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ON EXISTENCE OF DOUBLE COSET VARIETIES

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Abstract. Let G be a complex affine algebraic group and $\mathrm{H}, \mathrm{F} \subset \mathrm{G}$ be closed subgroups. The homogeneous space G/H can be equipped with the structure of a smooth quasiprojective variety. The situation is different for double coset varieties $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$. We give examples showing that the variety $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$ does not necessarily exist. We also address the question of existence of $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$ in the category of constructible spaces and show that under sufficiently general assumptions $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$ does exist as a constructible space.

1. Introduction.

Let G be a complex affine algebraic group and $\mathrm{H} \subseteq \mathrm{G}$ be a closed subgroup. By the Chevalley Theorem the set of left H-cosets can be equipped with a uniquely defined structure of a smooth quasiprojective variety such that G act morphically on G/H. Moreover, the projection $\mathrm{G} \to \mathrm{G}/\mathrm{H}$ is a geometric quotient for the action of H on G by right multiplication.

The construction of the homogeneous space G/H has a natural generalisation: one can take another subgroup $\mathrm{F} \subset \mathrm{G}$ and consider double cosets, i.e. the sets $\mathrm{F}g\mathrm{H}$, $g \in \mathrm{G}$. These cosets are orbits of the action of $\mathrm{F} \times \mathrm{H}$ on G given by $(f, h) \circ g = fgh^{-1}$. It is clear that this action, unlike the action of H on G by multiplication, can have orbits of different dimensions, thus it does not necessarily admit a geometric quotient. Because of this we consider a weaker quotient, namely, a categorical one.

The double coset variety $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$ is defined to be the underlying space of the categorical quotient $\mathrm{G} \to \mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$ with respect to the described action of $\mathrm{F} \times \mathrm{H}$, if this quotient exists. If the subgroups F and H are reductive then this variety exists and coincides with the spectrum $\mathrm{Spec}({}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}})$ of the algebra of regular functions on G invariant under the action of $\mathrm{F} \times \mathrm{H}$. Moreover, if G is also reductive then by a result of Luna [9] the action $\mathrm{F} \times \mathrm{H} : \mathrm{G}$ is stable ([1]), hence $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H}$ parametrises generic (closed) double cosets.

In this paper we consider the case when the subgroups F and H are not reductive. In this setting one cannot guarantee that $\mathrm{F}\backslash\!\backslash \mathrm{G}/\!/\mathrm{H} = \mathrm{Spec}({}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}})$;

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([1]) Reductivity of G is essential: consider the group B of upper-triangular matrices and its subgroup T of diagonal matrices; the action $\mathrm{T} \times \mathrm{T} : \mathrm{B}$ is not stable.

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moreover, $\mathrm{F}\backslash\!\backslash\mathrm{G}/\!/\mathrm{H}$ does not necessarily exist. To illustrate this we give the following examples:

I. A unipotent group G and a subgroup U of G such that the variety $\mathrm{U}\backslash\!\backslash\mathrm{G}/\!/\mathrm{U}$ does not exist.

II. A reductive group G and two subgroups F, H such that the variety $\mathrm{F}\backslash\!\backslash\mathrm{G}/\!/\mathrm{H}$ does not exist.

III. A semisimple group G and two subgroups F, H such that the algebra of $\mathrm{F}\times\mathrm{H}$-invariant regular functions $R = {}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}}$ is finitely generated and the natural morphism $\pi : \mathrm{G} \to \operatorname{Spec} R$ is surjective, but π is not a categorical quotient.

It is interesting to remark that though $\mathrm{U}\backslash\!\backslash\mathrm{G}/\!/\mathrm{U}$ considered in Example I does not exist as an algebraic variety, it does exist as a constructible space. Thus, here we observe the same phenomenon as in [1], [2], [4] and [5], namely, an action that admits no quotient in the category of algebraic varieties does admit one in the category of constructible spaces.

In Example III the categorical quotient $\mathrm{F}\backslash\!\backslash\mathrm{G}/\!/\mathrm{H}$ exists in the category of algebraic varieties; its underlying space is the blow-up of $\operatorname{Spec}({}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}})$ at one point; moreover, in this example the categorical quotient separates generic double cosets.

2. Preliminaries on categorical quotients. Let an algebraic group G act on an algebraic variety X. Recall that the categorical quotient of this action is a G-invariant (i.e., constant on G-orbits) morphism $\pi_{\mathrm{G}} : X \to Y$ such that every G-invariant morphism $\varphi : X \to Z$ factors uniquely through π_{G}, that is, there is a unique morphism $\tilde{\varphi}$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Z \\ & {\scriptstyle \pi_{\mathrm{G}}}\searrow \qquad \nearrow{\scriptstyle \tilde{\varphi}} & \\ & Y & \end{array}$$

The universal property of π_{G} implies that Y is defined uniquely up to isomorphism. Remark that π_{G} is necessarily surjective. By abuse of language we will sometimes call the variety $Y = X/\!/\mathrm{G}$ the categorical quotient.

If G is reductive and X is affine then the categorical quotient for the action $\mathrm{G} : X$ is $\pi_{\mathrm{G}} : X \to Y = \operatorname{Spec} \mathbb{C}[X]^{\mathrm{G}}$ with morphism π_{G} corresponding to inclusion $\mathbb{C}[X]^{\mathrm{G}} \subset \mathbb{C}[X]$; in this case π_{G} has an important additional property: it separates closed orbits. If G is not reductive then the quotient $X/\!/\mathrm{G}$ does not necessarily exist. Examples of actions not admitting a categorical quotient are given in [11, 4.3], [1], [2]. Let us point out one example that we will make use of.

Example 2.1 ([11, 4.3]). There is no categorical quotient for the action of a one-dimensional unipotent group U on the space $\mathrm{Mat}_{2\times 2}$ of 2×2-matrices given by the formula

$$\lambda \circ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Remark that we have $\mathbb{C}[\mathrm{Mat}_{2\times 2}]^{\mathrm{U}} = \mathbb{C}[a_{21}, a_{22}, \det]$ and the canonical morphism $\pi : \mathrm{Mat}_{2\times 2} \to \operatorname{Spec} \mathbb{C}[\mathrm{Mat}_{2\times 2}]^{\mathrm{U}} \cong \mathbb{A}^3$ separates the U-orbits of generic points having $a_{21} \neq 0$ or $a_{22} \neq 0$. The image of this morphism is $\mathbb{A}^3$ without the punctured line $\{a_{21} = a_{22} = 0, \det \neq 0\}$. Since the image of π is not open, by [2, Corollary 1.4] the action $\mathrm{U} : \mathrm{Mat}_{2\times 2}$ has no categorical quotient in the category of algebraic varieties.

The morphism π considered in Example 2.1 can be regarded as a quotient morphism after an appropriate modification to the definition of categorical quotient. It turns out that admitting only morphisms into *varieties* as categorical quotients is overly restrictive for certain actions $\mathrm{G} : X$. To work around this Bialynicki-Birula introduced in [4] the category of dense constructible subsets. This approach has been further developed in [2] to permit maps into constructible spaces as candidates for quotient morphisms. Recall that a constructible space is a topological space with a sheaf of functions admitting a finite cover by subsets that are isomorphic (as spaces with functions) to constructibe subsets of affine varieties. A morphism of constructible spaces is a morphism of spaces with functions. A *constructible quotient* is a categorical quotient in the category of constructible spaces. It is possible for an action $\mathrm{G} : X$ to have no quotient in the category of algebraic varieties, but to have a constructible quotient.

Example 2.2. Let a unipotent group G act on a vector space V. It follows from [2, Corollary 1.2] that the action $\mathrm{G} : V$ admits a constructible quotient, provided that $\mathbb{C}[V]^{\mathrm{G}}$ is finitely generated. If $\rho : V \to \operatorname{Spec} \mathbb{C}[V]^{\mathrm{G}}$ is the morphism corresponding to the inclusion $\mathbb{C}[V]^{\mathrm{G}} \subset \mathbb{C}[V]$ then the constructible quotient is $\rho : V \to \rho(V)$. In particular, the map π in Example 2.1 is a constructible quotient for the action $\mathrm{U} : \mathrm{Mat}_{2\times 2}$.

Let us point out a fact concerning quotients under two commuting actions; it will be used to identify double coset varieties with quotients of homogeneous spaces. Let $\mathrm{F} \times \mathrm{H}$ act on a variety X and $\pi_{\mathrm{F}} : X \to Y = X/\!/\mathrm{F}$ be the categorical quotient for the action of $\mathrm{F} : X$. The group H acts on Y as an abstract group: if $y = \pi_{\mathrm{F}}(x)$ then $h \circ y = \pi_{\mathrm{F}}(h \circ x)$. By [3, Theorem. 7.1.4] this action is regular. Moreover, existence of $Y/\!/\mathrm{H}$ is equivalent to existence of $X/\!/(\mathrm{F} \times \mathrm{H})$ and these two quotients coincide:

$$\begin{array}{ccc} X & \xrightarrow{\pi_{\mathrm{F}\times\mathrm{H}}} & X/\!\!/(\mathrm{F}\times\mathrm{H}) = Y/\!\!/\mathrm{H} \\ & {\scriptstyle \pi_{\mathrm{F}}}\searrow \quad \nearrow{\scriptstyle \pi_{\mathrm{H}}} & \\ & Y = X/\!\!/\mathrm{F} & \end{array}$$

The following statement will be used in Proposition 3.2.

LEMMA 2.3. *Let an algebraic group* G *act on an algebraic variety* Y*. Suppose that there is* $y_0 \in Y$ *that belongs to the closure of every* G*-orbit. Consider the action* $\mathrm{G} : X \times Y$*, where* X *is a normal variety and* G *acts trivially on the first factor. Let* $W \subseteq X \times Y$ *be a* G*-invariant open subset. Suppose that* W *contains* $X_0 \times Y$*, where* $X_0 \subseteq X$ *is a dense subset, and* $\mathrm{pr}(W) = X$*, where* pr *is the projection onto the first factor. Then the action* $\mathrm{G} : W$ *has* $\mathrm{pr} : W \to X$ *as a categorical quotient both in the category of algebraic varieties and in the category of constructible spaces.*

Proof. Let us fix a G-invariant morphism $\varphi : W \to Z$ into an algebraic variety Z (resp., into a constructible space) and show that it factors uniquely through pr.

STEP 1. We claim that φ extends to a continuous map on $W \cup X \times \{y_0\}$. Let us fix a point $(x', y_0) \notin X_0 \times \{y_0\}$ and an arbitrary sequence $\{x_n\} \subset X_0$ such that $x_n \to x'$. Now we show that the sequence $\varphi(x_n, y_0)$ converges. Since $\mathrm{pr}(W) = X$, there is a point (x', y) in W for some $y \in Y$. The points (x_n, y_0) and (x_n, y) belong to W, hence by G-invariance of φ we have $\varphi(x_n, y) = \varphi(x_n, y_0)$, thus $\lim_{n\to\infty} \varphi(x_n, y_0) = \lim_{n\to\infty} \varphi(x_n, y) = \varphi(x', y)$. Since a converging sequence can have only one limit, $\lim_{n\to\infty} \varphi(x_n, y_0)$ does not depend on the choice of $(x', y) \in W$. For the extended map φ we have $\varphi(x, y) = \varphi(x, y_0)$, so continuity of $\varphi|_{X\times\{y_0\}}$ implies continuity of φ on $W \cup X \times \{y_0\}$.

STEP 2. Now we show that $X \times \{y_0\}$ can be covered by open affine sets $X_i \times \{y_0\}$ such that the image of $\varphi : X_i \times \{y_0\} \to Z$ is contained in some affine subset of Z. Let $\{Z_i\}$ be an affine covering of Z and $\{U_i\}$ be an affine covering of X. The set $V_{ij} = \varphi^{-1}(\varphi(U_i) \cap Z_j)$ is open in U_i. Every set V_{ij} is a union of principal open subsets, $V_{ij} = \bigcup_k V_{ijk}$. The sets V_{ijk} make up the required covering of $X \times \{y_0\}$.

STEP 3. Since $\varphi(x, y) = \varphi(x, y_0)$, we have $\varphi = \tilde{\varphi} \circ \rho$, where $\tilde{\varphi} = \varphi|_{X\times\{y_0\}}$ and ρ is the map $W \to X \times \{y_0\}$, $\rho(x, y) = (x, y_0)$. Denote by $\imath$ the identification of X and $X \times \{y_0\}$: $\imath(x) = (x, y_0)$. We have $\varphi = \tilde{\varphi} \circ \imath \circ \pi$, so φ factors through π. It remains to verify that $\varphi|_{X\times\{y_0\}}$ is a morphism. The variety X is normal, hence the affine opens $X_i \times \{y_0\}$ constructed in Step 2 are normal varieties, too. Restrictions of φ to these opens are morphisms of affine varieties; if Z is a constructible space then $\varphi|_{X_i\times\{y_0\}}$ is a morphism into a constructible set, but it can be regarded as a morphism into an affine variety containing $\varphi(X_i \times \{y_0\})$. By the theorem on removable singularities the continuous extensions of $\varphi|_{X_i\times\{y_0\}}$ are morphisms. ∎

3. Existence and non-existence of double coset varieties

3.1. Consider a unipotent group G and a subgroup U:

$$\mathrm{G} = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \qquad \mathrm{U} = \begin{pmatrix} 1 & * & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix},$$

where $*$ denotes an arbitrary number. We claim that if we take $\mathrm{F} = \mathrm{H} = \mathrm{U}$ then the double coset variety $\mathrm{F}\backslash\!\backslash \mathrm{G} /\!/ \mathrm{H}$ does not exist. Remark that the group $\mathrm{F} \times \mathrm{H} = \mathrm{U} \times \mathrm{U}$ is unipotent, hence every double (U, U)-coset is closed [11, 1.3]; had F and H been reductive, this would have implied existence of the *geometric* quotient $\mathrm{G} \to \mathrm{G}/(\mathrm{F} \times \mathrm{H})$.

Proposition 3.1. *The action* $\mathrm{U} \times \mathrm{U} : \mathrm{G}$ *has no categorical quotient in the category of algebraic varieties. It admits a constructible quotient, and the constructible quotient parametrises generic double cosets.*

Proof. Consider the action of G on $\mathrm{Mat}_{4\times 2}$ by left multiplication. The subgroup U is the stabiliser of the matrix

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

Therefore, the homogeneous space G/U is isomorphic to $\mathbb{A}^4$ and can be identified with the variety of matrices

$$\begin{pmatrix} * & * \\ * & * \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

After this identification the action of U on G/U becomes the matrix multiplication; it is therefore isomorphic to the action of U on $\mathrm{Mat}_{2\times 2}$ by left multiplication. Example 2.1 shows that this action does not admit a categorical quotient. Thus, $\mathrm{U}\backslash\!\backslash \mathrm{G} /\!/ \mathrm{U} = (\mathrm{G}/\mathrm{U}) /\!/ \mathrm{U}$ does not exist.

From Example 2.2 it follows that the action $\mathrm{U} \times \mathrm{U} : \mathrm{G}$ has a constructible quotient $\pi : \mathrm{G} \to \mathrm{U}\backslash\!\backslash \mathrm{G} /\!/ \mathrm{U} \subset \mathbb{A}^3$ which separates generic double cosets. ∎

Remark. The constructible quotient $\pi : \mathrm{G} \to \mathrm{U}\backslash\!\backslash \mathrm{G}/\!/\mathrm{U}$ does not separate all closed double cosets. Indeed, all 2×2-matrices with $a_{21} = a_{22} = 0$ (we use the notation of Example 2.1) are fixed under the action of U and have $\det = 0$, hence their preimages in G are closed (U, U)-cosets, which are mapped by π to $0 \in \mathbb{A}^3$.

3.2. Take $\mathrm{G} = \mathrm{GL}_4$ and consider the action of G on 4×2-matrices. Let H be the stabiliser of the matrix M as in 3.1. The homogeneous space $W = \mathrm{G}/\mathrm{H}$ is identified with the variety of 4×2-matrices with non-zero columns. Let F be the subgroup of G consisting of the matrices

$$\begin{pmatrix} 1 & a & 0 & 0 \\ & 1 & 0 & 0 \\ & & s & 0 \\ & & & s \end{pmatrix}, \quad a \in \mathbb{C},\ s \in \mathbb{C}^\times.$$

The subgroup F acts on W via matrix multiplication.

Proposition 3.2. *The action* $\mathrm{F} \times \mathrm{H} : \mathrm{G}$ *does not admit a categorical quotient in the category of algebraic varieties, but has a constructible quotient.*

Proof. The group F is a direct product $\mathrm{F} = \mathrm{U} \times \mathrm{S}$ of one-dimensional unipotent group U and one-dimensional torus S. The categorical quotient for the action $\mathrm{S} : W$ is $\mathrm{pr} : W \to \mathrm{Mat}_{2\times2}$, which erases the bottom half of matrices of W. Indeed, one can apply Lemma 2.3 with the acting group S and $X = Y = \mathrm{Mat}_{2\times2}$ representing the top and bottom halves of matrices respectively, and with X_0 consisting of matrices with non-zero columns.

Thus, had the quotient $W/\!/\mathrm{F} = \mathrm{F}\backslash\!\backslash \mathrm{GL}_4/\!/\mathrm{H}$ existed, it would have been also $(W/\!/\mathrm{S})/\!/\mathrm{U} = \mathrm{Mat}_{2\times2}/\!/\mathrm{U}$, but, according to Example 2.1, the latter quotient does not exist.

By Lemma 2.3 and Example 2.2, the actions $\mathrm{S} : W$ and $\mathrm{U} : \mathrm{Mat}_{2\times2}$ both have a constructible quotient, thus $\mathrm{F}\backslash\!\backslash \mathrm{GL}_4/\!/\mathrm{H} = (W/\!/\mathrm{S})/\!/\mathrm{U}$ exists as a constructible space. ∎

3.3. This example is based on [2, 4.5]. Consider the following symmetric bilinear form on $\mathbb{C}^4$: $(e_1, e_4) = (e_2, e_3) = 1$ and the other pairings of basis vectors are zero. The cone

$$X = \{\mathbf{x} \in \mathbb{C}^4 \mid x_1x_4 + x_2x_3 = 0\} \setminus \{(0,0,0,0)\}$$

is the collection of non-zero isotropic vectors, therefore $X = \mathrm{SO}_4/\mathrm{H}$, where H is the stabiliser of a non-zero isotropic vector. As F we take the following unipotent subgroup of SO_4:

$$\begin{pmatrix} 1 & a & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -a \\ & & & 1 \end{pmatrix}, \qquad a \in \mathbb{C}.$$

The algebra $\mathbb{C}[X]^{\mathrm{F}}$ is freely generated by x_2 and x_4; indeed, these two functions are F-invariant and generic orbits meet the plane $\{x_1 = x_3 = 0\}$, so there are no other generators. It is clear that the canonical morphism $\pi : X \to \operatorname{Spec}\mathbb{C}[X]^{\mathrm{F}} = \mathbb{A}^2$ is surjective. Nevertheless, $\operatorname{Spec}\mathbb{C}[X]^{\mathrm{F}}$ is not a categorical quotient for $\mathrm{F} : X$ because the following morphism $\varphi : X \to \mathbb{P}^1$ does not factor through π:

$$\varphi(x_1, x_2, x_3, x_4) = (x_2 : x_4) = (x_1 : -x_3).$$

Indeed, $\pi(x_1, x_2, x_3, x_4) = (x_2, x_4)$, and from $\varphi = \tilde{\varphi} \circ \pi$ it would follow that $\tilde{\varphi}(x_2, x_4) = (x_2 : x_4)$ when $x_2 \neq 0$ or $x_4 \neq 0$, hence $\tilde{\varphi}$ is not continuous at $(0, 0)$, which is not possible.

Let us show that the action considered has a categorical quotient, though it does not coincide with $\operatorname{Spec}\mathbb{C}[X]^{\mathrm{F}}$.

Proposition 3.3. *The action* F : X *has a categorical quotient in the category of algebraic varieties.*

Proof. The quotient is the blow-up of the origin in $\mathbb{A}^2$,

$$\hat{\mathbb{A}}^2 = \{((x, y), (u : v)) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xv - yu = 0\}$$

with the morphism $\rho : X \to \hat{\mathbb{A}}^2$ given by the formula

$$\rho(x_1, x_2, x_3, x_4) = ((x_2, x_4), (x_1 : -x_3)) = ((x_2, x_4), (x_2 : x_4)).$$

Let us check that every F-invariant morphism $\varphi : X \to Z$ factors through ρ. By F-invariance of φ we have $\varphi(cx_1, x_2, cx_3, x_4) = \varphi(x_1, x_2, x_3, x_4)$. Indeed, if one of x_2, x_4 is not zero then (x_1, x_2, x_3, x_4) and (cx_1, x_2, cx_3, x_4) belong to one orbit of F; if $x_2 = x_4 = 0$ then

$$\begin{aligned} \varphi(x_1, 0, x_3, 0) &= \lim_{t\to 0} \varphi(x_1, tx_1, x_3, -tx_3) = \lim_{t\to 0} \varphi(cx_1, tx_1, cx_3, -tx_3) \\ &= \varphi(cx_1, 0, cx_3, 0). \end{aligned}$$

Define $\tilde{\varphi} : \hat{\mathbb{A}}^2 \to Z$ as the morphism taking $((x, y), (u : v))$ to $\varphi(u, x, -v, y)$. The reasoning above shows that $\tilde{\varphi}$ is well defined. Thus, $\varphi = \tilde{\varphi} \circ \rho$, i. e., φ factors through ρ. Since $\rho(X) = \hat{\mathbb{A}}^2$, the morphism $\tilde{\varphi}$ can be chosen uniquely. ∎

Remark. It is clear that $\rho : X \to \hat{\mathbb{A}}^2$ separates the orbits of points having $x_2 \neq 0$ or $x_4 \neq 0$. However, ρ does not separate all closed orbits: the points $z = (x_1, 0, x_3, 0)$ and $z' = (cx_1, 0, cx_3, 0)$ are F-fixed, but $\rho(z) = \rho(z')$.

Thus, the quotient $q : \mathrm{SO}_4 \to \mathrm{F}\backslash\backslash \mathrm{SO}_4/\!/\mathrm{H} = \hat{\mathbb{A}}^2$ separates generic double cosets, but fails to separate all closed double cosets.

3.4. Remark that in Examples I and II the actions $\mathrm{F} \times \mathrm{H} : \mathrm{G}$ have no categorical quotient in the category of algebraic varieties but do admit one in the category of constructible spaces. The following question has been raised by the referee.

QUESTION. Let G be a connected affine algebraic group and F, H be closed subgroups in G. Is it true that $\mathrm{F}\backslash\backslash\mathrm{G}/\!/\mathrm{H}$ exists as a constructible space?

The following proposition gives a partial answer to this question.

PROPOSITION 3.4. *Let* G *be a connected affine algebraic group and* $\mathrm{F}, \mathrm{H} \subset \mathrm{G}$ *be closed connected subgroups with trivial character groups. Suppose that the algebra* ${}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}}$ *is finitely generated and let* $\pi : \mathrm{G} \to \mathrm{Spec}({}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}})$ *be the canonical morphism. Then* $\mathrm{F}\backslash\backslash\mathrm{G}/\!/\mathrm{H}$ *exists as a constructible space and the map* $\pi : \mathrm{G} \to \pi(\mathrm{G})$ *is the constructible quotient for the action of* $\mathrm{F} \times \mathrm{H}$ *on* G.

Proof. By [10, Theorem 6], the underlying variety of G has a finite divisor class group. Additionally, F and H have trivial character groups, therefore every $\mathrm{F} \times \mathrm{H}$-invariant hypersurface $D \subset \mathrm{G}$ is the zero set of an invariant function $f_D \in {}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}}$. It now follows from [2, Corollary 1.2] that the action $\mathrm{F} \times \mathrm{H} : \mathrm{G}$ has $\pi : \mathrm{G} \to \pi(\mathrm{G})$ as a constructible quotient. ∎

REMARK. One can often give a positive answer to the question on finite generation of ${}^{\mathrm{F}}\mathbb{C}[\mathrm{G}]^{\mathrm{H}}$. Recall that if R is a reductive group, Z is an affine R-variety and $\mathrm{U} \subset \mathrm{R}$ is a maximal unipotent subgroup then the algebra $\mathbb{C}[Z]^{\mathrm{U}}$ is finitely generated [8, Chapter 3.2]. Thus, the constructible space $\mathrm{F}\backslash\backslash\mathrm{G}/\!/\mathrm{H}$ is guaranteed to exist if both groups F and H are maximal unipotent subgroups in bigger reductive subgroups $\mathrm{F}', \mathrm{H}' \subseteq \mathrm{G}$, or if one of them is semisimple and the other one is a maximal unipotent subgroup in a bigger reductive subgroup. Other results on finite generation of algebras of invariants can be found in [6].

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