mu, d)$ as above.

2) Consider a root system of type B_q with multiplicity $k = (k_1, k_2)$ where $k_2 = d/2$, $d \in \{1, 2, 4\}$, and $k_1 = (d/2)(p - q + 1) - 1/2$ for integer $p \ge q$ or arbitrary $k_1 \ge \frac{1}{2}(dq - 1)$. Then the associated Dunkl type Bessel functions $\xi \mapsto J_k^B(\xi, i\eta)$ are the characters of the hypergroup $(\Xi_q, \circ_\mu)$, where $\mu = k_1 + (q-1)k_2 + 1/2$ and the convolution $\circ_\mu$ is defined over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ depending on the value of d. In particular we have

THEOREM 2.1 ([R3]). *The Bessel function J_k^B satisfies the positive product formula*

$$J_k^B(\xi, z) J_k^B(\eta, z) = \int_{\Xi_q} J_k^B(\zeta, z)\, d(\delta_\xi \circ_\mu \delta_\eta)(\zeta), \qquad \xi, \eta \in \Xi_q,\ z \in \mathbb{C}^q. \tag{2.2}$$

We shall need the following anti-symmetry of Dunkl operators ([R2]).

PROPOSITION 2.2. *Let $k \geq 0$ and $1 \leq n \leq q$. Then for every $F \in \mathcal{S}(\mathbb{R}^q)$ and $G \in C_b^1(\mathbb{R}^q)$,*

$$\int_{\mathbb{R}^N} (T_n F)(x) G(x) \omega_k(x)\, dx = - \int_{\mathbb{R}^N} F(x) (T_n G)(x) \omega_k(x)\, dx.$$

PROPOSITION 2.3. *In our situation, i.e. for the Haar measure $\tilde{\omega}_\mu$,*

$$\int_{\Xi_q} (T_n f)(\eta) g(\eta)\, d\tilde{\omega}_\mu(\eta) = - \int_{\Xi_q} f(\eta)(T_n g)(\eta)\, d\tilde{\omega}_\mu(\eta), \qquad 1 \leq n \leq q.$$

Proof. Recall that $d\tilde{\omega}_\mu(\eta) = d_\mu h_\mu(\eta) d\eta$. Notice that h_μ coincides up to a constant factor with ω_k for $k = k(\mu, d)$. As ω_k is B_q-invariant, we therefore have

$$T_n f = T_n F|_{\Xi_q}, \qquad 1 \leq n \leq q,$$

where F (resp. G) is the B_q-invariant extension of f (resp. g) to $\mathbb{R}^q$. ∎

The Dunkl transform is defined on $L^1(\mathbb{R}^q, \omega_k)$ by

$$f \mapsto \hat{f}^k, \qquad \hat{f}^k(\xi) = c_k^{-1} \int_{\mathbb{R}^q} f(x) E_k(-i\xi, x) \omega_k(x)\, dx, \qquad \xi \in \mathbb{R}^q,$$

with the constant

$$c_k := \int_{\mathbb{R}^q} e^{-|x|/2} \omega_k(x)\, dx$$

(see [dJ1] and [dJ2] for more details). It shares many properties with the usual Fourier transform to which it reduces in case $k = 0$. In particular, the Dunkl transform (as normalized above) extends to an isometric isomorphism of $L^2(\mathbb{R}^q, \omega_k)$, and

$$(\widehat{T_\eta f})^k(\xi) = i\xi_n \hat{f}^k(\xi)$$

for differentiable f of sufficient decay.

The symmetric Bessel transform on Ξ_q is given by

$$\hat{f}(\eta) = \int_{\Xi_q} f(\xi) J_k^B(\xi, i\eta)\, d\tilde{\omega}_\mu(\xi).$$

As ω_k is B_q-invariant, we have $\hat{f}(\eta) = \mathrm{const} \cdot \hat{F}^k(\eta)$, where F denotes the B_q-invariant extension of f to $\mathbb{R}^q$ and $\hat{F}^k$ its Dunkl transform. Using the Plancherel theorem for the Dunkl transform and the identification of Ξ_q with its dual we obtain

$$\hat{f} = \hat{F}^k|_{\Xi_q} \quad \text{and} \quad d_\mu = \Big(\int_{\Xi_q} h_\mu(x) e^{-|x|^2/2}\, dx \Big)^{-1}.$$

The symmetric Bessel transform has the following properties:

(i) For $f \in L^1(\tilde{\omega}_\mu)$ we have

$$\|\hat{f}\|_{\infty,\mu} \leq \|f\|_{1,\mu}. \tag{2.3}$$

(ii) For $f \in \mathcal{S}(\varXi_q)$ we have

$$\widehat{p(T)f}(\eta) = p(-i\eta)\hat{f}(\eta), \quad p \in \mathcal{P}^G. \tag{2.4}$$

(iii) $\hat{}: f \mapsto \hat{f}$ is a topological automorphism of $\mathcal{S}(\varXi_q)$.

(iv) $\hat{}: f \mapsto \hat{f}$ is an isometric automorphism of $L^2(\tilde{\omega}_\mu)$ and we have the Parseval and Plancherel formulas: If $f, g \in L^1(\tilde{\omega}_\mu) \cap L^2(\tilde{\omega}_\mu)$, then

$$\int_{\varXi_q} f\bar{g}\, d\tilde{\omega}_\mu = \int_{\varXi_q} \hat{f}\,\overline{\hat{g}}\, d\tilde{\omega}_\mu, \quad \|\hat{f}\|_{2,\mu} = \|f\|_{2,\mu}.$$

(v) (Inversion formula) For $f \in L^1(\tilde{\omega}_\mu)$ such that $\hat{f} \in L^1(\tilde{\omega}_\mu)$ we have

$$f(\eta) = \int_{\varXi_q} \hat{f}(\xi) J_k^B(\xi, i\eta)\, d\tilde{\omega}_\mu(\xi). \tag{2.5}$$

For more details about these properties see [BH, Section 2.2, Chap 2]. (We shall apply the results of this section to our hypergroup $\varXi_q$ where its characters for $\mu \in \mathcal{M}_q$ are given by $\psi_\eta(\xi) = J_k^B(\xi, i\eta)$ and the Plancherel measure coincides with $\tilde{\omega}_\mu$ under the natural identification of $\varXi_q$ with its dual.)

2.3. Symmetric Bessel translation and symmetric Bessel convolution

Definition 2.4. For $\xi, \eta \in \varXi_q$ and a continuous function f on $\varXi_q$, we put

$$(\tau_\eta f)(\xi) = \int_{\varXi_q} f(\zeta)\, d(\delta_\xi \circ_\mu \delta_\eta)(\zeta),$$

and call τ_η the *symmetric Bessel translation* operator on $\varXi_q$.

The symmetric Bessel translation operator has the following properties.

Properties.

(1) τ_η is a continuous linear operator from $C_c(\varXi_q)$ into itself.

(2) For $\xi, \eta \in \varXi_q$ and $f \in C_c(\varXi_q)$, we have

$$(\tau_\eta f)(\xi) = (\tau_\xi f)(\eta), \quad \tau_\eta \circ \tau_\xi = \tau_\xi \circ \tau_\eta,$$
$$(\tau_0 f)(\xi) = f(\xi), \quad T_n \circ \tau_\eta = \tau_\eta \circ T_n.$$

(3) For all $\xi \in \varXi_q$, the operator τ_ξ can be extended to $L^p(\tilde{\omega}_\mu)$ $(p \geq 1)$ and for $f \in L^p(\tilde{\omega}_\mu)$ we have

$$\|\tau_\xi f\|_{p,\mu} \leq \|f\|_{p,\mu}.$$

(4) (Product formula) For all $\xi, \eta, \zeta \in \varXi_q$,

$$\tau_\xi(J_k^B(\cdot, \zeta))(\eta) = J_k^B(\xi, \zeta) J_k^B(\eta, \zeta).$$

(5) Let f, g be two measurable and positive functions on Ξ_q and let $\xi \in \Xi_q$. If either f or g is σ-finite with respect to $\tilde{\omega}_\mu$, then

$$\int\limits_{\Xi_q} (\tau_\xi f)(\eta) g(\eta)\, d\tilde{\omega}_\mu(\eta) = \int\limits_{\Xi_q} f(\eta)(\tau_\xi g)(\eta)\, d\tilde{\omega}_\mu(\eta).$$

(6) For all $\xi, \eta \in \Xi_q$ and $f \in L^1(\tilde{\omega}_\mu)$, we have

$$\widehat{\tau_\xi f}(\eta) = J_k^B(\xi, i\eta)\hat{f}(\eta).$$

Definition 2.5. Let f and g be two continuous functions on Ξ_q with compact support. Then we define the *convolution* of f and g by

$$(f \circ_\mu g)(\xi) = \int\limits_{\Xi_q} (\tau_\xi f)(\eta) g(\eta)\, d\tilde{\omega}_\mu(\eta), \quad \text{a.e. } \xi.$$

Properties.

(1) The convolution $\circ_\mu$ is associative and commutative.

(2) Let $p, q, r \in [1, \infty]$ be such that $1/p + 1/q = 1/r$. The map $(f, g) \mapsto f \circ_\mu g$, defined on $C_c(\Xi_q) \times C_c(\Xi_q)$, extends to a continuous map from $L^p(\tilde{\omega}_\mu) \times L^q(\tilde{\omega}_\mu)$ to $L^r(\tilde{\omega}_\mu)$, and

$$\|f \circ_\mu g\|_{r,\mu} \le \|f\|_{p,\mu} \|g\|_{q,\mu}.$$

(3) If $f \in L^1(\tilde{\omega}_\mu)$ and $g \in L^2(\tilde{\omega}_\mu)$, then

$$\widehat{f \circ_\mu g} = \hat{f} \cdot \hat{g}. \tag{2.6}$$

(4) If $\mathrm{supp}(f) \subset \{x : |x| \le a\}$ and $\mathrm{supp}(g) \subset \{x : b \le |x| \le c\}$ with $0 < a < b < c$, then

$$\mathrm{supp}(f \circ_\mu g) \subset \{x : b - a \le |x| \le c + a\}. \tag{2.7}$$

3. Hörmander multiplier theorem. Let $m : \Xi_q \to \mathbb{C}$ be a bounded measurable function. We define a linear transformation T_m on $L^2(\tilde{\omega}_\mu) \cap L^p(\tilde{\omega}_\mu)$ by

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi).$$

We shall say that m is an $L^p(\tilde{\omega}_\mu)$ *multiplier* if

$$\|T_m f\|_{p,\mu} \le A_p \|f\|_{p,\mu}. \tag{3.1}$$

The smallest A_p for which (3.1) holds will be called the norm of the multiplier. We denote by $\mathfrak{M}_p$ the class of $L^p(\tilde{\omega}_\mu)$ multipliers with the indicated norm. It is clearly a Banach algebra under pointwise multiplication.

Theorem 3.1. *Suppose that m is a bounded C^ℓ-function on $\Xi_q \setminus \{0\}$, where ℓ is the least integer such that $\ell > \mu q$, satisfying the Hörmander condition: For every differential monomial ∂_ξ^s, $s = (s_1, \ldots, s_q)$ with $|s| =$*

$s_1 + \cdots + s_q \leq \ell$, and every $0 < R < \infty$,

$$\Big(\int\limits_{R/2\leq|\xi|\leq 2R} |\partial_\xi^s m(\xi)|^2 \, d\tilde{\omega}_\mu(\xi) \Big)^{1/2} \leq C R^{\mu q - |s|}, \tag{3.2}$$

where C is a constant. Then $m \in \mathfrak{M}_p$, $1 < p < \infty$, that is,

$$\|T_m f\|_{p,\mu} \leq A_p \|f\|_{p,\mu}.$$

Remark 3.2. Theorem 3.1 gives a sufficient condition for a function m to be a G-invariant Dunkl multiplier of the root system B_q.

The condition (3.2) is satisfied if for ℓ an integer greater than μq, m is a C^ℓ-function on $\varXi_q \setminus \{0\}$ satisfying

$$|\partial_\xi^s m(\xi)| \leq C/|\xi|^{|s|} \quad \text{whenever } |s| \leq k.$$

The following theorem plays a crucial role in the proof of Theorem 3.1.

Theorem 3.3. *Let $h \in L^2(\tilde{\omega}_\mu)$ be such that its symmetric Bessel transform $\hat{h}$ is essentially bounded. Put $H(\xi,\eta) = (\tau_\xi h)(\eta)$ and suppose that*

$$\int\limits_{|\xi-\eta|\geq 2|\eta-\eta_0|} |H(\xi,\eta) - H(\xi,\eta_0)| \, d\tilde{\omega}_\mu(\xi) \leq C, \qquad \eta,\eta_0 \in \varXi_q.$$

Let T be a bounded linear transformation mapping $L^2(\tilde{\omega}_\mu)$ to itself, such that for $f \in L^1 \cap L^p(\tilde{\omega}_\mu)$, we have

$$(Tf)(\xi) = \int\limits_{\varXi_q} H(\xi,\eta) f(\eta) \, d\tilde{\omega}_\mu(\eta) \tag{3.3}$$

for a.e. $\xi \in \varXi_q$. Then there exists a constant A_p such that

$$\|Tf\|_{p,\mu} \leq A_{p,\mu} \|f\|_p, \qquad 1 < p \leq 2.$$

One can thus extend T to all of $L^p(\tilde{\omega}_\mu)$ by continuity. The constant A_p depends only on p, C, and the rank q. In particular it does not depend on the L^2 norm of h.

Remark 3.4. The assumption that $h \in L^2(\tilde{\omega}_\mu)$ is made for the purpose of having a direct definition of Tf on a dense subset of $L^p(\tilde{\omega}_\mu)$ (in this case $L^1 \cap L^p(\tilde{\omega}_\mu)$) and it could be replaced by other assumptions. In applications this hypothesis is of no consequence since it can be dispensed with by an appropriate limiting process; this is because the final bounds in Theorem 3.3 do not depend on the L^2 norm of h.

We first note that $(\varXi_q, \tilde{\omega}_\mu)$ is a space of homogeneous type ([S2, Ch. I]), that is, there is a fixed constant $C > 0$ such that

$$\tilde{\omega}_\mu(B_q(x, 2r)) \leq C \tilde{\omega}_\mu(B_q(x,r)), \qquad x \in \varXi_q,\ r > 0,$$

where $B_q(x,r)$ is the intersection of $\varXi_q$ with the closed ball of radius r centered at x. Then we can adapt to our context the classical technique.

We shall need the following lemma.

LEMMA 3.5. *Let f be a nonnegative integrable function on $\mathbb{R}^q$ and α be a positive constant. Then there exists a decomposition of $\mathbb{R}^q$ so that*

(i) $\mathbb{R}^q = F \cup \Omega$, $F \cap \Omega = \emptyset$.
(ii) $f(\xi) \le \alpha$ *almost everywhere on* F.
(iii) Ω *is the union of cubes,* $\Omega = \bigcup_k Q_k$, *whose interiors are disjoint, and for each* Q_k *there exist constants* A *and* C (*depending only on the dimension* q) *such that*

$$\tilde{\omega}_\mu(\Omega) \le \frac{A}{\alpha}\|f\|_{1,\mu}, \qquad \frac{1}{\tilde{\omega}_\mu(Q_k)} \int\limits_{Q_k} f(\xi)\, d\tilde{\omega}_\mu(\xi) \le C\alpha.$$

Proof. The proof is similar to that of Theorem 4 of [S1, p. 17]. In fact, it suffices to replace the Lebesgue measure by $\tilde{\omega}_\mu$. ∎

Proof of Theorem 3.3. First, for $f \in L^1 \cap L^2(\tilde{\omega}_\mu)$, we have

$$\widehat{Tf}(\zeta) = \hat{h}(\zeta)\hat{f}(\zeta).$$

The Plancherel theorem gives

$$\|Tf\|_{2,\mu} \le C\|f\|_{2,\mu},$$

which implies that T has a unique extension to all $L^2(\tilde{\omega}_\mu)$, where the above inequality still valid. For $\alpha > 0$, we obtain

$$\tilde{\omega}_\mu(\{\xi \in \Xi_q : |(Tf)(\xi)| > \alpha|\}) \le \frac{C^2}{\alpha^2} \int\limits_{\Xi_q} |f|^2\, d\tilde{\omega}_\mu, \qquad f \in L^2(\tilde{\omega}_\mu).$$

Thus T is of weak type $(2,2)$.

Now, to prove Theorem 3.3 it suffices to prove that T is of weak type $(1,1)$ and conclude by the Marcinkiewicz interpolation theorem. For this it suffices to replace, in the proof of Theorem 3 of [S2, p. 20], q by 2, $B_k^\star$ by Q_k and $B_k^{\star\star}$ by $Q_k^\star$, so $F^\star = (\bigcup B_k^{\star\star})^c$. ∎

Before proving Theorem 3.1, we need two lemmas.

In the first lemma, we prove a Bernstein inequality for the symmetric Bessel translation. An analogous result has been proved in [GS] for the generalized translation associated with the Bessel operator.

LEMMA 3.6 (Bernstein's inequality). *Let $\lambda > 0$ and $f \in L^1(\tilde{\omega}_\mu)$ be such that $\hat{f}$ is supported in $B_\lambda = \{|\xi| \le \lambda\}$. Then for all $x, y \in \Xi_q$,*

$$\|\tau_x f - \tau_y f\|_{1,\mu} \le C\lambda|x-y|\, \|f\|_{1,\mu}.$$

Proof. Let $h \in \mathcal{S}(\Xi_q)$ be a radial function satisfying $\hat{h} = 1$ in B_1 and let $h_\lambda(x) = \lambda^{\mu q} h(\lambda x)$; then $\hat{h}_\lambda(x) = \hat{h}(x/\lambda) = 1$ in B_λ.

An explicit formula for $\tau_x h$, due to Rösler [R1], with $h(y) = \tilde{h}(|y|)$, is

$$(\tau_x h)(y) = \int_{\Xi_q} \tilde{h}(A(x, -y, \eta))\, d\nu_x(\eta),$$

where ν_x is a probability measure supported in the convex hull $\mathrm{co}(Gx)$ and

$$A(x, y, \eta) = \sqrt{|x|^2 + |y|^2 - 2\langle y, \eta\rangle}.$$

Writing

$$\tau_x f - \tau_y f = h_\lambda \circ_\mu (\tau_x f - \tau_y f) = (\tau_x h_\lambda - \tau_y h_\lambda) \circ_\mu f,$$

we obtain

$$\|\tau_x f - \tau_y f\|_{1,\mu} \le \|\tau_x h_\lambda - \tau_y h_\lambda\|_{1,\mu} \|f\|_{1,\mu} = \|\tau_{\lambda x} h - \tau_{\lambda y} h\|_{1,\mu}\, \|f\|_{1,\mu}.$$

Let $x, y \in \Xi_q$. Then

$$\begin{aligned}
\|\tau_x h - \tau_y h\|_{1,\mu} &= \int_{\Xi_q} |(\tau_x h)(z) - (\tau_y h)(z)|\, d\tilde{\omega}_\mu(z) \\
&= \int_{\Xi_q} |(\tau_z h)(x) - (\tau_z h)(y)|\, d\tilde{\omega}_\mu(z) \\
&\le \int_{\Xi_q} \int_{\Xi_q} |\tilde{h}(A(z, -x, \eta)) - \tilde{h}(A(z, -y, \eta))|\, d\nu_z(\eta)\, d\tilde{\omega}_\mu(z) \\
&\le \int_{\Xi_q} \int_{\Xi_q} \int_0^1 |\tilde{h}'(A(z, -x + s(x - y), \eta))| \\
&\qquad \times \left| \frac{d}{ds} A(z, -x + s(x - y), \eta) \right| ds\, d\nu_z(\eta)\, d\tilde{\omega}_\mu(z).
\end{aligned}$$

As $\left|\frac{d}{ds} A(z, -x + s(x - y), \eta)\right| \le |x - y|$, it follows that

$$\|\tau_x h - \tau_y h\|_{1,\mu} \le |x - y| \int_{\Xi_q} \sup_{s\in[0,1]} \sup_{\eta\in\mathbb{R}^q} |\tilde{h}'(A(z, -x + s(x - y), \eta))|\, d\tilde{\omega}_\mu(z).$$

Thus

$$\|\tau_x f - \tau_y f\|_{1,\mu} \le C\lambda |x - y|\, \|f\|_{1,\mu}. \ \blacksquare$$

LEMMA 3.7. *Assume m satisfies the condition* (3.2). *Then there exists a locally integrable function $h \in \Xi_q \backslash \{0\}$ such that for all $\xi \in (\mathrm{supp}(f))^c$,*

$$(T_m f)(\xi) = \int_{\Xi_q} H(\xi, \eta) f(\eta)\, d\tilde{\omega}_\mu(\eta)$$

where H is given by $H(\xi, \eta) = (\tau_\xi h)(\eta)$, $\xi \ne \eta$.

Proof. In the whole proof C will denote constants depending only on q which may have different values in different formulas.

Let $\varphi \in \mathcal{D}(\Xi_q)$ be a function supported in $\{1/2 < |\xi| < 2\}$ such that

$$\sum_{j=-\infty}^{+\infty} \varphi_j(\xi) = 1, \quad \xi \neq 0,$$

where we put $\varphi_j(\xi) = \varphi(2^{-j}\xi)1_{\Xi_q}(\xi) = \varphi(2^{-j}\xi_1, \ldots, 2^{-j}\xi_q)1_{\Xi_q}(\xi_1, \ldots, \xi_q)$. Let $m_j(\xi) = m(\xi)\varphi_j(\xi)$. The support of m_j is contained in the spherical crown $\{2^{j-1} < |\xi| < 2^{j+1}\}$. Leibniz's formula gives, for $1 \leq n \leq q$,

$$\partial_\xi^s m_j(\eta) = \sum_{a+b=s} 2^{-j|b|} \partial_\xi^a m(\xi) \partial_\xi^b \varphi(2^{-j}\eta). \tag{3.4}$$

Using (3.2) with $R = 2^j$ and the fact that the derivatives of φ are bounded, we obtain

$$\int\limits_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \sum_{|s| \leq k} |2^{j|s|} \partial_\xi^s m_j(\xi)|^2 \, d\tilde{\omega}_\mu(\xi) \leq C \cdot 2^{j\mu q}.$$

Let h_j be the inverse symmetric Bessel transform of m_j. Plancherel's theorem and (2.4) yield

$$\|(-|\eta|^2)^{|s|} h_j(\eta)\|_{2,\mu} = \|\Delta_\kappa^{|s|} m_j\|_{2,\mu}, \quad |s| \leq \ell,$$

where $\Delta_\kappa = \sum_{n=1}^q T_n^2$ is the Dunkl laplacian.

We shall now prove the estimate

$$\|\Delta_\kappa^{|s|} m_j\|_{2,\mu} \leq C \cdot 2^{j(\mu q - |s|)}, \quad |s| \leq \ell. \tag{3.5}$$

The Dunkl operator is given by

$$T_n = \frac{\partial}{\partial x_n} + \sum_{\alpha \in R} a_{\alpha,n} \frac{\mathrm{id} - \sigma_\alpha}{\langle \alpha, \cdot \rangle}.$$

where $a_{\alpha,\xi} = \frac{1}{2}k(\alpha)\alpha_n$. Recall that for $\alpha \in R$, σ_α is the reflection in the hyperplane perpendicular to α. Since Ξ_q is a Weyl chamber (see [R3]), if $\eta \in \Xi_q$, then for all $\alpha \in R$, $\sigma_\alpha \eta \notin \Xi_q$; thus $m_j(\sigma_\alpha \eta) = 0$. Therefore the Dunkl operator is reduced to

$$(T_n m_j)(\eta) = \frac{\partial}{\partial \eta_n} m_j(\eta) + \sum_{\alpha \in R} a_{\alpha,n} \frac{1}{\langle \alpha, \eta \rangle} m_j(\eta).$$

Since $\partial/\partial \eta_n$ and $\mathrm{id}/\langle \alpha, \cdot \rangle$ do not commute, we cannot apply the binomial formula. A straightforward calculation gives, for $\eta \neq 0$ and $r \in \mathbb{N}$,

$$(T_n^r m_j)(\eta) = \frac{\partial^r}{\partial \eta_n^r} m_j(\eta) + \sum_{a+b<r} C_{\alpha,n,a,b} \frac{\partial^a}{\partial \eta_n^a}\left(\frac{1}{\langle \alpha, \eta \rangle}\right) \frac{\partial^b}{\partial \eta_n^b} m_j(\eta)$$
$$+ \left(\sum_{\alpha \in R} a_{\alpha,n} \frac{1}{\langle \alpha, \eta \rangle}\right)^r \cdot m_j(\eta).$$

Then by (3.4) and (3.2), we immediately obtain

$$\int_{\Xi_q} |(T_n^r m_j)(\eta)|^2 \, d\tilde{\omega}_\mu(\eta) \le C \cdot 2^{2j(\mu q - r)}.$$

By (2.4) and Plancherel's theorem, we obtain for $|s| \le \ell$,

$$\|\Delta_\kappa^{|s|} m_j\|_{2,\mu} = \|(-|\eta|^2)^{|s|} h_j\|_{2,\mu} \le C \cdot 2^{j(\mu q - |s|)}.$$

Applying this formula with $|s| = 0$ and $|s| = \ell$, we find that the series

$$\sum_{j=-\infty}^{-1} \|h_j\|_{2,\mu} \quad \text{and} \quad \sum_{j=0}^{+\infty} \|(-|\eta|^2)^{|s|} h_j\|_{2,\mu}$$

are convergent and $\sum_{j=-\infty}^{+\infty} |h_j(\eta)|$ is convergent for a.e. $\eta \neq 0$.

Now, using the fact that if $x \notin \operatorname{supp}(f)$, then $0 \notin \operatorname{supp}(\tau_x f)$, we obtain, by Cauchy–Schwarz's inequality,

$$\int_{\Xi_q} |(\tau_\xi f)(\eta)| \sum_{j=-\infty}^{-1} |h_j(\eta)| \, d\tilde{\omega}_\mu(\eta) \le \|\tau_\xi f\|_{2,\mu} \sum_{j=-\infty}^{-1} \|h_j\|_{2,\mu} < \infty,$$

$$\int_{\Xi_q} |(\tau_\xi f)(\eta)| \sum_{j=0}^{+\infty} |h_j(\eta)| \, d\tilde{\omega}_\mu(\eta) \le \left\| \frac{(\tau_\xi f)(\eta)}{(-|\eta|^2)^\ell} \right\|_{2,\mu} \sum_{j=0}^{+\infty} \|(-|\eta|^2)^\ell h_j(\eta)\|_{2,\mu} < \infty.$$

Putting $h = \sum_{j=-\infty}^{+\infty} h_j$, we can write

$$(T_m f)(\xi) = \int_{\Xi_q} h(\eta)(\tau_\xi f)(\eta) \, d\tilde{\omega}_\mu(\eta) = \int_{\Xi_q} H(\xi, \eta) f(\eta) \, d\tilde{\omega}_\mu(\eta).$$

This completes the proof. ∎

Proof of Theorem 3.1. The adjoint operator T_m^* is the multiplier operator associated with $\overline{m}$ and

$$(T_m^* f)(\xi) = \int_{\Xi_q} \overline{H(\eta, \xi)} f(\eta) \, d\tilde{\omega}_\mu(\eta).$$

From this and a duality argument, it suffices to show that the function H satisfies

$$\sum_{j=-\infty}^{+\infty} \int_{||\xi|-|\eta||>2|\eta-\eta_0|} |(\tau_\eta h_j)(\xi) - (\tau_{\eta_0} h_j)(\xi)| \, d\tilde{\omega}_\mu(\xi) \le C, \qquad \eta_0 \in \Xi_q. \tag{3.6}$$

To simplify we can assume that $\eta_0 = 0$; then we have $|\eta - \eta_0| = |\eta| = t$, and $\big||\xi| - |\eta|\big| > 2|\eta - \eta_0|$ can be replaced by $|\xi| > 2t$. So (3.6) becomes

$$\sum_{j=-\infty}^{+\infty} \int_{|\xi|>2t} |(\tau_\eta h_j)(\xi) - h_j(\xi)| \, d\tilde{\omega}_\mu(\xi) \le C. \tag{3.7}$$

Now to prove (3.7), we need the estimates

$$\int_{\Xi_q} |h_j(\xi)|\, d\tilde{\omega}_\mu(\xi) \le C, \qquad \int_{|\xi|>t} |h_j(\xi)|\, d\tilde{\omega}_\mu(\xi) \le C(2^j t)^{\mu q-|s|}. \tag{3.8}$$

Cauchy–Schwarz's inequality, Parseval's formula and (3.5) give

$$\begin{aligned}\int_{\Xi_q} |h_j(\xi)|\, d\tilde{\omega}_\xi(\eta) &\le \|(1+|\eta|^2)^{-\ell}\|_{2,\mu}\|(1+|\eta|^2)^{\ell} h_j\|_{2,\mu}\\ &\le C\cdot 2^{-j\mu q}\sum_{b=0}^{\ell}\binom{\ell}{b} 2^{jb}\|\Delta_\kappa^b m_j\|_{2,\mu} \le C.\end{aligned}$$

Note that this also shows that $|m_j| = |\hat{h}_j| \le C$ almost everywhere, hence $|\sum m_j| \le 2C$ since at most two m_j can be $\neq 0$ at any point. In the same way we obtain

$$\begin{aligned}&\int_{|\xi|>t} |h_j(\xi)|\, d\tilde{\omega}_\mu(\xi)\\ &\quad\le \|(1+2^j|\xi|^2)^{\ell} h_j(\xi)\|_{2,\mu}\Big(\int_{|\xi|>t} ((1+2^j|\xi|^2)^{\ell})^{-2\ell}\, d\tilde{\omega}_\mu(\xi)\Big)^{1/2}\\ &\quad\le C(2^j t)^{\mu q-|s|},\end{aligned}$$

proving (3.8).

Write

$$M_N = \sum_{j=-N}^{N} m_j, \qquad H_N = \sum_{j=-N}^{N} h_j.$$

Then $|M_N| \le 2C$, hence

$$\|H_N\|_{2,\mu} = \|M_N\|_{2,\mu} \le 2C.$$

Let us estimate

$$\int_{|\xi|>2t} |(\tau_\eta H_N)(\xi) - H_N(\xi)|\, d\tilde{\omega}_\mu(\xi), \qquad |\eta| \le t.$$

The second inequality of (3.8) gives

$$\int_{|\xi|\ge 2t} |(\tau_\eta h_j)(\xi) - h_j(\xi)|\, d\tilde{\omega}_\mu(\xi) \le C(2^j t)^{\mu q-|s|},$$

which is a good estimate when $2^j t \ge 1$. Further, the first inequality of (3.8) and Bernstein's inequality give

$$\int_{\Xi_q} |(\tau_\eta h_j)(\xi) - h_j(\xi)|\, d\tilde{\omega}_\mu(\xi) \le C\cdot 2^{j+1} t, \qquad |\eta| \le t,$$

since the spectrum of h_j is contained in the ball with radius 2^{j+1}. Thus, when $|\eta| \le t$,

$$\int_{|\xi|>2t} |(\tau_\eta H_N)(\xi) - H_N(\xi)|\, d\tilde{\omega}_\mu(\xi) \le C\sum_{j=-\infty}^{+\infty} \min(2^j t, (2^j t)^{\mu q-|s|})$$

and since the sum is obviously a bounded function of t, we get

$$\int_{|\xi|>2t} |(\tau_\eta H_N)(\xi) - H_N(\xi)|\, d\tilde{\omega}_\mu(\xi) \leq C, \qquad |\eta| \leq t.$$

As H_N converges to $h = \sum_{j=-\infty}^{+\infty} h_j$ and by continuity of τ_η, we obtain

$$\int_{|\xi|>2t} |(\tau_\eta h)(\xi) - h(\xi)|\, d\tilde{\omega}_\mu(\xi) \leq C, \qquad |\eta| \leq t.$$

This completes the proof of Theorem 3.1. ∎

4. Bilinear multiplier operator. Now consider m in $L^\infty(\mathbb{R}^q \times \mathbb{R}^q)$, smooth away from the origin and satisfying

$$|\partial_\xi^r \partial_\eta^s m(\xi,\eta)| \leq C_{r,s}(|\xi| + |\eta|)^{-(|r|+|s|)} \tag{4.1}$$

for all $r, s \in \mathbb{N}^q$. It is associated with the multiplier bilinear operator

$$C_m(f,g)(x) = \int_{\mathbb{R}^q}\int_{\mathbb{R}^q} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}d\xi\, d\eta$$

where f and g are Schwartz functions.

The known result of Coifman and Meyer [CM1] says that C_m is bounded from $L^{p_1} \times L^{p_2}$ into L^{p_3} whenever $1 < p_1, p_2, p_3 < \infty$ and $1/p_1 + 1/p_2 = 1/p_3$.

In this section, we are concerned with an analogous bilinear operator associated with the symmetric Bessel transform, defined on $\mathcal{S}(\Xi_q) \times \mathcal{S}(\Xi_q)$ by

$$B_m(f,g)(x) = \int_{\Xi_q}\int_{\Xi_q} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)J_k(\xi,ix)J_k(\eta,ix)\, d\tilde{\omega}_\mu(\xi)\, d\tilde{\omega}_\mu(\eta).$$

THEOREM 4.1. *Let m be a bounded C^∞-function on $\Xi_q^2 \setminus \{(0,0)\}$ satisfying* (4.1)*. Then B_m can be extended to a bounded operator from $L^{p_1}(\tilde{\omega}_\mu) \times L^{p_2}\tilde{\omega}_\mu)$ into $L^{p_3}(\tilde{\omega}_\mu)$ whenever $1 < p_1, p_2, p_3 < \infty$ and $1/p_1 + 1/p_2 = 1/p_3$.*

For the proof we adopt the same strategy as in [CM2]. The idea is to split the multiplier m into m_j, $j = 1, 2, 3$, where m_j is supported in an appropriate cone set. One then invokes Littlewood–Paley theory to establish the assertion for each m_j.

As a preliminary step we collect some standard facts from Littlewood–Paley theory, extended to the Dunkl Bessel setting.

To begin, let $\phi \in \mathcal{D}(\Xi_q)$ be supported in $\{1/2 \leq |\xi| \leq 2\}$ and such that

$$\sum_{j=-\infty}^{+\infty} \phi(2^{-j}\xi) = 1, \qquad \xi \neq 0.$$

For $f \in \mathcal{S}(\Xi_q)$ and $j \in \mathbb{Z}$, let

$$(S_j f)(x) = \int_{\Xi_q} \phi(2^{-j}\xi)\hat{f}(\xi)J_k(\xi, ix)\, d\tilde{\omega}_\mu(\xi).$$

Hence one can write

$$f(x) = \sum_{j=-\infty}^{\infty} (S_j f)(x) \quad \text{a.e. } x.$$

We define the Littlewood–Paley square function by

$$Sf = \Big(\sum_{j=-\infty}^{+\infty} |S_j f|^2\Big)^{1/2}.$$

As in the classical theory, using Theorem 3.1 and Khintchine's inequality [Ha], we get the following fundamental L^p-inequalities:

$$C'_p\|f\|_{p,\mu} \le \|Sf\|_{p,\mu} \le C_p\|f\|_{p,\mu} \tag{4.2}$$

for all $f \in \mathcal{S}(\Xi_q)$ and any $1 < p < \infty$. As a consequence of (4.2) we have

LEMMA 4.2. *Let $0 < a \le b < \infty$ and $1 < p < \infty$. Then there exists a constant $C_p > 0$ such that if $(f_j)_{j\in\mathbb{Z}}$ is a sequence of functions in $\mathcal{S}(\Xi_q)$ with* $\operatorname{supp}(\hat{f}_j) \subset \{a2^j \le |\xi| \le b2^j\}$ *and $\sum_j f_j$ is convergent in $L^p(\tilde{\omega}_\mu)$, then*

$$\Big\|\sum_{j=-\infty}^{\infty} f_j\Big\|_{p,\mu} \le C_p\Big\|\Big(\sum_{j=-\infty}^{\infty} |f_j|^2\Big)^{1/2}\Big\|_{p,\mu}.$$

Now, let $\varphi \in \mathcal{D}(\Xi_q)$. For $f \in \mathcal{S}(\Xi_q)$, $j \in \mathbb{Z}$ and $\lambda \in \Xi_q$, we define $f_{j,\lambda} = f_{\varphi,j,\lambda}$ by

$$\hat{f}_{j,\lambda}(\xi) = \varphi(2^{-j}\xi)J_k(\lambda, i2^j\xi)\hat{f}(\xi).$$

LEMMA 4.3. *Let $\varphi \in \mathcal{D}(\Xi_q)$ with $0 \notin \operatorname{supp}(\varphi)$ and let $\ell \in \mathbb{N}$ ($\ell > \mu q$). Then for all $1 < p < \infty$ there exists a constant $C_p > 0$ such that for all $\lambda \in \Xi_q$,*

$$\Big\|\Big(\sum_{j=0}^{\infty} |f_{j,\lambda}|^2\Big)^{1/2}\Big\|_{p,\mu} \le C_p(1+|\lambda|^2)^{\ell}\|f\|_{p,\mu} \quad \textit{for all } f \in \mathcal{S}(\Xi_q). \tag{4.3}$$

Proof. Consider the multiplier operator associated with

$$m_{N,\lambda}(\xi) = (1+|\lambda|^2)^{-\ell}\sum_{j=-N}^{N} \varepsilon_j\varphi(2^{-j}\xi)J_k(\lambda, i\xi 2^{-j}), \quad \varepsilon_j \in \{+1,-1\}, N \in \mathbb{N}.$$

Using (2.1) and the properties of the Dunkl kernel (see [D] or [R2]), we can easily see that $m_{N,\lambda}$ satisfies (3.2) uniformly in λ, N and in the choice of ε_j. Then an application of Khintchine's inequality and Theorem 3.1 gives (4.3). ∎

LEMMA 4.4. *Let $\varphi \in \mathcal{D}(\Xi_q)$ and $\ell \in \mathbb{N}$ (large enough). Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that for all $\lambda \in \Xi_q$,*

$$\Big\| \sup_j |f_{j,\lambda}| \Big\|_{p,\mu} \le C_p(1+|\lambda|^2)^\ell \|f\|_{p,\mu}. \tag{4.4}$$

Proof. One can write

$$f_{j,\lambda} = 2^{2j\mu q}\psi_\lambda(2^j\cdot) \circ_\mu f$$

where $\hat{\psi}_\lambda(\cdot) = J_k(\lambda, i\cdot)\varphi(\cdot)$, which satisfies the estimate

$$|\psi_\lambda(x)| \le C\frac{(1+|\lambda|^2)^\ell}{(1+|x|^2)^\ell}, \qquad \ell \in \mathbb{N}.$$

Thus

$$|f_{j,\lambda}| \le 2^{2j\mu q}\psi^e_\lambda(2^j\cdot) \circ_\mu f^e$$

where $h^e(x) = \sum_{g\in G}|h(gx)|$. So, (4.4) can be deduced from (2.1) and Theorems 6.1 and 6.2 of [TX]. ■

LEMMA 4.5. *Let m be a C^∞-function on $\Xi_q \times \Xi_q$, satisfying* (4.1)*. Then*

$$|T^r_{n,\xi}T^s_{\ell,\eta}m(\xi,\eta)| \le \frac{C_{r,s}}{(|\xi|+|\eta|)^{r+s}} \tag{4.5}$$

for any $r, s \in \mathbb{N}$ where

$$T_{n,\xi}f(\xi) = \frac{\partial}{\partial \xi_n}f(\xi) + \sum_{\alpha\in R}\kappa(\alpha)\alpha_n\frac{f(\xi)-f(\sigma_\alpha(\xi))}{\langle\alpha,\xi\rangle}.$$

Proof. Let us remark that for $f \in \mathcal{E}(\mathbb{R})$, we can write

$$(T_nf)(x) = \frac{\partial}{\partial x_n}f(x) + \sum_{\alpha\in R_+}k(\alpha)\alpha_n\int_0^1 \partial_x f(x-t\langle\alpha,x\rangle\alpha)\,dt.$$

Applying this successively to the function $m(\xi,\cdot)$ and using (4.1), we get

$$\left| |\xi|^{j+s}\frac{\partial^j}{\partial\xi_n^j}T^s_{n,\eta}m(\xi,\eta)\right| \le C_{j,s} \quad \text{for all } j,s\in\mathbb{N}. \tag{4.6}$$

However,

$$|\xi|^{r+s}T^r_{n,\xi}T^s_{\ell,\eta}m(\xi,\eta) = \sum_{j=0}^r\bigg(|\xi|^{j+s}\frac{\partial^j}{\partial\xi_n^j}a_j\sum_{g\in G}T^s_{\ell,\eta}m(g\xi,\eta)\bigg),$$

where a_j is a constant. So by (4.6) we obtain

$$\big||\xi|^{r+s}T^r_{n,\xi}T^s_{\ell,\eta}m(\xi,\eta)\big| \le C_{r,s}. \tag{4.7}$$

Similarly

$$\big||\eta|^{r+s}T^r_{n,\xi}T^s_{\ell,\eta}m(\xi,\eta)\big| \le C_{r,s}. \tag{4.8}$$

Combining (4.7) and (4.8) yields

$$|T^r_{n,\xi}T^s_{\ell,\eta}m(\xi,\eta)| \le \frac{C_{r,s}}{|\xi|^{r+s}+|\eta|^{r+s}} \le \frac{C_{r,s}}{(|\xi|+|\eta|)^{r+s}}.$$

Here $C_{r,s}$ is a constant depending on r,s. ■

Proof of Theorem 4.1. Given $\gamma\in\mathcal{D}(\Xi_q)$ supported in $\left[-\frac{1}{65},\frac{1}{65}\right]$ with $\gamma(x)=1$ in $\left[-\frac{1}{257},\frac{1}{257}\right]$, put

$$\begin{aligned} m_1(\xi,\eta) &= m(\xi,\eta)\gamma\bigg(\frac{\eta^2}{\xi^2+\eta^2}\bigg),\\ m_2(\xi,\eta) &= m(\xi,\eta)\bigg(1-\gamma\bigg(\frac{\eta^2}{\xi^2+\eta^2}\bigg)\bigg)\gamma\bigg(\frac{\xi^2}{\xi^2+\eta^2}\bigg),\\ m_3(\xi,\eta) &= m(\xi,\eta)\bigg(1-\gamma\bigg(\frac{\eta^2}{\xi^2+\eta^2}\bigg)\bigg)\bigg(1-\gamma\bigg(\frac{\xi^2}{\xi^2+\eta^2}\bigg)\bigg). \end{aligned}$$

We note that m_j, $j=1,2,3$, satisfy the condition (4.1), since every homogenous function of degree 0 does. We are therefore reduced to proving the boundedness of B_m for $\operatorname{supp}(m)\subset D_j$, $j=1,2,3$, where

$$\begin{aligned} D_1 &= \big\{(\xi,\eta): |\eta|\le \tfrac18|\xi|\big\},\\ D_2 &= \big\{(\xi,\eta): |\xi|\le \tfrac18|\eta|\big\},\\ D_3 &= \big\{(\xi,\eta): \tfrac1{16}|\xi|\le|\eta|\le 16|\xi|\big\}. \end{aligned}$$

Suppose that $\operatorname{supp}(m)\subset D_1$. We begin by decomposing m, using a dyadic partition of unity given by taking $\varphi\in\mathcal{D}(\Xi_q)$ supported in $\{1/2\le|\xi|\le 2\}$ such that

$$\sum_{j=-\infty}^{\infty}\varphi(2^{-j}\xi)=1,\qquad \xi\ne 0.$$

Then one can write $m=\sum_{j=-\infty}^{\infty}m_j$ with $m_j(\xi,\eta)=m(\xi,\eta)\varphi(2^{-j}\xi)$, so m_j is supported in $\{(\xi,\eta): 2^{j-1}\le|\xi|\le 2^{j+1},\ |\eta|\le\frac14\cdot 2^j\}$. Note that m_j satisfies (4.1) uniformly in j, and similarly for (4.5). Next put, for $j\in\mathbb{N}$,

$$h_j(y,z)=\int\limits_{\Xi_q}\int\limits_{\Xi_q} m_j(2^j\xi,2^j\eta)J_k(\xi,iy)J_k(\eta,iz)\,d\tilde\omega_\mu(\xi)\,d\tilde\omega_\mu(\eta). \tag{4.9}$$

Since for all $r,s\in\mathbb{N}^q$ we have

$$\begin{aligned} &(-1)^{|r|+|s|}|y|^{2|r|}|z|^{2|s|}h_j(y,z)\\ &\quad= 2^{2j(|r|+|s|)}\int\limits_{\Xi_q}\int\limits_{\Xi_q}\Delta_\kappa^{|r|}\Delta_\kappa^{|s|}m_j(2^j\xi,2^j\eta)J_k(\xi,iy)J_k(\eta,iz)\,d\tilde\omega_\mu(\xi)\,d\tilde\omega_\mu(\eta), \end{aligned}$$

using (4.5) we deduce that, for all $N\in\mathbb{N}$,

$$(1+|y|^2)^N(1+|z|^2)^N|h_j(y,z)|\le C_N. \tag{4.10}$$

Here C_N is a constant independent of j. A remarkable consequence of the estimate (4.10) is that (4.9) can be inverted; by the inversion formula,

$$m_j(\xi,\eta)=\int_{\Xi_q}\int_{\Xi_q} h_j(y,z)J_k(y,i2^{-j}\xi)J_k(z,i2^{-j}\eta)\,d\tilde{\omega}_\mu(y)\,d\tilde{\omega}_\mu(z). \tag{4.11}$$

Now, let $\theta,\chi\in\mathcal{D}(\Xi_q)$ be respectively supported in $\{x:7/16\le|x|\le 3\}$ and $\{x:|x|\le 5/16\}$, with $\theta(x)=1$ in $\{x:1/2\le|x|\le 2\}$ and $\chi(x)=1$ in $\{x:|x|\le 1/4\}$. For $j\in\mathbb{Z}$ and $y,z\in\Xi_q$ define the functions $f_{j,y},g_{j,z}$ $\in\mathcal{S}(\Xi_q)$ by

$$\hat{f}_{j,y}(\xi)=\theta(2^{-j}\xi)\hat{f}(\xi)J_k(y,i2^{-j}\xi),\qquad \xi\in\Xi_q,$$
$$\hat{g}_{j,z}(\eta)=\chi(2^{-j}\eta)\hat{g}(\eta)J_k(z,i2^{-j}\eta),\qquad \eta\in\Xi_q.$$

In view of (4.11), we obtain

$$B_m(f,g)(x)=\int_{\Xi_q}\int_{\Xi_q}\sum_{j=-\infty}^{\infty} h_j(y,z)f_{j,y}(x)g_{j,z}(x)\,d\tilde{\omega}_\mu(y)\,d\tilde{\omega}_\mu(z).$$

We note that $\sum_j h_j(y,z)f_{j,y}(x)g_{j,z}(x)$ converges in $L^2(\tilde{\omega}_\mu)$. Indeed, by (4.10),

$$\sum_{j=-\infty}^{\infty}\|h_j(y,z)f_{j,y}g_{j,z}\|_{2,\mu}\le C\sum_{j=-\infty}^{\infty}\|f_{j,y}g_{j,z}\|_{2,\mu}.$$

However, from (2.3), (2.4) and (2.6),

$$\|f_{j,y}g_{j,z}\|_{2,\mu}\le C\|f_{j,y}\|_\infty\|g_{j,z}\|_{2,\mu}\le C\|\hat{f}_{j,y}\|_{1,\mu}\|\hat{g}\|_{2,\mu}.$$

Moreover,

$$\|\hat{f}_{j,y}\|_{1,\mu}\le C\int_{\{\frac{7}{16}\cdot 2^j\le|\xi|\le 3\cdot 2^j\}}|\hat{f}(\xi)|\,d\tilde{\omega}_\mu(\xi),$$

which implies the convergence of $\sum_{j=-\infty}^{\infty}\|\hat{f}_{j,y}\|_{1,\mu}$ and $\sum_{j=-\infty}^{\infty}\|f_{j,y}g_{j,z}\|_{2,\alpha}$.

Next, in view of (2.6) and (2.7), $\widehat{f_{j,y}g_{j,z}}=\hat{f}\circ_\mu\hat{g}$ is supported in $\{2^{j-3}\le|\xi|\le 2^{j+2}\}$. Then, by using (4.10), Hölder's inequality, and Lemmas 4.2–4.4, we get

$$\begin{aligned}\Big\|\sum_{j=-\infty}^{\infty}h_j(y,z)f_{j,y}g_{j,z}\Big\|_{r,\mu}&\le C\,\Big\|\Big(\sum_{j=-\infty}^{+\infty}|h_j(y,z)f_{j,y}g_{j,z}|^2\Big)^{1/2}\Big\|_{r,\mu}\\&\le\frac{C_N}{(1+|y|^2)^N(1+|z|^2)^N}\Big\|\Big(\sum_{j=-\infty}^{\infty}|f_{j,y}|^2\Big)^{1/2}\Big\|_{p,\mu}\Big\|\sup_j|g_{j,z}|\Big\|_{q,\mu}\\&\le C_N\frac{(1+|y|^2)^\ell(1+|z|^2)^\ell}{(1+|y|^2)^N(1+|z|^2)^N}\|f\|_{p,\mu}\|g\|_{q,\mu}.\end{aligned}$$

Since N and ℓ are arbitrary large integers, we can choose $N>2\mu q+\ell$ to obtain

$$\begin{aligned}\|B_m(f,g)\|_{r,\mu} &\le C_N \|f\|_{p,\mu}\|g\|_{q,\mu} \int\limits_{\Xi_q} \frac{(1+|y|^2)^\ell(1+|z|^2)^\ell}{(1+|y|^2)^N(1+|z|^2)^N}\, d\tilde{\omega}_\mu(y)\, d\tilde{\omega}_\mu(z)\\ &\le C\|f\|_{p,\mu}\|g\|_{q,\mu}.\end{aligned}$$

Thus the boundedness of B_m is proved.

As D_1 and D_2 are symmetric with respect to the origin, if $\operatorname{supp}(m) \subset D_2$ the boundedness of B_m follows by similar arguments.

Now suppose that $\operatorname{supp}(m) \subset D_3$. Proceeding as in the first case and considering $\theta, \chi \in \mathcal{D}(\Xi_q)$ respectively supported in $\{x : 1/3 \le |x| \le 3\}$ and $\{x : 1/48 \le |x| \le 48\}$ with $\theta(x) = 1$ in $\{x : 1/2 \le |x| \le 2\}$ and $\chi(x) = 1$ in $\{x : 1/32 \le |x| \le 32\}$, by Cauchy–Schwarz's inequality, Hölder's inequality and Lemma 4.3, we get

$$\begin{aligned}&\Big\| \sum_{j=-\infty}^{\infty} h_j(y,z) f_{j,y} g_{j,z}\Big\|_{r,\mu}\\ &\quad\le \frac{C_N}{(1+|y|^2)^N(1+|z|^2)^N} \Big\|\Big(\sum_{j=-\infty}^{\infty} |f_{j,y}|^2\Big)^{1/2}\Big\|_{p,\mu} \Big\|\Big(\sum_{j=-\infty}^{\infty} |g_{j,z}|^2\Big)^{1/2}\Big\|_{q,\mu}\\ &\quad\le C_N \frac{(1+|y|^2)^\ell(1+|z|^2)^\ell}{(1+|y|^2)^N(1+|z|^2)^N} \|f\|_{p,\mu}\|g\|_{q,\mu}.\end{aligned}$$

Again, choosing $N > 2\mu q + \ell$ we deduce the boundedness of B_m, which completes the proof of the theorem. ∎

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