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DEFORMED MESH ALGEBRAS OF DYNKIN TYPE $\mathbb{C}_n$

BY

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Abstract. In our recent paper (J. Algebra 345 (2011)) we prove that the deformed preprojective algebras of generalized Dynkin type $\mathbb{L}_n$ (in the sense of our earlier work in Trans. Amer Math. Soc. 359 (2007)) are exactly (up to isomorphism) the stable Auslander algebras of simple plane singularities of Dynkin type $\mathbb{A}_{2n}$. In this article we complete the picture by showing that the deformed mesh algebras of Dynkin type $\mathbb{C}_n$ are isomorphic to the canonical mesh algebras of type $\mathbb{C}_n$, and hence to the stable Auslander algebras of simple plane curve singularities of type $\mathbb{A}_{2n-1}$. Moreover, we describe the minimal (periodic) bimodule projective resolutions of the canonical mesh algebras of type $\mathbb{C}_n$.

Introduction and the main results. Throughout this article, K will denote a fixed algebraically closed field. By an *algebra* we mean an associative finite-dimensional K-algebra with identity, which we moreover assume to be basic and connected. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules and by Ω_A the syzygy operator which assigns to a module M in $\operatorname{mod} A$ the kernel of a minimal projective cover $P_A(M) \to M$ of M in $\operatorname{mod} A$. Then a module M in $\operatorname{mod} A$ is called *periodic* if $\Omega_A^n(M) \cong M$ for some $n \geq 1$. Further, the category of finite-dimensional A-A-bimodules over an algebra A is canonically equivalent to the module category $\operatorname{mod} A^e$ over the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_K A$ of A. Then the algebra A is called a *periodic algebra* if A is a periodic module in $\operatorname{mod} A^e$. It is known that any periodic algebra A is selfinjective, and that every module M in $\operatorname{mod} A$ without non-zero projective direct summands is periodic. Periodic algebras play currently a prominent rôle in representation theory of algebras and have attracted much attention (see the survey article [12]). In particular, it has been proved recently in [9] that all selfinjective algebras of finite representation type (different from K) are periodic.

Important examples of periodic algebras are the deformed mesh algebras of generalized Dynkin types $\mathbb{A}_n$ ($n \geq 2$), $\mathbb{B}_n$ ($n \geq 2$), $\mathbb{C}_n$ ($n \geq 3$), $\mathbb{D}_n$ ($n \geq 4$), $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$, and $\mathbb{L}_n$ ($n \geq 1$) (defined in [5], [12]), for which the third

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syzygy permutes the isomorphism classes of simple modules. This class of algebras contains the deformed preprojective algebras of generalized Dynkin types $\mathbb{A}_n$ ($n \geq 2$), $\mathbb{D}_n$ ($n \geq 4$), $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and $\mathbb{L}_n$ ($n \geq 1$), which occur naturally in very different contexts. For these, the third syzygy of any simple module is isomorphic to its shift by the Nakayama functor (see [5]). Mesh algebras of Dynkin types include in particular the stable Auslander algebras of Arnold's simple hypersurface singularities [1]. In fact, it is an exciting open problem whether the stable Auslander algebra of any simple hypersurface singularity over an arbitrary closed field K is a deformed mesh algebra of the corresponding Dynkin type. We now briefly explain the related context.

A *hypersurface singularity* over K is a quotient algebra R of the K-algebra $K[[x_0, x_1, \dots, x_n]]$ of formal power series in $n+1$ variables by the principal ideal (f) generated by a non-zero element f of the square $\mathfrak{m}^2$, where $\mathfrak{m} = (x_0, x_1, \dots, x_n)$ is the unique maximal ideal of $K[[x_0, x_1, \dots, x_n]]$. That is, $R = K[[x_0, x_1, \dots, x_n]]/(f)$. We denote by $\mathrm{CM}(R)$ the category of finitely generated *maximal Cohen–Macaulay* R-modules, that is, the finitely generated R-modules M whose depth $\operatorname{depth}(M)$ is the Krull dimension $\dim R$ of R. Then $\mathrm{CM}(R)$ is a Krull–Schmidt category, that is, every object in $\mathrm{CM}(R)$ has a decomposition into a direct sum of indecomposable objects, which is unique up to isomorphism.

The hypersurface singularity R is called of *finite Cohen–Macaulay type* if $\mathrm{CM}(R)$ has only finitely many pairwise non-isomorphic indecomposable objects. We note that by a result of Auslander [4] every hypersurface singularity R of finite Cohen–Macaulay type is an isolated singularity, and then, by an observation of Greuel and Kröning [14], $R \cong K[[x_0, x_1, \dots, x_n]]/(F)$ for a polynomial F in $K[x_0, x_1, \dots, x_n]$ (so F defines a hypersurface in the affine space K^{n+1} having an isolated singularity at the origin).

The hypersurface singularities of finite Cohen–Macaulay type have a beautiful characterization via the deformation theory. Namely, the concept of finite deformation type of a hypersurface singularity was introduced by Arnold [1] (in characteristic 0) and by Greuel and Kröning [14] (in positive characteristic). Roughly speaking, an isolated hypersurface singularity R is of finite deformation type if R can be deformed only into finitely many non-isomorphic singularities (see [1], [14]). Independently, the simple hypersurface singularities (ADE singularities) have been investigated and classified in [1], [2], [17], [18], [20] (see [14, Section 1] for their normal forms). Then, for a hypersurface singularity R, the following statements are equivalent:

- R is simple;
- R is of finite deformation type;
- R is of finite Cohen–Macaulay type,

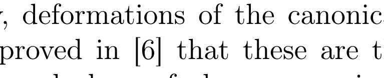
by results established in [1], [4], [7], [14], [17], [20]. We note that in characteristic $\neq 2, 3, 5$ the simple hypersurface singularities are isomorphic to Arnold's simple hypersurface singularities from [1]. In general, the normal forms of simple curve singularities (dimension 1) were classified by Kiyek and Steinke [17], the normal forms of simple surface singularities (dimension 2) were classified by Artin [2], and the normal forms of simple hypersurface singularities of dimensions ≥ 3 are obtained from those of dimensions 1 and 2 by double suspensions (see [14], [20]).

Let R be a simple hypersurface singularity. Then $\mathrm{CM}(R)$ is a Frobenius category, with R the unique (up to isomorphism) indecomposable projective object, and we may consider the stable category $\underline{\mathrm{CM}}(R)$ of $\mathrm{CM}(R)$ modulo the ideal consisting of all morphisms which factor through direct sums of copies of R. Since R is of finite Cohen–Macaulay type, we may choose a finite complete set $M_1, \dots, M_n$ of pairwise non-isomorpic indecomposable non-projective objects in $\mathrm{CM}(R)$, and consider the endomorphism algebra

$$\underline{\mathcal{A}}(R) = \mathrm{End}_{\underline{\mathrm{CM}}(R)}(M_1 \oplus \cdots \oplus M_n),$$

called the *stable Auslander algebra* of R. It is known that $\underline{\mathcal{A}}(R)$ is a finite-dimensional selfinjective algebra over K, and it would be interesting to know when $\underline{\mathcal{A}}(R)$ is a periodic algebra (see [12, Problem 8]). We also mention that the Auslander–Reiten quiver of the category $\mathrm{CM}(R)$ is isomorphic to the Auslander–Reiten quiver of $\mathrm{CM}(R^*)$ for an Arnold's simple hypersurface singularity R^* of dimension 1 or 2, canonically associated to R (by results of [8], [18], [20]), and hence the Gabriel quivers of $\underline{\mathcal{A}}(R)$ and $\underline{\mathcal{A}}(R^*)$ coincide. Moreover, the stable Auslander algebras of Arnold's simple singularities of dimensions 1 and 2 are mesh algebras of generalized Dynkin types $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{B}_n$ $(n \geq 2)$, $\mathbb{C}_n$ $(n \geq 3)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$, and $\mathbb{L}_n$ $(n \geq 1)$ (see [12]). Therefore, it is natural to ask when the stable Auslander algebra $\underline{\mathcal{A}}(R)$ of an arbitrary simple hypersurface singularity R is a deformed mesh algebra of generalized Dynkin type (as introduced in [5], [12]).

In our recent paper [6] we established a complete classification of the isomorphism classes of deformed preprojective algebras of generalized Dynkin types

$\mathbb{L}_n$:  (n vertices), $n \geq 1$,

or equivalently, deformations of the canonical mesh algebras of types $\mathbb{L}_n$. Moreover, we proved in [6] that these are the isomorphism classes of the stable Auslander algebras of plane curve singularities of Dynkin types $\mathbb{A}_{2n}$ $(n \geq 1)$. We recall from [17] that, if the characteristic of K is different from 2, then $R = R_n^{(n)} = K[[x, y]]/(x^2 + y^{2n+1})$ is the unique simple plane singularity of type $\mathbb{A}_{2n}$, up to isomorphism. For K of characteristic 2, the plane curve singularities

$$R_n^{(r)} = K[[x,y]]/(x^2 + y^{2n+1} + xy^{n+r}), \quad r \in \{1,\dots,n-1\},$$

together with $R_n^{(n)}$, form a complete set of representatives for the isomorphism classes of simple curve singularities of type $\mathbb{A}_{2n}$.

On the other hand, by [17], for an integer $n \geq 1$, and K of any characteristic, there is only one simple curve singularity of Dynkin type $\mathbb{A}_{2n-1}$ (up to isomorphism), namely $R_n = K[[x,y]]/(x^2 + xy^n)$, and this is isomorphic to Arnold's simple plane singularity $K[[x,y]]/(x^2+y^{2n})$ if K is of characteristic $\neq 2$. Moreover, $\underline{\mathcal{A}}(R_1) \cong K \times K$, $\underline{\mathcal{A}}(R_2)$ is the canonical mesh algebra $\Lambda(\mathbb{B}_2)$ of Dynkin type $\mathbb{B}_2 = \mathbb{C}_2$: $\bullet \overset{(1,2)}{\rule{1cm}{0.4pt}} \bullet$, and, for $n \geq 3$, $\underline{\mathcal{A}}(R_n)$ is the canonical mesh algebra $\Lambda(\mathbb{C}_n)$ of Dynkin type

$$\mathbb{C}_n : \quad \bullet \overset{(2,1)}{\text{——}} \bullet \text{——} \bullet - \dots - \bullet \text{——} \bullet \quad (n \text{ vertices}).$$

Hence, it is natural to ask if the algebras $\Lambda(\mathbb{C}_n)$, $n \geq 3$, have no proper deformations in the sense of [12]. We note that this is obvious for $\Lambda(\mathbb{B}_2)$, and clearly for $K \times K$.

Theorem A. *Let $n \geq 3$ be an integer and $\Lambda^f = \Lambda^f(\mathbb{C}_n)$ be a deformed mesh algebra of type $\mathbb{C}_n$. Then Λ^f is isomorphic to the canonical mesh algebra $\Lambda(\mathbb{C}_n)$.*

The following theorem is a direct consequence of Theorem A and [6, Theorems 2 and 3].

Theorem B. *Let A be an algebra and $m \geq 2$ an integer. The following statements are equivalent:*

(i) *A is isomorphic to the stable Auslander algebra of a simple curve singularity of Dynkin type $\mathbb{A}_m$.*
(ii) *A is isomorphic to one of the algebras:*
   (a) *a deformed mesh algebra of type $\mathbb{C}_n$, if $m = 2n-1$ is odd;*
   (b) *a deformed preprojective algebra of type $\mathbb{L}_n$, if $m = 2n$ is even.*

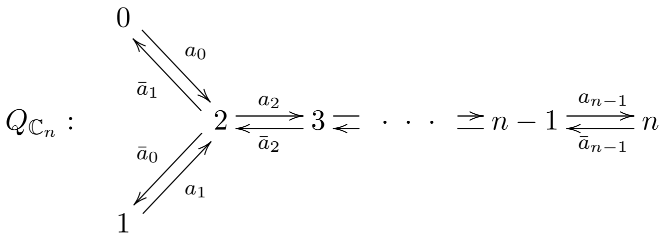
Theorem C. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be a canonical mesh algebra of type $\mathbb{C}_n$, $n \geq 3$. Then Λ is a periodic algebra of period* 6.

We note that it has been proved in [5, Proposition 2.3] that all deformed preprojective algebras of type $\mathbb{L}_n$ are periodic algebras but the proof presented there does not allow us to determine their periods.

We also mention that the canonical mesh algebra $\Lambda(\mathbb{B}_2) = \Lambda(\mathbb{C}_2)$ is a special biserial symmetric algebra, derived equivalent to the symmetric Nakayama algebra with three simple modules and of Loewy length 4, and consequently it is a periodic algebra of period 6 (see 4.2 in [11]).

For basic background on the relevant representation theory we refer to [3], [19]; and for background on sigularities and Cohen–Macaulay modules to [15], [21].

1. Deformed mesh algebras of type $\mathbb{C}_n$. Let n be an integer ≥ 3. Following [12, Section 7], we denote by $\Lambda(\mathbb{C}_n)$ the *canonical mesh algebra* of type $\mathbb{C}_n$ given by the quiver



and the relations

$$a_0\bar{a}_0 = 0, \quad a_1\bar{a}_1 = 0, \quad \bar{a}_1 a_0 + \bar{a}_0 a_1 + a_2\bar{a}_2 = 0,$$
$$\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0 \quad \text{for } i \in \{2,\dots,n-2\}, \quad \bar{a}_{n-1}a_{n-1} = 0.$$

We note that $\Lambda(\mathbb{C}_n)$ is a symmetric algebra. Further, consider the local commutative algebra

$$R(\mathbb{C}_n) = K\langle x,y\rangle/(xy, yx, (x+y)^{n-1}),$$

which is isomorphic to the algebra $e_2\Lambda(\mathbb{C}_n)e_2$, where e_2 is the primitive idempotent in $\Lambda(\mathbb{C}_n)$ associated to the vertex 2 of $Q_{\mathbb{C}_n}$. For an element f from the square $\operatorname{rad}^2 R(\mathbb{C}_n)$ of the radical $\operatorname{rad} R(\mathbb{C}_n)$ of $R(\mathbb{C}_n)$, we denote by $\Lambda^f(\mathbb{C}_n)$ the algebra given by the quiver $Q_{\mathbb{C}_n}$ and the relations

$$a_0\bar{a}_0 = 0, \quad a_1\bar{a}_1 = 0, \quad (\bar{a}_1 a_0 + \bar{a}_0 a_1)^{n-1} = 0,$$
$$\bar{a}_1 a_0 + \bar{a}_0 a_1 + a_2\bar{a}_2 + f(\bar{a}_1 a_0, \bar{a}_0 a_1) = 0,$$
$$\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0 \quad \text{for } i \in \{2,\dots,n-2\}, \quad \bar{a}_{n-1}a_{n-1} = 0.$$

Then $\Lambda^f(\mathbb{C}_n)$ is called a *deformed mesh algebra* of type $\mathbb{C}_n$ (see [12, Section 7]). Observe that $\Lambda^f(\mathbb{C}_n)$ is obtained from $\Lambda(\mathbb{C}_n)$ by deforming the relation at the exceptional vertex 2 of $Q_{\mathbb{C}_n}$, and $\Lambda^f(\mathbb{C}_n) = \Lambda(\mathbb{C}_n)$ if $f = 0$.

Proof of Theorem A. Let $n \geq 3$ be a positive integer, f an element of $\operatorname{rad}^2 R(\mathbb{C}_n)$, $\Lambda = \Lambda(\mathbb{C}_n)$ and $\Lambda^f = \Lambda^f(\mathbb{C}_n)$. We will show that the algebras Λ and Λ^f are isomorphic. This will be done via a change of generators in Λ^f.

Observe first that f is of the form

$$f = \Big(\sum_{i=1}^{n-2} \lambda_i x^{i+1} + \sum_{j=1}^{n-3} \mu_j y^{j+1}\Big) + (xy, yx, (x+y)^{n-1})$$

for some elements $\lambda_1, \dots, \lambda_{n-3}, \lambda_{n-2}, \mu_1, \dots, \mu_{n-3}$ of K. Hence Λ^f is given by the quiver $Q_{\mathbb{C}_n}$ and the relations

$$a_0\bar{a}_0 = 0, \quad a_1\bar{a}_1 = 0, \quad (\bar{a}_1 a_0 + \bar{a}_0 a_1)^{n-1} = 0,$$
$$\bar{a}_1 a_0 + \bar{a}_0 a_1 + a_2\bar{a}_2 + \sum_{i=1}^{n-2} \lambda_i (\bar{a}_1 a_0)^{i+1} + \sum_{j=1}^{n-3} \mu_j (\bar{a}_0 a_1)^{j+1} = 0,$$
$$\bar{a}_i a_i + a_{i+1}\bar{a}_{i+1} = 0 \quad \text{for } i \in \{2, \dots, n-2\}, \quad \bar{a}_{n-1} a_{n-1} = 0.$$

Let $\Gamma := K[z]/(z^n)$; this algebra is isomorphic to the subalgebra of $e_2\Lambda^f e_2$ generated by $\bar{a}_1 a_0$, and it is also isomorphic to the subalgebra generated by $\bar{a}_0 a_1$. We write the deformed relation as $g(\bar{a}_1 a_0) + h(\bar{a}_0 a_1) + a_2\bar{a}_2 = 0$ with g, h in Γ, namely if we write $\bar{z} = z + (z^n)$ and we set $\lambda_0 = 1$ and $\mu_0 = 1$, then we take

$$g(\bar{z}) := \sum_{i=0}^{n-2} \lambda_i \bar{z}^{i+1}, \quad h(\bar{z}) := \sum_{j=0}^{n-3} \mu_j \bar{z}^{j+1}.$$

Then we have

$$g(\bar{z}) = \bar{z}u(\bar{z}), \quad h(\bar{z}) = \bar{z}v(\bar{z}),$$

where

$$u(\bar{z}) = \Big(\sum_{i=0}^{n-2} \lambda_i \bar{z}^i\Big), \quad h(\bar{z}) = \Big(\sum_{j=0}^{n-3} \mu_j \bar{z}^j\Big).$$

Since $\lambda_0 = 1 = \mu_0$, it is clear that $u(\bar{z})$ and $v(\bar{z})$ are invertible in Γ.

Now we change generators in Λ^f. We replace a_0 by $\psi(a_0) \in \Lambda^f$ and a_1 by $\psi(a_1) \in \Lambda^f$, where

$$\psi(a_0) := u(a_0\bar{a}_1)a_0, \quad \psi(a_1) := v(a_1\bar{a}_0)a_1.$$

We keep all other arrows as they are. Let u^* and v^* be the inverses of u and v, respectively, in Γ. Then

$$a_0 = u^*(a_0\bar{a}_1)\psi(a_0), \quad a_1 = v^*(a_1\bar{a}_0)\psi(a_1).$$

Therefore this is an invertible change of generators. Moreover, using this we can write down the relations in terms of the new generators. First

$$\psi(a_0)\bar{a}_0 = u(a_0\bar{a}_1)a_0\bar{a}_0 = 0, \quad \psi(a_1)\bar{a}_1 = v(a_1\bar{a}_0)a_1\bar{a}_1 = 0.$$

Next, we have

$$g(\bar{a}_1 a_0) = \bar{a}_1\Big(\sum_{i=0}^{n-2} \lambda_i (a_0\bar{a}_1)^i\Big)a_0 = \bar{a}_1 u(a_0\bar{a}_1)a_0 = \bar{a}_1\psi(a_0),$$
$$h(\bar{a}_0 a_1) = \bar{a}_0\Big(\sum_{j=0}^{n-3} \mu_j (a_1\bar{a}_0)^j\Big)a_1 = \bar{a}_0\psi(a_1).$$

Therefore

$$0 = g(\bar{a}_1 a_0) + h(\bar{a}_0 a_1) + a_2\bar{a}_2 = \bar{a}_1\psi(a_0) + \bar{a}_0\psi(a_1) + a_2\bar{a}_2.$$

This is precisely the branch relation in the undeformed algebra $\Lambda(\mathbb{C}_n)$.

All other relations remain unchanged. Hence with these new generators, Λ^f satisfies the relations of Λ, and consequently the algebras Λ and Λ^f are isomorphic. ∎

2. Periodicity of mesh algebras of type $\mathbb{C}_n$. The first part in this section is more general, here A is an arbitrary algebra. Let $e_0, e_1, \ldots, e_n$ be a set of pairwise orthogonal primitive idempotents of A with $1_A = e_0 + e_1 + \cdots + e_n$. Then $e_i \otimes e_j$ for $i, j \in \{0, 1, \ldots, n\}$ form a set of pairwise orthogonal primitive idempotents of the enveloping algebra $A^e = A^{\mathrm{op}} \otimes A$ with $1_{A^e} = \sum_{0\le i,j\le n} e_i \otimes e_j$. Hence $P(i,j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_jA$ for $i, j \in \{0, 1, \ldots, n\}$ form a complete set of pairwise non-isomorphic indecomposable projective right A^e-modules (A-A-bimodules). Moreover, the right A-modules $S_i = e_iA/e_i \operatorname{rad} A$ for $i \in \{0, 1, \ldots, n\}$ give a complete set of pairwise non-isomorphic simple right A-modules.

The following result by Happel [16, Lemma 1.5] describes the terms of a minimal projective bimodule resolution of an algebra.

Proposition 2.1. *Let A be an algebra. Then A admits in* $\operatorname{mod} A^e$ *a minimal projective resolution of the form*

$$\cdots \to \mathbb{P}_r \to \mathbb{P}_{r-1} \to \cdots \to \mathbb{P}_1 \to \mathbb{P}_0 \to A \to 0,$$

where

$$\mathbb{P}_r = \bigoplus_{0\le i,j\le n} P(i,j)^{\dim_K \operatorname{Ext}^r_A(S_i,S_j)}.$$

Let Λ be a symmetric algebra of the form $\Lambda = KQ/I$, where Q is a finite connected quiver and I is an admissible ideal in the path algebra KQ of Q. We assume that Λ is graded by the powers of the radical and I is generated by homogeneous relations. We denote by $|b|$ the degree of a homogeneous element b in Λ. We also write $(X)_d$ for the set of elements in a subset X of Λ which are homogeneous of degree d. For an arrow a of Q, we denote by ia and ta the starting and ending vertex of a, respectively. Moreover, we denote by e_i the primitive idempotent of Λ corresponding to a vertex i of Q and by ω_i a fixed non-zero element of the socle of $e_i\Lambda$. We fix a K-basis $\mathcal{B}$ of Λ consisting of homogeneous elements such that each $v \in \mathcal{B}$ belongs to $e_i\Lambda e_j$ for some vertices i, j, and moreover assume that the basis $\mathcal{B}$ contains the primitive idempotents e_i, the arrows of Q, and the fixed elements ω_i. Then we may take the non-degenerate symmetric associative K-bilinear form

$(-,-) : \Lambda \times \Lambda \to K$ such that, for $b_1, b_2 \in \mathcal{B}$ and $b_1 = e_i b_1$, we have

$$(b_1, b_2) := \text{the coefficient of } \omega_i \text{ in } b_1 b_2$$

when $b_1 b_2$ is expressed in terms of $\mathcal{B}$. Consider also the dual K-basis $\mathcal{B}^* = \{b^*;\, b \in \mathcal{B}\}$ of Λ such that $(b, c^*) = \delta_{bc}$ for $b, c \in \mathcal{B}$. Since the relations generating I and elements of $\mathcal{B}$ are homogeneous, for $b_1 = e_i b_1$ and $b_2 \in \mathcal{B}$, (b_1, b_2) can only be non-zero if $b_2 = b_2 e_i$ and $|b_1| + |b_2| = |\omega_i|$. In particular, if $b \in e_i \mathcal{B} e_j$ then $b^* \in e_j \mathcal{B} e_i$, and b^* is homogeneous of degree $|\omega_i| - |b|$.

Assume now that σ is an algebra automorphism of Λ which permutes the primitive idempotents e_i, $i \in Q_0$, and the arrows $a \in Q_1$ of Λ. Consider the projective right Λ^e-module

$$P = \bigoplus_{i \in Q_0} \Lambda e_i \otimes \sigma(e_i)\Lambda$$

and the homomorphism of right Λ^e-modules $R : P \to \Lambda^e$ given by

$$R(e_i \otimes \sigma(e_i)) := \sum_{a \in Q_0,\, ia = i} a \otimes \sigma(e_i) + \sum_{c \in Q_0,\, tc = i} e_i \otimes \sigma(c).$$

Moreover, we define the elements in P

$$\xi_i := \sum_{b \in e_i \mathcal{B}} (-1)^{|b|} (b \otimes \sigma(b^*)), \qquad i \in Q_0.$$

Then we have the following proposition (similar to [13, Proposition 2.3], there is also some variation in [10]).

Proposition 2.2. *Let i be a vertex of the quiver Q of Λ. Then:*

(i) $R(\xi_i) = 0$.
(ii) $\xi_i \Lambda = \sigma(e_i)\Lambda$ *as right Λ-modules.*
(iii) $\Lambda \xi_i = \Lambda e_i$ *as left Λ-modules.*

Proof. (i) We have the equalities

$$\begin{aligned} R(\xi_i) &= \sum_{b \in e_i \mathcal{B}} (-1)^{|b|} R(b \otimes \sigma(b^*)) = \sum_{j \in Q_0} \sum_{b \in e_i \mathcal{B} e_j} (-1)^{|b|} R(b \otimes \sigma(b^*)) \\ &= \sum_{j \in Q_0} \sum_{b \in e_i \mathcal{B} e_j} (-1)^{|b|} b R\big(e_j \otimes \sigma(e_j)\big) \sigma(b^*) \\ &= \sum_{j \in Q_0} \sum_{b \in e_i \mathcal{B} e_j} \sum_{a \in Q_0,\, ia = j} (-1)^{|b|} ba \otimes \sigma(b^*) \\ &\quad + \sum_{j \in Q_0} \sum_{b \in e_i \mathcal{B} e_j} \sum_{c \in Q_0,\, tc = j} (-1)^{|b|} b \otimes \sigma(c)\sigma(b^*). \end{aligned}$$

We fix some degree and an arrow $a : j \to k$ of Q. We must show that the terms $(-1)^{|b|} ba \otimes \sigma(b^*)$ cancel when b runs through all elements b in $e_i \mathcal{B} e_j$

of degree d. Let $\{x_1, \dots, x_s\} = (e_i\mathcal{B}e_j)_d$ and $\{y_1, \dots, y_t\} = (e_i\mathcal{B}e_k)_{d+1}$. The claim will follow if we show the equality

$$\sum_{l=1}^{s} x_l a \otimes \sigma(x_l^*) = \sum_{m=1}^{t} y_m \otimes \sigma(a)\sigma(y_m^*).$$

For $l \in \{1, \dots, s\}$, $x_l a \in e_i\Lambda e_k$ and has degree $d + 1$, so we can write

$$x_l a = \sum_{m=1}^{t} b_{ml} y_m$$

for some elements $b_{1l}, \dots, b_{tl} \in K$. We now find the elements ay_m^* for $m \in \{1, \dots, t\}$. Let z be an element of $\mathcal{B}$ with $(z, ay_m^*) \neq 0$. Then $(za, y_m^*) = (z, ay_m^*) \neq 0$, and hence $za \in (e_i\mathcal{B}e_k)_{d+1}$, because $y_m^* \in (e_k\mathcal{B}e_i)_{|\omega_i|-d-1}$. This shows that $z \in (e_i\mathcal{B}e_j)_d$, and consequently $z = x_l$ for some $l \in \{1, \dots, s\}$. Thus we obtain

$$(x_l, ay_m^*) = (x_l a, y_m^*) = \Big(\sum_{j=1}^{t} b_{jl} y_j, y_m^*\Big) = b_{ml}.$$

Therefore, $ay_m^* = \sum_{l=1}^{s} b_{ml} x_l^*$ for any $m \in \{1, \dots, t\}$. Finally, we obtain the equalities

$$\begin{aligned}\sum_{m=1}^{t} y_m \otimes \sigma(a)\sigma(y_m^*) &= \sum_{m=1}^{t} y_m \otimes \sigma(ay_m^*) = \sum_{m=1}^{t} \Big[y_m \otimes \sigma\Big(\sum_{l=1}^{s} b_{ml} x_l^*\Big)\Big] \\ &= \sum_{m=1}^{t} \Big[y_m \otimes \Big(\sum_{l=1}^{s} b_{ml}\sigma(x_l^*)\Big)\Big] \\ &= \sum_{l=1}^{s} \Big[\Big(\sum_{m=1}^{t} b_{ml} y_m\Big) \otimes \sigma(x_l^*)\Big] = \sum_{l=1}^{s} x_l a \otimes \sigma(x_l^*),\end{aligned}$$

as required.

(ii) It follows from our assumption on σ that $\sigma(e_i) = e_j$ for a vertex $j \in Q_0$. Then, for any $b \in e_i\mathcal{B}$, we have $\sigma(b^*) \in \Lambda\sigma(e_i) = \Lambda e_j$, and hence $\xi_i = \xi_i e_j$. Consider the epimorphism of right Λ-modules $\pi_i : e_j\Lambda \to \xi_i\Lambda$ given by $\pi_i(x) = \xi_i x$ for any $x \in e_j\Lambda$. Further, the socle of $e_j\Lambda$ is simple and spanned (over K) by ω_j. Moreover, since $\omega_i = \omega_i e_i$, we conclude that $\omega_i^* = e_i$. Observe also that ω_j annihilates the radical of Λ. Then we obtain the equalities

$$\begin{aligned}\pi_i(\omega_j) = \xi_i\omega_j &= \sum_{b\in e_i\mathcal{B}} (-1)^{|b|}(b \otimes \sigma(b^*))\omega_j = \sum_{b\in e_i\mathcal{B}} (-1)^{|b|}(b \otimes (\sigma(b^*)\omega_j)) \\ &= (-1)^{|\omega_i|}(\omega_i \otimes \sigma(\omega_i^*)\omega_j) = (-1)^{|\omega_i|}(\omega_i \otimes \sigma(e_i)\omega_j) = (-1)^{|\omega_i|}(\omega_i \otimes \omega_j),\end{aligned}$$

and so $\pi_i(\omega_j) \neq 0$. This shows that π_i is an isomorphism of right Λ-modules.

(iii) Since $b = e_i b$ for any $b \in e_i\mathcal{B}$, we have $\xi_i = e_i\xi_i$. Consider the epimorphism of left Λ-modules $\theta_i : \Lambda e_i \to \Lambda\xi_i$ given by $\theta_i(x) = x\xi_i$ for any $x \in \Lambda e_i$. The socle of the left Λ-module Λe_i is simple and spanned (over K) by ω_i, and clearly ω_i annihilates the radical of Λ. Then we obtain the equalities

$$\begin{aligned}\theta_i(\omega_i) = \omega_i\xi_i &= \sum_{b\in e_i\mathcal{B}} (-1)^{|b|}\omega_i\big(b\otimes\sigma(b^*)\big)\\ &= \sum_{b\in e_i\mathcal{B}} (-1)^{|b|}(\omega_i b)\otimes\sigma(b^*) = \omega_i\otimes\sigma(e_i^*),\end{aligned}$$

and so $\theta_i(\omega_i) \neq 0$. This proves that θ_i is an isomorphism of left Λ-modules. ∎

Now we apply these to the algebra $\Lambda = \Lambda(\mathbb{C}_n)$; this has a basis with all properties needed for the previous result, and we fix such a basis $\mathcal{B}$.

PROPOSITION 2.3. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be the canonical mesh algebra of type $\mathbb{C}_n$ $(n \geq 3)$ and let σ be the automorphism of Λ of order 2 which interchanges the idempotents e_0 and e_1, and the adjacent arrows. Then the first few terms of a minimal projective bimodule resolution of Λ are*

$$\mathbb{P}_3 \xrightarrow{S} \mathbb{P}_2 \xrightarrow{R} \mathbb{P}_1 \xrightarrow{d} \mathbb{P}_0 \xrightarrow{u} \Lambda \to 0,$$

where

$$\mathbb{P}_0 = \bigoplus_{i\in Q_0} \Lambda e_i\otimes e_i\Lambda, \qquad \mathbb{P}_1 = \bigoplus_{a\in Q_1} \Lambda e_{ia}\otimes e_{ta}\Lambda,$$

$$\mathbb{P}_2 = \mathbb{P}_3 = \bigoplus_{a\in Q_0} \Lambda e_i\otimes\sigma(e_i)\Lambda,$$

$$u(e_i\otimes e_i) = e_i \quad \text{for } i\in\{0,1,\dots,n\},$$

$$d(e_{ia}\otimes e_{ta}) = a\otimes e_{ta} - e_{ia}\otimes a \quad \text{for } a\in Q_1,$$

$$R(e_i\otimes\sigma(e_i)) = \sum_{a\in Q_0,\, ia=i} a\otimes\sigma(e_i) + \sum_{c\in Q_0,\, tc=i} e_i\otimes\sigma(c) \quad \text{for } i\in Q_0,$$

$$S(e_i\otimes\sigma(e_i)) = \xi_i := \sum_{b\in e_i\mathcal{B}} (-1)^{|b|} b\otimes\sigma(b^*) \quad \text{for } i\in Q_0.$$

Proof. We denote by σ the automorphism of order 2 of $Q = Q_{\mathbb{C}_n}$ induced by the automorphism σ of Λ. For $i \in Q_0 = \{0,1,\dots,n\}$, we denote by $P_i = e_i\Lambda$ and $S_i = e_i\Lambda/e_i \operatorname{rad}\Lambda$ the associated indecomposable projective right Λ-module and simple right Λ-module, respectively. Then the first few terms of a minimal projective resolution of a simple module S_i in $\operatorname{mod}\Lambda$ are given by the exact sequence

$$0 \to S_{\sigma(i)} \to P_{\sigma(i)} \to \bigoplus_{a\in Q_0,\, ia=i} P_{ta} \to P_i \to S_i \to 0.$$

Hence the required presentations for $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ are then consequences of Proposition 2.1. A simple checking shows that the sequence

$$\mathbb{P}_2 \xrightarrow{R} \mathbb{P}_1 \xrightarrow{d} \mathbb{P}_0 \xrightarrow{u} \Lambda \to 0$$

is exact. We claim that $\Omega^3_{\Lambda^e}(\Lambda) = \operatorname{Ker} R$ is the Λ-Λ-bimodule generated by the elements ξ_i for $i \in Q_0 = \{0, 1, \dots, n\}$. It follows from Proposition 2.2(i) that

(1) $\xi_0, \xi_1, \dots, \xi_n$ belong to $\operatorname{Ker} R$;
(2) $\xi_i \Lambda \cong \sigma(e_i)\Lambda = e_{\sigma(i)}\Lambda$ as right Λ-modules for all $i \in \{0, 1, \dots, n\}$;
(3) $\Lambda\xi_i = \Lambda e_i$ as left Λ-modules for all $i \in \{0, 1, \dots, n\}$.

This shows that $S : \mathbb{P}_2 \to \operatorname{Ker} R$ is a minimal projective cover of the Λ-Λ-bimodule $\operatorname{Ker} R = \Omega^3_{\Lambda^e}(\Lambda)$. ∎

Moreover, the above shows that $\Omega^3_{\Lambda^e}(\Lambda)$ is isomorphic to ${}_1\Lambda_\gamma$ where $\gamma \in \operatorname{Aut}(\Lambda)$ satisfies $a\xi_i = \xi_j\gamma(a)$ for $a \in e_i\Lambda e_j$.

Corollary 2.4. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be the canonical mesh algebra of type $\mathbb{C}_n$ $(n \geq 3)$. Then there exists an isomorphism of Λ-Λ-bimodules $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\gamma$ for an algebra automorphism γ of Λ.*

We keep the basis $\mathcal{B}$ from before, and we determine γ as above.

Lemma 2.5. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be the canonical mesh algebra of type $\mathbb{C}_n$ $(n \geq 3)$ and let γ be the algebra automorphism of Λ as above, such that $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\gamma$ as Λ-Λ-bimodules. Then:*

(i) *$\gamma(e_i) = \sigma(e_i)$ for all $i \in \{0, 1, \dots, n\}$.*
(ii) *For any arrow $a : i \to j$ in $Q_{\mathbb{C}_n}$, there is a unique element $b = b^{(a)}$ in $\mathcal{B}$ such that $ab^{(a)} = \varepsilon_a\omega_i$, where $\varepsilon_a = \pm 1$.*
(iii) *For any arrow a in $Q_{\mathbb{C}_n}$, we have*

$$\gamma(a) = -\varepsilon_a((b^{(a)})^*).$$

Proof. (i) We have $e_i\xi_i = \xi_i = \xi_i\sigma(e_i)$, and hence $\gamma(e_i) = \sigma(e_i)$ for any $i \in \{0, 1, \dots, n-1\}$.

(ii) Let $a : i \to j$ be an arrow in $Q_{\mathbb{C}_n}$. Then $a\xi_j = \xi_i\gamma(a)$, and the right hand side has a term $\omega_i \otimes \gamma(a)$. Hence, we only need to identify all terms $(-1)^{|b|}ab \otimes \sigma(b^*)$ from $a\xi_j$ where ab involves ω_i. If this is the case, then $b \in e_j\mathcal{B}e_i$ with $|b| + 1 = |\omega_i| = 2n - 2$, so b is in the second socle $\operatorname{soc}_2(e_j\Lambda) = \operatorname{soc}(e_j\Lambda/\operatorname{soc}(e_j\Lambda))$ of $e_j\Lambda$ and ends in i. We note that the set $e_j\mathcal{B}e_i$ has only one element of degree $|\omega_i| - 1 = 2n - 3$. Thus b is unique, and we denote it by $b^{(a)}$. Moreover, $ab^{(a)} = \varepsilon_a\omega_i$ for some $\varepsilon_a \in \{-1, 1\}$.

(iii) Let a be an arrow of $Q_{\mathbb{C}_n}$. Since the homogeneous element $b^{(a)}$ is of odd degree $2n - 3$, we obtain from (ii) the formula

$$\gamma(a) = -\varepsilon_a((b^{(a)})^*). \ ∎$$

PROPOSITION 2.6. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be the canonical mesh algebra of type $\mathbb{C}_n$ $(n \geq 3)$, and γ be the automorphism of Λ such that*

(i) $\gamma(e_i) = \sigma(e_i)$ *for any* $i \in \{0, 1, \dots, n-1\}$;
(ii) $\gamma(a_0) = -a_1$, $\gamma(a_1) = a_0$, $\gamma(\bar{a}_0) = \bar{a}_1$, $\gamma(\bar{a}_1) = -\bar{a}_0$;
(iii) $\gamma(a_k) = (-1)^k a_k$ *and* $\gamma(\bar{a}_k) = (-1)^k \bar{a}_k$ *for any* $k \in \{2, \dots, n-1\}$.

Then $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\gamma$ *as* Λ-Λ-*bimodules.*

Proof. In order to apply Lemma 2.5, we fix some explicit elements of the socle $\mathrm{soc}(\Lambda)$ of Λ, and also basis elements of the second socle $\mathrm{soc}_2(\Lambda) = \mathrm{soc}(\Lambda/\mathrm{soc}(\Lambda))$ of Λ. Let $\alpha = \bar{a}_1 a_0$, $\beta = \bar{a}_0 a_1$, $\eta = a_2 \bar{a}_2$. Then we have the relations

$$\alpha\beta = 0, \quad \beta\alpha = 0, \quad \alpha^{n-1} = -\beta^{n-1} \neq 0, \quad \eta^{n-1} = 0, \quad \eta^{n-2} \neq 0.$$

Moreover, we take the socle elements of $e_i\Lambda$, $i \in \{0, 1, \dots, n-1\}$:

$$\begin{aligned}
\omega_0 &= a_0 \alpha^{n-2} \bar{a}_1, \\
\omega_1 &= a_1 \beta^{n-2} \bar{a}_0, \\
\omega_2 &= \alpha^{n-1}, \\
\omega_3 &= \bar{a}_2 \alpha^{n-2} a_2, \\
\omega_4 &= \bar{a}_3 \bar{a}_2 \alpha^{n-3} a_2 a_3, \\
&\vdots \\
\omega_k &= \bar{a}_{k-1} \bar{a}_{k-2} \dots \bar{a}_2 \alpha^{n-k+1} a_2 a_3 \dots a_{k-1} \quad \text{for } k \in \{3, \dots, n\}.
\end{aligned}$$

Next we fix basis vectors of $\mathrm{soc}_2(\Lambda)$, and we find their dual elements. We note that the dual element is always of degree 1, and it must be $\pm a$, where a is an arrow, and the sign is given by the requirement that $ba = \omega_i$ if $b \in e_i\mathcal{B}$. Furthermore, for each chosen b in $\mathrm{soc}_2(\Lambda)$, we list the arrow a such that $b = b^{(a)}$, and the sign $\varepsilon_a \in \{-1, 1\}$ with $ab = \varepsilon_a \omega_i$:

$e_0\Lambda$	$e_1\Lambda$	$e_2\Lambda$			$e_3\Lambda$	
$a_0\alpha^{n-2}$	$a_1\beta^{n-2}$	$\alpha^{n-2}\bar{a}_1$	$\beta^{n-2}\bar{a}_0$	$\alpha^{n-2}a_2$	$\bar{a}_2\alpha^{n-2}$	$\bar{a}_2\alpha^{n-3}a_2a_3$
$\bar{a}_1$	$\bar{a}_0$	a_0	$-a_1$	$-\bar{a}_2$	a_2	$\bar{a}_3$
$\bar{a}_1$	$\bar{a}_0$	a_0	a_1	$\bar{a}_2$	a_2	$\bar{a}_3$
1	-1	1	1	1	-1	1

$e_4\Lambda$		$e_n\Lambda$
$\bar{a}_3\bar{a}_2\alpha^{n-3}a_2$	$\bar{a}_3\bar{a}_2\alpha^{n-4}a_2a_3a_4$	$\bar{a}_{n-1}\dots\bar{a}_2\alpha a_2\dots a_{n-2}$
a_3	$-\bar{a}_4$	a_{n-1}
a_3	$\bar{a}_4$	a_{n-1}
1	1	$(-1)^n$

$e_k\Lambda$	
$(\bar{a}_{k-1}\dots\bar{a}_2\alpha^{n-k+1}a_2\dots a_{k-2}$	$\bar{a}_{k-1}\dots\bar{a}_2\alpha^{n-k}a_2\dots a_{k-1}a_k$
a_{k-1}	$(-1)^{k+1}\bar{a}_k$
a_{k-1}	$\bar{a}_k$
$(-1)^k$	1

Now a straightforward calculation shows that the algebra automorphism γ of Λ with $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\gamma$ as Λ-Λ-bimodules, discussed in Lemma 2.5, is defined by the imposed conditions (i)–(iii). ∎

COROLLARY 2.7. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be the canonical mesh algebra of type $\mathbb{C}_n$ $(n \geq 3)$. Then there exists an algebra automorphism γ of Λ such that $\Omega^6_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_{\gamma^2}$ as Λ-Λ-bimodules with $\gamma^2(a) = -a$ for $a \in \{a_0, a_0, \bar{a}_0, \bar{a}_1\}$ and $\gamma^2(a) = a$ for the remaining arrows a of $Q_{\mathbb{C}_n}$.*

The following lemma completes the proof of Theorem C.

LEMMA 2.8. *Let $\Lambda = \Lambda(\mathbb{C}_n)$ be the canonical mesh algebra of type $\mathbb{C}_n$ $(n \geq 3)$ and γ the algebra automorphism of Λ described in Proposition 2.6. Then*

(i) *γ is not inner;*
(ii) *γ^2 is inner.*

Proof. (i) Assume for a contradiction that γ is inner. Then there exists an invertible element $c \in \Lambda$ such that $\gamma(x) = cxc^{-1}$ for any $x \in \Lambda$. In particular, we conclude that

$$e_0 = e_0e_0e_0 = e_0\gamma(e_1)e_0 = e_0(ce_1c^{-1})e_0 = (e_0ce_1)(e_1c^{-1}e_0)$$

and this belongs to $\operatorname{rad}\Lambda$, a contradiction.

(ii) Let $c := -e_0 - e_1 + \sum_{i=2}^n e_i \in \Lambda$. Then $c^2 = 1_\Lambda$, hence c is a unit with $c = c^{-1}$. We have

$$ca_0c^{-1} = (-e_0)a_0e_2 = -a_0 = \gamma^2(a_0)$$

and similarly $cac^{-1} = -a$ for a one of $a_1, \bar{a}_0, \bar{a}_1$, and clearly c commutes with all other arrows. Therefore, $\gamma^2(x) = cxc^{-1}$ for any $x \in \Lambda$. ∎

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