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COMPLETELY CONTINUOUS OPERATORS

BY

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Abstract. A Banach space X has the Dunford–Pettis property (DPP) provided that every weakly compact operator T from X to any Banach space Y is completely continuous (or a Dunford–Pettis operator). It is known that X has the DPP if and only if every weakly null sequence in X is a Dunford–Pettis subset of X. In this paper we give equivalent characterizations of Banach spaces X such that every weakly Cauchy sequence in X is a limited subset of X. We prove that every operator $T : X \to c_0$ is completely continuous if and only if every bounded weakly precompact subset of X is a limited set.

We show that in some cases, the projective and the injective tensor products of two spaces contain weakly precompact sets which are not limited. As a consequence, we deduce that for any infinite compact Hausdorff spaces K_1 and K_2, $C(K_1) \otimes_\pi C(K_2)$ and $C(K_1) \otimes_\epsilon C(K_2)$ contain weakly precompact sets which are not limited.

1. Definitions and notation. Our notation and terminology are standard. We denote the canonical unit vector basis of c_0 by (e_n) and the canonical unit vector basis of ℓ_1 by (e_n^*). Throughout this paper, X, Y, E, and F will denote real Banach spaces. The set of all operators from X to Y will be denoted by $L(X,Y)$, and the set of all completely continuous operators will be denoted by $CC(X,Y)$. The w^*-w continuous (resp. w^*-w continuous compact) maps from X^* to Y will be denoted by $L_{w^*}(X^*,Y)$ (resp. $K_{w^*}(X^*,Y)$). An operator is *completely continuous* (or *Dunford–Pettis*) if it maps weakly Cauchy sequences to norm convergent sequences, and *weakly completely continuous* (or *Dieudonné*) if it maps weakly Cauchy sequences to weakly convergent sequences.

A bounded subset A of X is called a *Dunford–Pettis* (DP) *subset* (resp. *limited subset*) of X if each weakly null sequence in X^* (resp. w^*-null sequence in X^*) tends to 0 uniformly on A. A Banach space X has the *Dunford–Pettis property* (DPP) if every weakly compact operator T with domain X is completely continuous. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ for all weakly null sequences (x_n) in X and (x_n^*) in X^* [19]. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP [3], [39], [27]. The reader can check Diestel [19], [18], Diestel and Uhl [22], and Andrews [2]

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for a guide to the extensive classical literature dealing with the DPP, equivalent formulations of the preceding definitions, and undefined notation and terminology.

A subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence. Every DP set is weakly precompact (see, e.g., [51, p. 377], [2], [36]). Since any limited set is a DP set, any limited set is weakly precompact. An operator T is *weakly precompact* (or *almost weakly compact*) if $T(B_X)$ is weakly precompact.

A bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence (x_n) in X tends to 0 uniformly on A, i.e.,

$$\lim_n \sup\{|x^*(x_n)| : x^* \in A\} = 0.$$

A Banach space X has the DPrcP (resp. X has the *property* RDP*) if every DP subset of X is relatively compact (resp. relatively weakly compact) [6], [29], and X has the *Gelfand–Phillips* (GP) *property* (resp. the (BD) *property*) if every limited subset of X is relatively compact (resp. relatively weakly compact) [14], [22]. Emmanuele and Bator [33], [4] showed that $\ell_1 \not\hookrightarrow X$ iff any L-subset of X^* is relatively compact iff X^* has the DPrcP.

The space c_0 does not have the property RDP* and has the DPP. Note that $(s_n) = (\sum_{i=1}^{n} e_i)$ is a DP subset of c_0 which is not relatively weakly compact. The space ℓ_2 has the property RDP* and does not have the DPP.

If X has the DPrcP, then X has the (GP) property (since any limited set is a DP set). The space c_0 has the (GP) property (since it is separable [14]), and does not have the DPrcP (since it does not have the property RDP*).

If X has the (GP) property, then X has the (BD) property. Schlumprecht constructed a $C(K)$ space which has the (BD) property, but does not have the (GP) property ([57, Proposition 5.1.7, p. 144]).

If X contains no copies of ℓ_1, then X has the (BD) property [14]. Schlumprecht gave an example of a Banach space Y such that Y contains no copies of ℓ_1 and Y does not have the (GP) property ([57, Theorem 5.2.4, p. 149]).

If X has the DPrcP, then X has the property RDP*. Let X be the Bourgain–Delbaen space which is infinite-dimensional, has the Schur property, and X^* is weakly sequentially complete [13]. Note that DP subsets of X^* are weakly precompact, and thus relatively weakly compact. Hence X^* has the property RDP* and does not have the DPrcP (since X contains copies of ℓ_1, [4], [33]).

If X has the property RDP*, then X has the (BD) property. Note that c_0 has the (BD) property (since c_0 contains no copies of ℓ_1 [14]), and c_0 does not have the property RDP*.

A series $\sum x_n$ of elements of X is *weakly unconditionally convergent* (wuc) if $\sum |x^*(x_n)| < \infty$ for each $x^* \in X^*$. A bounded subset A of X (resp. A

of X^*) is called a V^*-*subset* of X (resp. a V-*subset* of X^*) provided that

$$\lim_n \sup\{|x_n^*(x)| : x \in A\} = 0$$

$$(\text{resp. } \lim_n \sup\{|x^*(x_n)| : x^* \in A\} = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp. wuc series $\sum x_n$ in X). A Banach space X has *property* (V) if every V-subset of X^* is relatively weakly compact, and X has *property* (V^*) if every V^*-subset of X is relatively weakly compact [47]. Pełczyński [47] proved that $C(K)$ spaces have property (V).

A topological space S is called *scattered* (or *dispersed*) if every nonempty closed subset of S has an isolated point [60]. A compact Hausdorff space K is scattered if and only if $\ell_1 \not\hookrightarrow C(K)$ [48].

The *Banach–Mazur distance* $d(E,F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\,\|T^{-1}\|)$, where the infinum is taken over all isomorphisms T from E onto F. A Banach space E is called an $\mathcal{L}_\infty$-*space* (resp. $\mathcal{L}_1$-*space*) [10] if there is a $\lambda \geq 1$ so that every finite-dimensional subspace of E is contained in another subspace N with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer n. Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$) spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) (Proposition 1.26 of [10]). The dual of an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp. $\mathcal{L}_1$-space) (Proposition 1.27 of [10]). $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP (Corollary 1.30 of [10]).

It is known that if E and F are $\mathcal{L}_\infty$-spaces, then $E \otimes_\epsilon F$ is an $\mathcal{L}_\infty$-space. If E and F are $\mathcal{L}_1$- spaces, then $E \otimes_\pi F$ is an $\mathcal{L}_1$-space (Theorem 34.9 of [17]). Further, if E is an $\mathcal{L}_1$-space and F is an $\mathcal{L}_\infty$-space, then $E \otimes_\epsilon F$ and $E \otimes_\pi F$ have the DPP [9], [30].

2. Completely continuous operators and limited sets. The following theorem is similar in spirit to Theorem 1 of [19] and gives equivalent conditions for a Banach space X with the property that every bounded weakly precompact subset of X is limited. We note that if X is a Schur space, then $L(X, c_0) = CC(X, c_0)$.

THEOREM 1. *The following statements are equivalent:*

(1) $L(X, c_0) = CC(X, c_0)$.
(2) $L(X, Y) = CC(X, Y)$ *for every Banach space* Y *so that* B_{Y^*} *is* w^**-sequentially compact.*
(3) $|x_n^*(x_n)| \to 0$ *whenever* (x_n) *is weakly null in* X *and* (x_n^*) *is* w^**-null in* X^*.
(4) $|x_n^*(x_n)| \to 0$ *whenever* (x_n) *is weakly Cauchy in* X *and* (x_n^*) *is* w^**-null in* X^*.
(5) $|x_n^*(x_n)| \to 0$ *whenever* (x_n) *is weakly null in* X *and* (x_n^*) *is* w^**-Cauchy in* X^*.

(6) *Every operator* $T \in L_{w^*}(X^*, c_0)$ *is* w^**-norm sequentially continuous.*
(7) *Every operator* $T \in L_{w^*}(X^*, Y)$ *is* w^**-norm sequentially continuous for every Banach space* Y *so that* B_{Y^*} *is* w^**-sequentially compact.*
(8) *Every operator* $T \in L_{w^*}(Y^*, X)$ *has a* w^**-norm sequentially continuous adjoint for every Banach space* Y*.*
(9) *Every operator* $T \in L_{w^*}(l_1, X)$ *has a* w^**-norm sequentially continuous adjoint.*
(10) *Every weakly precompact operator* $T : Y \to X$ *has a* w^**-norm sequentially continuous adjoint for every Banach space* Y*.*
(11) *If* $T \in L(X, Y)$ *and* $T^*(B_{Y^*})$ *is* w^**-sequentially compact, then* T *is completely continuous.*

Proof. We proved that (1), (2), and (3) are equivalent in Theorem 2.2 of [35].

To see that (3)⇒(4), let (x_n) be weakly Cauchy in X and (x_n^*) be w^*-null in X^*. Suppose by contradiction that $|x_n^*(x_n)| \not\to 0$. Without loss of generality assume that $|x_n^*(x_n)| > \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$. Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|x_{n_2}^*(x_{n_1})| < \epsilon/2$. We can do this since (x_n^*) is w^*-null. Continue inductively. Choose $n_k > n_{k-1}$ so that $|x_{n_k}^*(x_{n_{k-1}})| < \epsilon/2$. By hypothesis, $|x_{n_k}^*(x_{n_k} - x_{n_{k-1}})| \to 0$. However,

$$|x_{n_k}^*(x_{n_k} - x_{n_{k-1}})| \geq |x_{n_k}^*(x_{n_k})| - |x_{n_k}^*(x_{n_{k-1}})| > \epsilon/2,$$

a contradiction. Clearly (4)⇒(3). Hence (1)–(4) are equivalent.

To see that (3)⇒(5), let (x_n) be weakly null in X and (x_n^*) be w^*-Cauchy in X^*, and suppose that $|x_n^*(x_n)| \not\to 0$. Without loss of generality assume that $|x_n^*(x_n)| > \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$. Let $n_1 = 1$ and choose $n_2 > n_1$ so that $|x_{n_1}^*(x_{n_2})| < \epsilon/2$. We can do this since (x_n) is weakly null. Continue inductively. Choose $n_{k+1} > n_k$ so that $|x_{n_k}^*(x_{n_{k+1}})| < \epsilon/2$. By hypothesis, $|(x_{n_{k+1}}^* - x_{n_k}^*)(x_{n_{k+1}})| \to 0$. Since

$$|(x_{n_{k+1}}^* - x_{n_k}^*)(x_{n_{k+1}})| \geq |x_{n_{k+1}}^*(x_{n_{k+1}})| - |x_{n_k}^*(x_{n_{k+1}})| > \epsilon/2,$$

we have a contradiction.

To see that (5)⇒(1), let $T : X \to c_0$ be an operator, (x_n) be a weakly null sequence in X, and suppose that $T(x_n) \not\to 0$. Without loss of generality $\|T(x_n)\| \geq \epsilon$ for each $n \in \mathbb{N}$, for some $\epsilon > 0$. Choose (y_n^*) in B_{ℓ_1} so that $y_n^*(T(x_n)) = \|T(x_n)\| \geq \epsilon$. Since B_{ℓ_1} is w^*-sequentially compact, we can assume that (y_n^*) is w^*-Cauchy. Then $(T^*(y_n^*))$ is w^*-Cauchy in X^*, and thus $T^*(y_n^*)(x_n) = y_n^*(T(x_n)) \to 0$, a contradiction.

To see that (4)⇒(7), suppose that $T \in L_{w^*}(X^*, Y)$, B_{Y^*} is w^*-sequentially compact and let (x_n^*) be a w^*-null sequence in X^*. If $(T(x_n^*))$ is not norm null, we can assume without loss of generality that $(T(x_n^*))$ is a weakly null normalized basic sequence in Y. Let (f_n^*) be a bounded sequence in Y^* such that $f_i^*(T(x_n^*)) = \delta_{ij}$. Without loss of generality (f_n^*) is w^*-Cauchy, thus $(T^*(f_n^*))$ is weakly Cauchy in X. By (4), $T^*(f_n^*)(x_n^*) \to 0$, a contradiction.

Note that (7)⇒(6). To see that (6)⇒(3), let (x_n) be weakly null in X and (x_n^*) be w^*-null in X^*. Define $T : X^* \to c_0$ by $T(x^*) = (x^*(x_n))$. Note that $T^*(e_n^*) = x_n$, $T^*(\ell_1) \subseteq X$, and thus T is w^*-w continuous. Hence T is w^*-norm sequentially continuous. Therefore

$$|x_n^*(x_n)| \leq \sup_i |x_n^*(x_i)| = \|T(x_n^*)\| \to 0.$$

Hence, (7), (6), (3), and (4) are equivalent.

To see that (4)⇒(8), let $T \in L_{w^*}(Y^*, X)$. Note that T is weakly compact, and thus weakly precompact. If (x_n^*) is w^*-null in X^*, then $(T^*(x_n^*))$ is weakly null in Y. If $(T^*(x_n^*))$ is not norm null, by the Bessaga–Pełczyński selection principle we may assume that $(T^*(x_n^*))$ is basic. Let (y_n^*) be a bounded sequence in Y^* so that $y_i^*(T^*(x_j^*)) = \delta_{ij}$. Since T is weakly precompact, we may assume without loss of generality that $(T(y_n^*))$ is weakly Cauchy. Then $x_n^*(T(y_n^*)) \to 0$, and we have a contradiction.

To see that (8)⇒(3), suppose that (x_n) is weakly null in X and (x_n^*) is w^*-null in X^*. Let $T : X^* \to c_0$ be defined by $T(x^*) = (x^*(x_n))$. Note that T, and thus T^*, is w^*-w continuous. By (8), $T^{**} = T$ is w^*-norm sequentially continuous. Then $\|T(x_n^*)\| \to 0$, hence $|x_n^*(x_n)| \leq \|T(x_n^*)\| \to 0$.

We note that (8)⇒(9). To see that (9)⇒(3), suppose that (x_n) is weakly null in X and (x_n^*) is w^*-null in X^* and continue as in (8)⇒(3).

To see that (4)⇒(10), let $T : Y \to X$ be an almost weakly compact operator. Suppose (x_n^*) is a w^*-null sequence in X^* and $\|T^*(x_n^*)\| > \epsilon$ for some $\epsilon > 0$. Choose (y_n) in B_Y so that $x_n^*(T(y_n)) > \epsilon/2$. Since T is almost weakly compact, we may assume that $(T(y_n))$ is weakly Cauchy. By (4), $x_n^*(T(y_n)) \to 0$, a contradiction.

To see that (10)⇒(3), suppose that (x_n) is weakly null in X and (x_n^*) is w^*-null in X^*. Define $T : \ell_1 \to X$ by $T(y) = \sum y_n x_n$ for $y = (y_n) \in \ell_1$. Since $T(B_{\ell_1}) = \overline{\mathrm{co}}\{x_n\}$ is weakly compact, T is weakly compact. Then T is weakly precompact, and thus T^* is w^*-norm sequentially continuous. Note that $T^*(x^*) = (x^*(x_n))$, $x^* \in X^*$. Therefore $|x_n^*(x_n)| \leq \|T^*(x_n^*)\| \to 0$. Hence, (4), (8), (9), (10), and (3) are equivalent.

To see that (5)⇒(11), let $T : X \to Y$ be an operator such that $T^*(B_{Y^*})$ is w^*-sequentially compact. Suppose (x_n) is weakly null in X and $\|T(x_n)\| > \epsilon$ for some $\epsilon > 0$. Choose (y_n^*) in B_{Y^*} so that $y_n^*(T(x_n)) > \epsilon$. Without loss of generality, $(T^*(y_n^*))$ is w^*-convergent, thus w^*-Cauchy. By (5), $T^*(y_n^*)(x_n) \to 0$, a contradiction.

To see that (11)⇒(3), suppose that (x_n) is weakly null in X and (x_n^*) is w^*-null in X^*. Define $T : X \to c_0$ by $T(x) = (x_n^*(x))$. Note that B_{ℓ_1}, and thus $T^*(B_{\ell_1})$, is w^*-sequentially compact. Then T is completely continuous, and thus $|x_n^*(x_n)| \le \|T(x_n)\| \to 0$. ■

Corollary 2.

(i) *If X is infinite-dimensional and $L(X, c_0) = CC(X, c_0)$, then X has the DPP and $\ell_1 \hookrightarrow X$.*
(ii) *If K is scattered, then $L(C(K), c_0) \neq CC(C(K), c_0)$.*
(iii) *If X is infinite-dimensional and X^* has the Schur property, then $L(X, c_0) \neq CC(X, c_0)$.*

Proof. (i) Let (x_n^*) be a normalized w^*-null sequence in X^* (use the Josefson–Nissenzweig theorem). Let (x_n) be a sequence in B_X so that $x_n^*(x_n) > 1/2$ for each n. If $\ell_1 \not\hookrightarrow X$, by Rosenthal's ℓ_1 theorem we can assume without loss of generality that (x_n) is weakly Cauchy. By Theorem 1, $x_n^*(x_n) \to 0$, a contradiction. Hence $\ell_1 \hookrightarrow X$.

(ii) If K is scattered, then $\ell_1 \not\hookrightarrow C(K)$. Apply (i).

(iii) If X^* has the Schur property, then $\ell_1 \not\hookrightarrow X$. Otherwise $\ell_1 \hookrightarrow X$, $L_1 \hookrightarrow X^*$, and thus X^* does not have the Schur property. Apply (i). ■

Corollary 3.

(i) *If $L(X, c_0) = CC(X, c_0)$ and $Y \overset{c}{\hookrightarrow} X$, then $L(Y, c_0) = CC(Y, c_0)$.*
(ii) *If $c_0 \overset{c}{\hookrightarrow} X$, then $L(X, c_0) \neq CC(X, c_0)$.*
(iii) *If $c_0 \hookrightarrow X$ and X has the* (BD) *property, then $L(X, c_0) \neq CC(X, c_0)$.*

Proof. (i) Suppose (y_n) is weakly null in Y and (y_n^*) is w^*-null in Y^*. If $P : X \to Y$ is a projection, then $(P^* y_n^*)$ is w^*-null in X^*. By Theorem 1, $P^* y_n^*(y_n) \to 0$, hence $y_n^*(y_n) \to 0$. Another application of Theorem 1 concludes the proof.

(ii) Note that $L(c_0, c_0) \neq CC(c_0, c_0)$, as the identity map on c_0 is not completely continuous. Apply (i).

(iii) If $c_0 \hookrightarrow X$ and X has the (BD) property, then $c_0 \overset{c}{\hookrightarrow} X$, by Corollary 1.3.3 of [57]. Apply (ii). ■

We remark that the property $L(X, c_0) = CC(X, c_0)$ is inherited by complemented subspaces, but not by quotients. For instance, $L(\ell_1, c_0) = CC(\ell_1, c_0)$, ℓ_2 is a quotient of ℓ_1 ([46, Proposition 3]), and the natural injection from ℓ_2 to c_0 is continuous and not completely continuous. In this respect, this property is similar to the DPP. (The same example shows that the DPP is not inherited by quotients.) It is known that if X^* has the DPP, then X has the DPP [19]. This result is no longer true for the property $L(X, c_0) = CC(X, c_0)$. For instance, $L(\ell_1, c_0) = CC(\ell_1, c_0)$, and the identity map on c_0 is not completely continuous.

COROLLARY 4.

(i) *If* X *has the* (BD) (*resp.* (GP)) *property and* $L(Y, c_0) = CC(Y, c_0)$, *then every operator* $T : Y \to X$ *is weakly completely continuous* (*resp. completely continuous*).

(ii) *If* X *has the property* RDP* (*resp.* X *has the* DPrcP) *and* Y *has the DPP, then every operator* $T : Y \to X$ *is weakly completely continuous* (*resp. completely continuous*).

Proof. (i) Suppose X has (BD). Let $T : Y \to X$ be an operator and let (y_n) be a weakly Cauchy sequence in Y. If (x_n^*) is w^*-null in X^*, then $(T^* x_n^*)$ is w^*-null in Y^*, and Theorem 1 implies that $x_n^* T(y_n) = T^* x_n^*(y_n) \to 0$. Then $(T(y_n))$ is limited in X, and thus relatively weakly compact. Hence $(T(y_n))$ is weakly convergent, and T is weakly completely continuous.

Suppose X has the (GP) property and (y_n) is weakly null in Y. The previous argument shows that $(T(y_n))$ is limited, thus relatively compact. Hence $T(y_n) \to 0$.

(ii) Suppose X has RDP*. Let $T : Y \to X$ be an operator and let (y_n) be weakly Cauchy in Y. If (x_n^*) is weakly null in X^*, then $(T^* x_n^*)$ is weakly null in Y^*. Since Y has the DPP, $x_n^* T(y_n) = T^* x_n^*(y_n) \to 0$. Then $(T(y_n))$ is a DP subset of X, thus relatively weakly compact. Hence $(T(y_n))$ is weakly convergent.

Suppose X has the DPrcP and (y_n) is weakly null in Y. Then $(T(y_n))$ is a DP set, thus relatively compact. Hence $T(y_n) \to 0$. ∎

A Banach space X has the *Grothendieck property* if weak* and weak convergences of sequences in X^* coincide. The Grothendieck property is stable under complemented subspaces. Grothendieck spaces with the DPP are important in the theory of Banach spaces and vector measures (see [22, p. 179]). Some of the known Grothendieck spaces with the DPP are L_∞, injective spaces (e.g. ℓ_∞), $B(\Sigma)$, where Σ is a σ-algebra, certain $C(K)$ spaces, and the space H^∞ of all bounded analytic functions on the open unit disk [11], [12].

Recall that K is called *extremally disconnected* or *Stonean* (resp. σ-*Stonean*) if the closure of each open set (resp. F_σ-set) is open, and K is called an F-*space* if the closures of disjoint open F_σ-sets are disjoint. The Stone–Čech compactification of a discrete topological space [39, p. 167], as well as the order complete spaces $\ell_\infty(\Gamma)$ (Γ infinite), $L_\infty(\mu)$ (μ σ-finite), are $C(K)$ spaces with K Stonean [22, p. 156]. Every Stonean space is σ-Stonean and every σ-Stonean space is an F-space. Grothendieck ([22, Corollary VI.2.12], [39]) proved that if K is Stonean, then $C(K)$ has the Grothendieck property. The same result is due to Andô [1] for K σ-Stonean (a fact also noted by Semadeni [59]), and Seever [58] for K an F-space.

Corollary 5.

(i) ([57]) *If X has the DPP and the Grothendieck property, then $L(X, c_0) = CC(X, c_0)$. Consequently, a bounded subset of X is weakly precompact if and only if it is limited.*

(ii) *If μ is a positive σ-finite measure, then every operator $T : L_\infty(\mu) \to c_0$ is completely continuouus.*

(iii) *If K is scattered, then every operator $T : C(K)^* \to c_0$ is completely continuous. Further, every operator $T : C(K)^{**} \to c_0$ is completely continuous.*

(iv) *If X is an $\mathcal{L}_1$-space, then every operator $T : X^* \to c_0$ is completely continuous.*

(v) *If X is an $\mathcal{L}_\infty$-space, then every operator $T : X^{**} \to c_0$ is completely continuous.*

Proof. (i) Let (x_n) be weakly null in X and (x_n^*) be w^*-null in X^*. Then (x_n^*) is weakly null, and thus $|x_n^*(x_n)| \to 0$ (since X has the DPP). By Theorem 1, $L(X, c_0) = CC(X, c_0)$. Further, every bounded weakly precompact subset of X is limited (by Theorem 1(4)). Since every limited set is weakly precompact [14], the conclusion follows.

(ii) Note that $L_\infty(\mu)$ has the Grothendieck property [22, p. 156]. Since $L_\infty(\mu)$ is a $C(K)$ space, it has the DPP (Corollary VI.2.6 of [22]).

(iii) If K is scattered, then $\ell_1 \not\hookrightarrow C(K)$. Since $C(K)$ has the DPP, $C(K)^*$ has the Schur property [19], [37], and thus every operator $T : C(K)^* \to c_0$ is completely continuous. Further, $C(K)^{**}$ is isomorphic to $\ell_\infty(I)$ for some set I [48]. Then $C(K)^{**}$ has the DPP and the Grothendieck property. Apply (i).

(iv) If X is an $\mathcal{L}_1$-space, then X^* is an injective space [41], and thus X^* has the Grothendieck property. Since X^* has the DPP, the result follows by (i).

(v) If X is an $\mathcal{L}_\infty$-space, then X^* is an $\mathcal{L}_1$-space. Apply (iv). ∎

The previous result shows that for any compact Hausdorff space K, every operator $T : C(K)^{**} \to c_0$ is completely continuous, since $C(K)$ spaces are $\mathcal{L}_\infty$-spaces.

If $X = C(K)$ has the Grothendieck property, then a bounded subset of X is weakly precompact if and only if it is limited. It is known that ℓ_∞ contains limited sets which are not relatively weakly compact (see Example 1.1.8 of [57]). For instance, let $(s_n) = (\sum_{i=1}^n e_i)$. Note that (s_n) is bounded, $(s_n) \subseteq c_0$, and (s_n) is weakly precompact (in fact, (s_n) is a DP set). Further, (s_n) is not relatively weakly compact (since $(1, 1, 1, \ldots)$ is not in c_0). Haydon [40] has given an example of a $C(K)$ space which is a Grothendieck space and does not contain ℓ_∞. Such a space must contain limited sets which are not relatively weakly compact [57, pp. 27–28].

COROLLARY 6. *If X does not have the Schur property and B_{X^*} is w^*-sequentially compact, then $L(X,c_0)\neq CC(X,c_0)$.*

Proof. Let (x_n) be a weakly null normalized sequence in X. Without loss of generality we may assume that (x_n) is basic. Let (x_n^*) be a bounded sequence in X^* such that $x_i^*(x_j)=\delta_{ij}$. We may assume that (x_n^*) is w^*-convergent, and thus w^*-Cauchy. Apply Theorem 1. ■

REMARK. The previous corollary shows that if X is separable and not a Schur space, then $L(X,c_0)\neq CC(X,c_0)$. Consequently, X must contain a bounded and weakly precompact subset which is not limited. Bourgain and Diestel [14] showed that a limited subset of a separable space must be relatively compact. Therefore if X is weakly sequentially complete, separable, and not a Schur space, then X must contain a relatively weakly compact subset which is not limited. Specifically, if $X=L_p[0,1]$, $1\le p<\infty$, or $X=\ell_p$, $1\le p<\infty$, then X contains a relatively weakly compact subset which fails to be limited, and $L(X,c_0)\neq CC(X,c_0)$. The next result deals with analogous results for spaces of Bochner integrable functions.

Let (S,Σ,μ) be a non-purely atomic measure space and X be a Banach space. Let $L_p(\mu,X)$ denote the Banach space of X-valued p-Bochner μ-integrable functions with its usual norm, $1\le p<\infty$. Emmanuele [32] proved that if X contains a copy of c_0, then $L_p(\mu,X)$ contains a complemented copy of c_0. Thus, if $c_0\hookrightarrow X$, there is an operator $T:L_p(\mu,X)\to c_0$ which is not completely continuous (by Corollary 3(ii)). The following result is inspired by Emmanuele's result.

THEOREM 7. *If X is not a Schur space, then there is an operator $T: L_p(\mu,X)\to c_0$ which is not completely continuous.*

Proof. By Theorem 1, it is enough to show that $L_p(\mu,X)$ contains a weakly null sequence which is not limited. Since $L_p(\mu,X)$ contains a complemented copy of $L_p([0,1],X)$ (Proposition 1.6.5 of [16]), it suffices to consider the case of $[0,1]$ with the Lebesgue measure. Let (x_n) be a weakly null normalized sequence in X. Without loss of generality suppose that (x_n) is basic. Let (x_n^*) be a bounded sequence in X^* such that $\langle x_n^*,x_m\rangle=\delta_{nm}$. Let (r_n) be the sequence of Radamacher functions, and define $f_n\in L_p([0,1],X)$ by $f_n(t)=r_n(t)x_n$ and $f_n^*\in L_p([0,1],X)^*$ by $f_n^*(t)=r_n(t)x_n^*$, for each n.

We claim that (f_n) is a weakly null, nonlimited sequence in $L_p([0,1],X)$. Let $f^*\in L_p([0,1],X)^*$. Suppose that $\Psi:[0,1]\to X^*$ is a w^*-measurable function such that $\|\Psi(\cdot)\|$ is measurable, $\|\Psi(\cdot)\|\in L_q([0,1])$ (where q is the conjugate of p), and $\langle f^*,f\rangle=\int_0^1\langle\Psi(t),f(t)\rangle\,dt$ for all $f\in L_p([0,1],X)$ (see Theorem 1.5.4 of [16]). Since

$$\langle f^*,f_n\rangle=\int_0^1 r_n(t)\langle\Psi(t),x_n\rangle\,dt,$$

(x_n) is weakly null, and $|\langle \Psi(t), x_n\rangle| \le \|\Psi(t)\|$ for all $n \in \mathbb{N}$ and $t \in [0,1]$, the Lebesgue dominated convergence theorem implies that $\langle f^*, f_n\rangle \to 0$. Thus (f_n) is weakly null.

Now suppose that $f \in L_p([0,1], X)$ is a simple function and note that

$$\int_0^1 r_n(t) f(t)\, dt \to 0,$$

because (r_n) is w^*-null in $L_1([0,1])^* = L_\infty([0,1])$. Then, since the simple functions are dense in $L_p([0,1], X)$, $\int_0^1 r_n(t) f(t)\, dt \to 0$ for each $f \in L_p([0,1], X)$. This implies that

$$|\langle f_n^*, f\rangle| = \Big|\int_0^1 \langle r_n(t) x_n^*, f(t)\rangle\, dt\Big| = \Big|\int_0^1 \langle x_n^*, r_n(t) f(t)\rangle\, dt\Big| = \Big|\Big\langle x_n^*, \int_0^1 r_n(t) f(t)\, dt\Big\rangle\Big|$$
$$\le \|x_n^*\| \Big\|\int_0^1 r_n(t) f(t)\, dt\Big\| \le C \Big\|\int_0^1 r_n(t) f(t)\, dt\Big\| \to 0$$

for each $f \in L_p([0,1], X)$. Hence (f_n^*) is w^*-null. Further, we have

$$\langle f_n^*, f_n\rangle = \int_0^1 r_n^2(t) \langle x_n^*, x_n\rangle\, dt = 1$$

for all n, and thus (f_n) is not limited. ∎

Maurey and Pisier [44], [50] proved that if (Ω, Σ, μ) is an arbitrary measure space and $1 < p < \infty$, then $L_p(\mu, X)$ contains a copy of ℓ_1 iff X contains a copy of ℓ_1. Thus, if X does not contain a copy of ℓ_1, then there is an operator $T : L_p(\mu, X) \to c_0$ which is not completely continuous (by Corollary 2).

Corollary 8. *Suppose that (S, Σ, μ) is a finite non-purely atomic measure space. If X has the DPP and X contains no copy of ℓ_1, then there is an operator $T : L_1(\mu, X) \to c_0$ which is not weakly compact.*

Proof. By Theorem 7, there is an operator $T : L_1(\mu, X) \to c_0$ which is not completely continuous. Since X has the DPP and X contains no copy of ℓ_1, $L_1(\mu, X)$ has the DPP (cf. [2]). Hence, T is not weakly compact. ∎

Theorem 1, and Lohman's lemma below, will be used to prove some permanence properties of Banach spaces X with the property that every operator $T : X \to c_0$ is completely continuous. We start by recalling Lohman's lemma.

Lemma 9 (Lohman's Lemma, [42]). *Let X be a Banach space and let Y be a closed linear subspace of X such that $\ell_1 \not\hookrightarrow Y$. Then each weakly Cauchy sequence in X/Y has a subsequence that is the image of a weakly Cauchy sequence in X under the natural quotient map $Q : X \to X/Y$.*

THEOREM 10. *Suppose that $L(X, c_0) = CC(X, c_0)$ and let Y be a closed linear subspace of X such that $\ell_1 \not\hookrightarrow Y$. Then $L(X/Y, c_0) = CC(X/Y, c_0)$.*

Proof. By Theorem 1, it is enough to show that every weakly null sequence in X/Y is limited. Suppose that (u_n) is a weakly null sequence in X/Y and (u_n^*) is a w^*-null sequence in $(X/Y)^*$. Let $Q : X \to X/Y$ be the natural quotient map. By Lohman's lemma, we can assume without loss of generality that $u_n = Q(x_n)$, where (x_n) is a weakly Cauchy sequence in X. By Theorem 1, (x_n) is limited in X. Since $(Q^* u_n^*)$ is w^*-null in X^*,

$$Q^* u_n^*(x_n) = u_n^*(Q(x_n)) = u_n^*(u_n) \to 0.$$

Then $L(X/Y, c_0) = CC(X/Y, c_0)$. ∎

We recall that if X is a Banach space and Y is a linear subspace of X, then

$$Y^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in Y\}$$

is a subspace of X^*.

COROLLARY 11. *Suppose that $L(X^*, c_0) = CC(X^*, c_0)$ and Y is a closed linear subspace of X such that $\ell_1 \not\hookrightarrow Y^\perp$. Then $L(Y^*, c_0) = CC(Y^*, c_0)$.*

Proof. Use the fact that there is an isometric isomorphism $\phi : X^*/Y^\perp \to Y^*$ that identifies $X^*/Y^\perp$ with Y^*, given by

$$\langle \phi(x^* + Y^\perp), y\rangle = x^*(y)$$

for all $x^* \in X^*$, $y \in Y$, i.e. $\phi(x^* + Y^\perp) = x^*$ restricted to Y (see Theorem 1.10.16 of [45]). By Theorem 10, every operator $T : X^*/Y^\perp \to c_0$ is completely continuous. Then every operator $T : Y^* \to c_0$ is completely continuous. ∎

3. Completely continuous operators on tensor products. In this section we study completely continuous operators on the projective and injective tensor product of two Banach spaces. First, we give some results about completely continuous operators on the projective tensor product of X and Y.

We begin by noting that there are examples so that all operators from the projective tensor product to c_0 are completely continuous. Specifically, since $\ell_1 \otimes_\pi \ell_1 \simeq \ell_1$ ([53, p. 43]), $L(\ell_1 \otimes_\pi \ell_1, c_0) = CC(\ell_1 \otimes_\pi \ell_1, c_0)$. As a counter-demonstration, Corollary 2 above can easily be applied to establish the existence of non-completely continuous operators in many cases. For example, Theorem 2.2 of [8] shows that if K_1 and K_2 are infinite compact Hausdorff spaces, then $C(K_1) \otimes_\pi C(K_2)$ has the DPP if and only if both K_1 and K_2 are scattered ($\ell_1 \not\hookrightarrow C(K_1)$ and $\ell_1 \not\hookrightarrow C(K_2)$). Therefore, if either K_1 or K_2 is not scattered, Corollary 2 immediately shows that $L(C(K_1) \otimes_\pi$

$C(K_2), c_0) \neq CC(C(K_1) \otimes_\pi C(K_2), c_0)$. Corollary 14 below strengthens this observation.

Further, Theorem 3 of [38] shows that if E does not have the Schur property, F contains a copy of ℓ_1, and $L(E, F^*) = CC(E, F^*)$, then $E \otimes_\pi F$ does not have the DPP. Another application of Corollary 2 shows that in this case $L(E \otimes_\pi F, c_0) \neq CC(E \otimes_\pi F, c_0)$.

Lemma 12 ([8]). *Suppose that $L(E, F^*) = CC(E, F^*)$, (x_n) is a weakly null sequence in E and (y_n) is a bounded sequence in F. Then the sequence $(x_n \otimes y_n)$ is weakly null in $E \otimes_\pi F$.*

Theorem 13. *Suppose that X does not have the Schur property, Y is infinite-dimensional and $L(X, Y^*) = CC(X, Y^*)$. Then $L(X \otimes_\pi Y, c_0) \neq CC(X \otimes_\pi Y, c_0)$.*

Proof. Let (x_n) be a weakly null normalized sequence in X. Without loss of generality assume that (x_n) is basic and let (x_n^*) be a bounded sequence in X^* so that $x_i^*(x_j) = \delta_{ij}$. Use the Josefson–Nissenzweig theorem [18, Chapter XII] to choose a w^*-null sequence (y_n^*) of norm one in Y^*. Suppose (y_n) is a sequence in B_Y such that $y_n^*(y_n) > 1/2$ for each n. Define $T : X \otimes_\pi Y \to c_0$ by

$$T(x \otimes y) = (x_i^*(x) y_i^*(y))_i, \qquad x \in X,\ y \in Y.$$

Note that T is a well-defined, linear and continuous operator. By Lemma 12, $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$. For each $n \in \mathbb{N}$, $\|T(x_n \otimes y_n)\| \geq 1/2$, hence T is not completely continuous. ∎

It is known that every linear operator from any $C(K)$ space into the dual of another $C(K)$ space is weakly compact, and thus completely continuous [47].

Corollary 14. *$L(C(K_1) \otimes_\pi C(K_2), c_0) \neq CC(C(K_1) \otimes_\pi C(K_2), c_0)$ for all infinite compact Hausdorff spaces K_1 and K_2.*

Proof. Since K_1 is infinite, c_0 embeds in $C(K_1)$ ([15], [16, p. 16]), and thus $C(K_1)$ does not have the Schur property. Apply Theorem 13. ∎

Observe that we have $L(\ell_\infty, c_0) = CC(\ell_\infty, c_0)$ (by Corollary 5(i)), but $L(\ell_\infty \otimes_\pi \ell_\infty, c_0) \neq CC(\ell_\infty \otimes_\pi \ell_\infty, c_0)$ (by Corollary 14). Moreover, $\ell_\infty \otimes_\pi \ell_\infty$ does not have the DPP [8].

Corollary 15. *Suppose X does not have the Schur property and $L(X, X^*) = CC(X, X^*)$. Then $L(X \otimes_\pi X, c_0) \neq CC(X \otimes_\pi X, c_0)$.*

Corollary 16.

(i) *Suppose that X has the DPP and property (V), and does not have the Schur property. If Y is infinite-dimensional and $\ell_1 \overset{c}{\not\hookrightarrow} Y$, then there is an operator $T : X \otimes_\pi Y \to c_0$ which is not completely continuous.*

(ii) *If K is an infinite compact Hausdorff space, Y is infinite-dimensional, and $\ell_1 \overset{c}{\not\hookrightarrow} Y$, then there is an operator $T : C(K) \otimes_\pi Y \to c_0$ which is not completely continuous.*

Proof. (i) Since $\ell_1 \overset{c}{\not\hookrightarrow} Y$, we have $c_0 \not\hookrightarrow Y^*$ [7]. If $T : X \to Y^*$ is an operator, then T is is unconditionally converging [7]. Property (V) and the DPP imply that T is weakly compact [47], and thus completely continuous. Apply Theorem 13.

(ii) follows from (i), since $C(K)$ has the DPP [39] and property (V) [47], and does not have the Schur property. ■

Corollary 5 [38] shows that if E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces and $E \otimes_\pi F$ has the DPP, then either E and F have the Schur property or E^* and F^* have the Schur property. We note that if E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then $L(E, F^*) = CC(E, F^*)$ and $L(F, E^*) = CC(F, E^*)$ (Theorems 3.7 and 2.17 of [20]). Bourgain and Delbaen [13] gave an example of a separable infinite-dimensional $\mathcal{L}_\infty$-space with the Schur property.

Corollary 17.

(i) *If E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces and $L(E \otimes_\pi F, c_0) = CC(E \otimes_\pi F, c_0)$, then E and F have the Schur property.*

(ii) *If E and F are infinite-dimensional $\mathcal{L}_1$-spaces, then $L(E^* \otimes_\pi F^*, c_0) \neq CC(E^* \otimes_\pi F^*, c_0)$.*

(iii) *If E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then*

$$L(E^{**} \otimes_\pi F^{**}, c_0) \neq CC(E^{**} \otimes_\pi F^{**}, c_0).$$

Proof. (i) Note that $F \otimes_\pi E \simeq E \otimes_\pi F$. Theorem 13 implies that E and F have the Schur property.

(ii) If E and F are infinite-dimensional $\mathcal{L}_1$-spaces, then E^* and F^* are infinite-dimensional $\mathcal{L}_\infty$-spaces and E and F contain complemented copies of ℓ_1 (Proposition 1.24 of [10]). Suppose $L(E^* \otimes_\pi F^*, c_0) = CC(E^* \otimes_\pi F^*, c_0)$. By (i), E^* and F^* have the Schur property. Thus E and F contain no copies of ℓ_1 (see the proof of Corollary 2(iii)), a contradiction which concludes the proof.

(iii) If E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then E^* and F^* are infinite-dimensional $\mathcal{L}_1$-spaces. Apply (ii). ■

We observe that if E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then $L(E^{**}, c_0) = CC(E^{**}, c_0)$, $L(F^{**}, c_0) = CC(F^{**}, c_0)$, but $L(E^{**} \otimes_\pi F^{**}, c_0) \neq CC(E^{**} \otimes_\pi F^{**}, c_0)$ (by Corollaries 5(v) and 17).

Next we give some results about completely continuous operators on the injective tensor product of X and Y. We consider $X \otimes_\epsilon Y$ canonically embedded in $L_{w^*}(X^*, Y)$. The space $L_{w^*}(X^*, Y)$ was studied by many authors; see for instance [25], [28], [29], [31], [36], [54].

Lust [43] and Ryan [54] showed that $L_{w^*}(X^*, Y)$ has the Schur property if and only if X and Y have the Schur property. Thus, if X and Y have the Schur property, then every operator $T : L_{w^*}(X^*, Y) \to c_0$ is completely continuous.

It is known that $(E \otimes_\epsilon F)^* = I(E, F^*)$, the space of integral operators from E to F^* (Corollary VII.2.12 of [22]). The following lemma is essentially contained in [38] (see the proof of Theorem 12 there). We include the argument for the convenience of the reader.

LEMMA 18 ([38]). *If (x_n) is weakly null in E and (y_n) is bounded in F, then $(x_n \otimes y_n)$ is weakly null in $E \otimes_\epsilon F$.*

Proof. If $T \in I(E, F^*) = (E \otimes_\epsilon F)^*$, then T is completely continuous (Theorem VIII.2.9 of [22]; note that the natural inclusion $I : L_\infty(\mu) \to L_1(\mu)$ is weakly compact, hence completely continuous). So $\langle T, x_n \otimes y_n\rangle \le C\|T(x_n)\| \to 0$, and thus $(x_n \otimes y_n)$ is weakly null in $E \otimes_\epsilon F$. ∎

We recall the following well-known isometries [52]:

(1) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$ and $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \mapsto T^*)$;
(2) $W(X, Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \mapsto T^{**})$.

THEOREM 19. *Suppose that X and Y are infinite-dimensional. If $L(X, c_0) \ne CC(X, c_0)$ or $L(Y, c_0) \ne CC(Y, c_0)$, then there is an operator $T : K_{w^*}(X^*, Y) \to c_0$ which is not completely continuous.*

Proof. Suppose that $L(X, c_0) \ne CC(X, c_0)$. Let (x_n) be a weakly null sequence in X which is not limited. Choose a w^*-null sequence (x_n^*) in X^* so that $x_n^*(x_n) = 1$ for each n. Use the Josefson–Nissenzweig theorem to choose a w^*-null sequence (y_n^*) of norm one in Y^*. Suppose (y_n) is a sequence in B_Y such that $y_n^*(y_n) > 1/2$ for each n.

We claim that $(x_n \otimes y_n)$ is weakly null and not limited in $K_{w^*}(X^*, Y)$. By Lemma 18, $(x_n \otimes y_n)$ is weakly null in $X \otimes_\epsilon Y$, and thus in $K_{w^*}(X^*, Y)$.

Consider the sequence $(x_n^* \otimes y_n^*)$ in the dual of $K_{w^*}(X^*, Y)$, where $\langle x_n^* \otimes y_n^*, T\rangle = \langle T(x_n^*), y_n^*\rangle$, $T \in K_{w^*}(X^*, Y)$. If $T \in K_{w^*}(X^*, Y)$, then T^* is w^*-w continuous and compact. Then $\|T^*(y_n^*)\| \to 0$, and thus

$$\langle x_n^* \otimes y_n^*, T\rangle = \langle T^*(y_n^*), x_n^*\rangle \le \|T^*(y_n^*)\| \to 0.$$

Therefore $(x_n^* \otimes y_n^*)$ is w^*-null in $(K_{w^*}(X^*, Y))^*$. Since $\langle x_n^* \otimes y_n^*, x_n \otimes y_n\rangle > 1/2$ for each n, $(x_n \otimes y_n)$ is not limited in $K_{w^*}(X^*, Y)$. Apply Theorem 1.

If $L(Y, c_0) \ne CC(Y, c_0)$, use the previous argument and the isometry (1) stated above. ∎

COROLLARY 20. *Suppose X and Y are infinite-dimensional. If $L(X, c_0) \ne CC(X, c_0)$ or $L(Y, c_0) \ne CC(Y, c_0)$, then $L(X \otimes_\epsilon Y, c_0) \ne CC(X \otimes_\epsilon Y, c_0)$.*

Proof. Suppose that $L(X,c_0) \neq CC(X,c_0)$. Let (x_n), (x_n^*), (y_n), and (y_n^*) be as in the proof of Theorem 19. The previous argument shows that $(x_n \otimes y_n)$ is weakly null in $X \otimes_\epsilon Y$. Further, $(x_n \otimes y_n)$ is not limited in $K_{w^*}(X^*, Y)$. Hence $(x_n \otimes y_n)$ is not limited in $X \otimes_\epsilon Y$. Apply Theorem 1. ∎

COROLLARY 21. *Suppose X and Y are infinite-dimensional. If Y has a complemented subspace F such that $L(F,c_0) \neq CC(F,c_0)$ (or X has a complemented subspace E such that $L(E,c_0) \neq CC(E,c_0)$), then there is an operator $T : X \otimes_\epsilon Y \to c_0$ which is not completely continuous.*

Proof. Suppose F is complemented in Y and $L(F,c_0) \neq CC(F,c_0)$. By Corollary 20, there is an operator $L : X \otimes_\epsilon F \to c_0$ which is not completely continuous. If I is the identity on X and $P : Y \to F$ is a projection, then $I \otimes_\epsilon P : X \otimes_\epsilon Y \to X \otimes_\epsilon F$ is a projection. Apply Corollary 3(i). ∎

COROLLARY 22. *Suppose that X and Y are infinite-dimensional. If $L(X^*,c_0) \neq CC(X^*,c_0)$ or $L(Y,c_0) \neq CC(Y,c_0)$, then there is an operator $T : K(X,Y) \to c_0$ which is not completely continuous.*

Proof. Apply Theorem 19 and the isometry $K(X,Y) \simeq K_{w^*}(X^{**}, Y)$ (see [52]). ∎

Consider the the space $\ell_1[X]$ of all unconditionally convergent series in X with the norm

$$\|(x_n)\| = \sup\Big\{ \sum |x^*(x_n)| : x^* \in B_{X^*} \Big\}.$$

COROLLARY 23. *Suppose that X is infinite-dimensional and $L(X,c_0) \neq CC(X,c_0)$. Then there is an operator $T : \ell_1[X] \to c_0$ which is not completely continuous.*

Proof. It is known that $\ell_1[X]$ is isometrically isomorphic to $K(c_0, X)$ [29]. Since $L(X,c_0) \neq CC(X,c_0)$, there is an operator $T : K(c_0,X) \to c_0$ which is not completely continuous (by Corollary 22). ∎

We recall the following results about the complementability of c_0 in $X \otimes_\epsilon Y$, $K_{w^*}(X^*,Y)$, and $C(K,X)$.

THEOREM 24.

(i) ([55]–[57]) *Suppose X and Y are infinite-dimensional. If $c_0 \hookrightarrow X$, then $X \otimes_\epsilon Y$ contains an isomorphic copy of c_0 which is complemented in $X \otimes_\epsilon Y$.*

(ii) ([28], [34]) *Suppose X and Y are infinite-dimensional. If $c_0 \hookrightarrow X$ or $c_0 \hookrightarrow Y$, then $K_{w^*}(X^*,Y)$ contains a copy of c_0 which is complemented in $K_{w^*}(X^*,Y)$.*

(iii) ([15]) *If K is an infinite compact Hausdorff space and X is an infinite-dimensional Banach space, then $C(K,X)$ contains a complemented copy of c_0.*

Corollary 20 shows that if X and Y are infinite-dimensional, and $L(X, c_0) \neq CC(X, c_0)$ or $L(Y, c_0) \neq CC(Y, c_0)$, then $L(X \otimes_\epsilon Y, c_0) \neq CC(X \otimes_\epsilon Y, c_0)$. Our next result shows that there are infinite-dimensional spaces X and Y so that $L(X, c_0) = CC(X, c_0)$, $L(Y, c_0) = CC(Y, c_0)$, and $L(X \otimes_\epsilon Y, c_0) \neq CC(X \otimes_\epsilon Y, c_0)$. For example, we can take X and Y to be Grothendieck $C(K)$ spaces; $X = C(K_1)^{**}$, $Y = C(K_2)^{**}$, with K_1, K_2 infinite compact Hausdorff spaces; or $X = E^{**}$, $Y = F^{**}$, with E and F infinite-dimensional $\mathcal{L}_\infty$-spaces.

Corollary 25.

(i) *Suppose X and Y are infinite-dimensional. If $c_0 \hookrightarrow X$ or $c_0 \hookrightarrow Y$, then there is a non-completely continuous operator $T : X \otimes_\epsilon Y \to c_0$. Further, there is a non-completely continuous operator $S : K_{w^*}(X^*, Y) \to c_0$.*

(ii) *If K_1 and K_2 are infinite compact Hausdorff spaces, then there is a non-completely continuous operator $T : C(K_1) \otimes_\epsilon C(K_2) \to c_0$.*

(iii) *If K is an infinite compact Hausdorff space and X is an infinite-dimensional space, then there is a non-completely continuous operator $T : C(K, X) \to c_0$.*

(iv) *If X and Y are infinite-dimensional and X (or Y) is an $\mathcal{L}_1$-space, then there is a non-completely continuous operator $T : X^* \otimes_\epsilon Y^* \to c_0$.*

(v) *If X and Y are infinite-dimensional and X (or Y) is an $\mathcal{L}_\infty$-space, then there is a non-completely continuous operator $T : X^{**} \otimes_\epsilon Y^{**} \to c_0$.*

Proof. (i) Note that $L(c_0, c_0) \neq CC(c_0, c_0)$, since the identity map $i : c_0 \to c_0$ is not completely continuous. Apply Theorem 24 and Corollary 3(ii).

(ii) If K is infinite, $C(K)$ contains a copy of c_0. Apply (i).

(iii) Apply Theorem 24 and Corollary 3(ii). Alternatively, apply (i) and use the isometry $C(K, X) \simeq C(K) \otimes_\epsilon X$ (cf. [52]).

(iv) If X is an infinite-dimensional $\mathcal{L}_1$-space, then X contains a complemented copy of ℓ_1 (Proposition 1.24 of [10]), and thus $c_0 \hookrightarrow X^*$. Apply (i).

(v) If X is an infinite-dimensional $\mathcal{L}_\infty$-space, then X^* is an infinite-dimensional $\mathcal{L}_1$-space. Apply (iv). ∎

Remark. Dobrakov [23] showed that if X is a Schur space, then $C(K, X)$ has the DPP. Andrews [2] proved that if X^* is a Schur space, then $L_1(\mu, X)$ has the DPP if μ is finite. Bourgain [9] proved that for any countable measure μ and any compact Hausdorff space K, the spaces $L_1(\mu, C(K))$ and $C(K, L_1(\mu))$ have the DPP. By Corollary 25, there are non-completely continuous operators $T : C(K, X) \to c_0$ and $S : C(K, L_1(\mu)) \to c_0$.

If X^* is a Schur space, then there is a non-completely continuous operator $T : L_1(\mu, X) \to c_0$. To see this, note that X^* being a Schur space implies that $\ell_1 \not\hookrightarrow X$ (see the proof of Corollary 2(iii)), and thus $L(X, c_0) \neq CC(X, c_0)$. Now, $L_1(\mu, X) \simeq L_1(\mu) \otimes_\pi X$ contains complemented copies of X [53]. Hence Corollary 3 (i) implies that there is a non-completely continuous operator $T : L_1(\mu, X) \to c_0$. A similar argument shows that if K is scattered ($\ell_1 \not\hookrightarrow C(K)$), then there is a non-completely continuous operator $T : L_1(\mu, C(K)) \to c_0$. Thus we obtain examples of injective and projective tensor products with the DPP so that not every operator defined on them is completely continuous.

We recall that a Banach space X has the DPrcP (resp. the (GP)) if every DP (resp. limited) subset of X is relatively compact. We note that X has the DPrcP (resp. the (GP)) if and only if every weakly null DP (resp. limited) sequence in X is norm null [26], [35]. If X is a Banach space and $\mathcal{P}(\mathbb{N})$ is the power class of $\mathbb{N}$, then $\mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$ denotes the Banach space of all countably additive vector measures $\mu : \mathcal{P}(\mathbb{N}) \to X$ with the sup norm $\|\mu\| = \sup\{\|\mu(E)\| : E \in \mathcal{P}(\mathbb{N})\}$. Drewnowski proved that if (m_n) is a sequence in $\mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$ and $(m_n(A))$ is norm null for all $A \in \mathcal{P}(\mathbb{N})$, then $\|m_n\| \to 0$ (Lemma 3 of [24]). Using this result, we have the following characterization of the DPrcP (resp. the (GP)) of $\mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$.

THEOREM 26. *X has the DPrcP (resp.* (GP)*) if and only if* $\mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$ *has the DPrcP (resp.* (GP)*).*

Proof. We present the proof of the result relative to the DPrcP; the other case is similar. Suppose that X has the DPrcP and let (m_n) be a weakly null, DP sequence in $\mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$. Then $(m_n(A))$ is weakly null, DP sequence in X for all $A \in \mathcal{P}(\mathbb{N})$. Since X has the DPrcP, $(m_n(A))$ is norm null for all $A \in \mathcal{P}(\mathbb{N})$. By Lemma 3 of [24], (m_n) is norm null, and thus $\mathrm{ca}(\mathcal{P}(\mathbb{N}), X)$ has the DPrcP. The other implication is straightforward. ∎

4. Unconditionally converging operators, property (V) and V-sets. An operator $T : X \to Y$ is called *unconditionally converging* if T maps weakly unconditionally convergent (wuc) series in X into unconditionally convergent series in Y. Unconditionally converging operators and completely continuous operators are fundamentally linked as a result of the following theorem of Pełczyński [47].

THEOREM 27. *Let K be a compact Hausdorff space and let Y be an arbitrary Banach space. Then for every linear operator $T : C(K) \to Y$ the following conditions are equivalent:*

(i) *T is unconditionally converging.*
(ii) *T is weakly compact.*
(iii) *T is completely continuous.*

Moreover, if an operator $T : X \to Y$ fails to be unconditionally converging, then T is an isomorphism on a copy of c_0 (cf. [7]). Consequently, any completely continuous operator is necessarily unconditionally converging.

For K infinite, $C(K)$ spaces always contain copies of c_0 ([16, p. 16], [15]). However, they may not contain complemented copies of c_0. Phillips's theorem shows that c_0 is not complemented in ℓ_∞.

Recall that $C(K)$ *has the Grothendieck property if and only if it contains no complemented copy of* c_0 ([15]). We present the following generalization of this result.

Theorem 28. *Suppose that X has property (V). Then X has the Grothendieck property if and only if c_0 is not complemented in X.*

Proof. If X has the Grothendieck property, then c_0 is not complemented in X, since c_0 does not have the Grothendieck property.

Conversely, suppose that X does not have the Grothendieck property. Let (x_n^*) be a w^*-null sequence in X^* with no weakly null subsequence. Let $T : X \to c_0$ be defined by $T(x) = (x_n^*(x))$. Since $(T^*(e_n^*)) = (x_n^*)$ is not relatively weakly compact, T is not weakly compact. Since X has property (V), T is not unconditionally converging [47].

Let $\sum x_n$ be wuc in X so that $\sum T(x_n)$ is not unconditionally convergent. Then T is an isomorphism on a copy of c_0. Let $\epsilon > 0$ and let (n_k) and (m_k) be intertwining sequences of positive integers so that

$$\Big\| \sum_{i=m_k}^{n_k} T(x_i) \Big\| > \epsilon$$

for each k. Hence $\| \sum_{i=m_k}^{n_k} x_i \| > \epsilon/\|T\|$ for each k. Let $(y_k) = (\sum_{i=m_k}^{n_k} x_i)$. Note that $\sum y_n$ is wuc and $\inf \|y_k\| > 0$. By a result of Bessaga and Pełczyński (Lemma 3 on p. 160 of [7]), we may assume without loss of generality that (y_n) is equivalent to (e_n). Since $\|T(y_n)\| \nrightarrow 0$, $x_n^*(y_n) \nrightarrow 0$. Hence (y_n) is not limited in X. By Theorem 1.3.2 of [57], (y_n) has a subsequence whose closed linear span is complemented in X. ∎

Corollaries 5(i) and 3(ii), and the result mentioned before Theorem 28, immediately produce the following result.

Theorem 29. *A Banach space $C(K)$ satisfies*

$$L(C(K), c_0) = CC(C(K), c_0)$$

if and only if it has the Grothendieck property.

It follows that $L(C(K), c_0) \neq CC(C(K), c_0)$ whenever K contains a nontrivial converging sequence; indeed, if (t_n) converges in K to $t \in K$, then $\delta_{t_n} \to \delta_t$ in the weak* topology but not weakly, if we assume that t_n are all different. This, in particular, covers Corollary 2(ii).

Phillips's theorem shows that c_0 is not complemented in ℓ_∞. The following result generalizes this theorem. A subset K of X^* is called *relatively w^*-sequentially* compact if every sequence from K has a subsequence which converges to a point in the w^*-topology of X^*.

THEOREM 30. *If K is a relatively w^*-sequentially compact subset of X^{**}, then K is a V-subset of X^{**}.*

Proof. Suppose not. Let $\sum x_n^*$ be a wuc series in X^* so that $\sup\{|x^{**}(x_n^*)| : x^{**} \in K\} \not\to 0$ as $n \to \infty$. Without loss of generality, suppose that $\delta > 0$ and $|x_n^{**}(x_n^*)| > \delta$ for each n, where $x_n^{**} \in K$. Moreover, suppose that $x_n^{**} \to x^{**}$ in the w^*-topology.

If $A \subseteq \mathbb{N}$, then w^*-$\lim_n \sum_{i\in A,\, i\le n} x_i^*$ exists. Denote this limit by x_A^*, and define $\mu_n : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by

$$\mu_n(A) = x_n^{**} x_A^*$$

for $A \subseteq \mathbb{N}$.

Since $\sum_n x_n^*$ is wuc, μ_n is countably additive for each n. Additionally, the w^*-convergence of (x_n^{**}) guarantees that $(\mu_n(A))$ converges for every $A \subseteq \mathbb{N}$. An application of the Vitali–Hahn–Saks–Nikodym theorem ensures that (μ_n) is uniformly strongly additive ([22], p. 23). Therefore $\mu_n(i) \xrightarrow{i} 0$ uniformly in n. This produces an immediate contradiction since $|\mu_n(n)| > \delta$ for each n. ∎

COROLLARY 31.

(i) *If (x_n^{**}) is a w^*-null sequence in X^{**}, then (x_n^{**}) is a V-subset of X^{**}.*
(ii) ([19]) *If X^* has property (V), then X^* has the Grothendieck property.*
(iii) *If X^* has the DPP and property (V), then every operator $T : X^* \to c_0$ is completely continuous.*

Proof. (i) Apply Theorem 30.

(ii) Let (x_n^{**}) be w^*-null in X^{**}. By (i), (x_n^{**}) is a V-subset of X^{**}. Since X^* has property (V), (x_n^{**}) is weakly null. Hence X^* has the Grothendieck property.

(iii) Let $T : X^* \to c_0$ be an operator. By (ii), X^* has the Grothendieck property. Then T is completely continuous (by Corollary 5). ∎

COROLLARY 32. *If $B_{X^{**}}$ is w^*-sequentially compact, then every operator $T : c_0 \to X^*$ is compact.*

Proof. Suppose that $\sum x_n^*$ is wuc in X^*. By Theorem 30, $B_{X^{**}}$ is a V-set. Then $\|x_n^*\| \le \sup\{|x^{**}(x_n^*)| : x^{**} \in B_{X^{**}}\} \to 0$. This implies that any weakly unconditionally convergent series in X^* is unconditionally convergent.

Let $T : c_0 \to X^*$ be an operator. The preceding argument shows that $\sum T(e_n)$ is unconditionally convergent, and thus norm subseries convergent. Hence T is compact [18, p. 113, Problem 2]. ∎

As noted before Theorem 28, c_0 is not complemented in ℓ_∞. Corollary 2, p. 169 of [39] shows that a separable infinite-dimensional space cannot be complemented in $C(K)$ with K Stonean. In fact, the results in this section easily produce stronger results.

COROLLARY 33.

(i) *Suppose Y has the DPP and the Grothendieck property. If X is a complemented subspace of Y and B_{X^*} is w^*-sequentially compact, then X is finite-dimensional.*

(ii) *If $T : \ell_\infty \to c_0$ is an operator, then $T_{|c_0}$ is compact.*

Proof. (i) Suppose that $P : Y \to X$ is a projection. Let (x_n^*) be a sequence in B_{X^*}. Then there is a subsequence $(x_{n_i}^*)$ of (x_n^*) which is w^*-convergent, and thus $(P^*(x_{n_i}^*))$ is weakly convergent (Y has the Grothendieck property). Consequently, P^* and P are weakly compact. Since Y has the DPP, P is completely continuous. Thus $P^2 = P$ is compact (as the composition of a completely continuous operator and a weakly compact operator), B_X is compact, and X is finite-dimensional.

(ii) Every operator from ℓ_∞ to c_0 is completely continuous by Corollary 31(iii) and every bounded sequence in c_0 has a weakly Cauchy subsequence. ∎

COROLLARY 34.

(i) *Every bounded and weakly precompact subset of X^{**} is a V-subset of X^{**}.*

(ii) ([47]) *If K is a bounded and weakly precompact subset of X^*, then K is a V-subset of X^*.*

(iii) ([47]) *If X has property (V), then X^* is weakly sequentially complete.*

Proof. (i) Suppose that K is is a bounded weakly precompact subset of X^{**}. Let (x_n^{**}) be a sequence in K, and without loss of generality suppose that (x_n^{**}) is weakly Cauchy. Therefore there is an $M > 0$ so that (x_n^{**}) is a w^*-Cauchy net in $MB_{X^{**}}$, a w^*-compact set. Thus (x_n^{**}) is w^*-convergent. Apply Theorem 30.

(ii) Let η be the natural embedding of a space in its bidual. Then $\eta(K)$ is bounded and weakly precompact in X^{***}. By (i), $\eta(K)$ is a V-subset of X^{***}. Note that if $\sum x_n$ is wuc in X, then $\sum \eta(x_n)$ is wuc in X^{**}. Consequently, K is a V-subset of X^*.

(iii) Let (x_n^*) be a weakly Cauchy sequence in X^*. By (ii), (x_n^*) is a V-set, thus relatively weakly compact. ∎

Corollary 35.

(i) ([47]) *If K is a bounded and weakly precompact subset of X, then K is a V^*-subset of X.*
(ii) ([47]) *If X has property (V^*), then X is weakly sequentially complete.*

Proof. (i) Let η be the natural embedding of X into X^{**}. Since $\eta(K)$ is a bounded and weakly precompact subset of X^{**}, $\eta(K)$ is a V-subset of X^{**} (by Corollary 34(i)). Then K is a V^*-subset of X.

(ii) Let (x_n) be a weakly Cauchy sequence in X. By (i), (x_n) is a V^*-set, thus relatively weakly compact. ■

Theorem 36.

1. *Suppose that $T : X \to Y$ is an operator. The following are equivalent:*
   (i) *$T^*(B_{Y^*})$ is a V-subset of X^*.*
   (ii) *$\|T(x_n)\| \to 0$ whenever $\sum x_n$ is wuc in X.*
   (iii) *T is unconditionally convergent.*

   Moreover, a bounded subset K of X^* is a V-set iff there is a Banach space Y and an unconditionally converging operator $T : X \to Y$ such that $K \subseteq T^*(B_{Y^*})$.
2. *Suppose that $T : Y \to X$ is an operator. The following are equivalent:*
   (i) *$T(B_Y)$ is a V^*-subset of X.*
   (ii) *$\|T^*(x_n^*)\| \to 0$ whenever $\sum x_n^*$ is wuc in X^*.*
   (iii) *T^* is unconditionally converging.*

   Further, a bounded subset K of X is a V^*-set iff there is a Banach space Y and an operator $T : Y \to X$ such that $K \subseteq T(B_Y)$ and T^* is unconditionally converging.

Proof. 1. (i)⇔(ii). This follows directly from the equality

$$\|T(x_n)\| = \sup\{|\langle T^*(y^*), x_n\rangle| : y^* \in B_{Y^*}\}.$$

(ii)⇒(iii). Suppose that T is not unconditionally converging. Let $\sum x_n$ be a wuc series in X, $\epsilon > 0$, π a permutation of $\mathbb{N}$, and (m_i) and (n_i) intertwining sequences of positive integers such that $\|\sum_{n=m_i}^{n_i} T(x_{\pi(n)})\| > \epsilon$ for each i. If $y_i = \sum_{n=m_i}^{n_i} x_{\pi(n)}$, then $\sum y_i$ is wuc. This is a contradiction with (ii).

(iii)⇒(ii) is clear.

Now, let K be a bounded V-subset of X^*. Let $Y = B(K)$ be the Banach space of all bounded functions on K endowed with the sup norm, and define $T : X \to Y$ by $T(x)(x^*) = x^*(x)$, $x \in X$, $x^* \in K$. If $\sum x_n$ is wuc in X, then $\|T(x_n)\| = \sup\{|x^*(x_n)| : x^* \in K\} \to 0$, since K is a V-set. The previous argument implies that T is unconditionally converging. Note that $T^*(\delta_{x^*}) = x^*$, where δ_{x^*} is the point mass at $x^* \in K$. Hence, $K \subseteq T^*(B_{Y^*})$.

If $K \subseteq T^*(B_{Y^*})$ for some unconditionally converging operator $T : X \to Y$, then $T^*(B_{Y^*})$, thus K, is a V-subset of X^*.

2. (i)⇔(ii). This follows directly from the equality

$$\|T^*(x_n^*)\| = \sup\{|\langle T(y), x_n^*\rangle| : y \in B_Y\}.$$

(ii)⇒(iii) and (iii)⇒(i) are clear.

Now, suppose that K is a bounded subset of X. Let $B(K)$ be the Banach space of all bounded functions on K with sup norm, and let $E : X^* \to B(K)$ be the evaluation map defined by $E(x^*)(x) = x^*(x)$, $x^* \in X^*$, $x \in K$. If $\sum x_n^*$ is wuc in X^*, then $\|E(x_n^*)\| = \sup\{|E(x_n^*)(x)| : x \in K\} = \sup\{|x_n^*(x)| : x \in K\}$. Hence K is a V^*-set iff E is unconditionally converging.

We note that $B(K) \simeq \ell_\infty(K) = \ell_1(K)^*$. Let $T : \ell_1(K) \to X$ be defined by $T(f) = \sum_{x\in K} f(x)x$. Set $Y = \ell_1(K)$, check that $E = T^*$, and note that $K \subseteq T(B_Y)$. ■

The following corollary extends Proposition 1 in Pełczyński [47].

Corollary 37. *Suppose that X is a Banach space.*

1. *The following are equivalent:*
   (i) *Every V-subset of X^* is weakly precompact* (*resp. relatively weakly compact, relatively compact*).
   (ii) *If $T : X \to Y$ is an unconditionally converging operator, then T^* is weakly precompact* (*resp. weakly compact, compact*).

   Further, if every V-subset of X^* is weakly precompact, then every unconditionally convergent operator $T : X \to Y$ is weakly precompact.
2. ([36, Theorem 3.11])
   (i) *Every V^*-subset of X is weakly precompact* (*resp. relatively weakly compact, relatively compact*).
   (ii) *If $T : Y \to X$ is an operator such that T^* is unconditionally converging, then T is weakly precompact* (*resp. weakly compact, compact*).

Proof. We will show that the statements are equivalent in the weakly precompact case. The arguments for all the remaining implications in the theorem follow the same pattern.

1. (i)⇒(ii). Suppose that $T : X \to Y$ is unconditionally converging. By Theorem 36, $T^*(B_{Y^*})$ is a V-subset of X^*, and thus weakly precompact. Then T^* is weakly precompact.

(ii)⇒(i). Let K be a V-subset of X^*. Let Y be a Banach space and $T : X \to Y$ be an unconditionally converging operator such that $K \subseteq T^*(B_{Y^*})$ (use Theorem 36). Since $T^*(B_{Y^*})$ is weakly precompact, K is weakly precompact.

Now suppose that every V-subset of X^* is weakly precompact and $T : X \to Y$ is unconditionally converging. Then T^* is weakly precompact. By Corollary 2 of [5], T is weakly precompact.

2. (i)⇒(ii) Suppose that $T : Y \to X$ is an operator and T^* is unconditionally converging. By Theorem 36, $T(B_Y)$ is a V^*-subset of X, and thus weakly precompact. Then T is weakly precompact.

(ii)⇒(i) Let K be a V^*-subset of X. Suppose Y is a Banach space and $T : Y \to X$ is an operator such that $K \subseteq T(B_Y)$ and T^* is unconditionally converging. Since $T(B_Y)$ is weakly precompact, K is weakly precompact. ■

We remark that Corollary 37 implies that if every V-subset of X^* is relatively compact, then X^* has the Schur property.

THEOREM 38. *Let A be a bounded subset of X^* which is not a V-set. For each $\epsilon > 0$, there exists a sequence (x_n^*) in A and a wuc series $\sum x_n$ in X such that*

$$\langle x_n^*, x_n\rangle = 1 \quad \textit{and} \quad \sum_{m\neq n} |\langle x_n^*, x_m\rangle| < \epsilon \quad \textit{for each } n.$$

Proof. Suppose that A is a bounded subset of X^* which is not a V-set. Let (x_n^*) be a sequence in A and $\sum z_n$ be a wuc series in X such that $x_n^*(z_n) \nrightarrow 0$. Without loss of generality assume that $\|x_n^*\| \le 1$, $\|z_n\| \le 1$, and $x_n^*(z_n) = 1$ for each $n \in \mathbb{N}$. Let $\mathcal{P}$ be the power set of $\mathbb{N}$. For each n, define $\mu_n : \mathcal{P} \to \mathbb{R}$ by

$$\mu_n(A) = \sum_{i\in A} x_n^*(z_i).$$

Note that for each n, μ_n is well defined since $\sum z_i$ is wuc. Further, (μ_n) is a bounded sequence of countably additive scalar measures. Fix $\epsilon > 0$. By Rosenthal's lemma, there are subsequences of (z_n) and of (x_n^*), which we still denote (z_n) and (x_n^*), such that

$$|\mu_n|\Big(\bigcup_{m\neq n}\{m\}\Big) = \sum_{m\neq n} |x_n^*(z_m)| < \epsilon$$

for each n. ■

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