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$D_p(3)$ $(p \geq 5)$ *CAN BE CHARACTERIZED BY ITS ORDER COMPONENTS*

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Abstract. Let G be a finite group, and $M = D_p(3)$ $(p \geq 3)$. It is proved that $G \cong M$ if G and M have the same order components.

1. Introduction. Let G be a finite group. The *prime graph* of G, denoted by $\Gamma(G)$, is defined in [32] as follows: its vertices are the primes dividing the order of G, and two vertices p and q are joined by an edge if and only if there is an element in G of order pq. The set of all connected components of the graph $\Gamma(G)$ is denoted by $T(G) = \{\pi_i(G) \mid i = 1, \ldots, t(G)\}$ where $t(G)$ is the number of connected components of $\Gamma(G)$, and if G is of even order, we always assume that 2 belongs to $\pi_1(G)$ (see [3]). The order components of non-abelian simple groups with disconnected prime graph have been obtained in [3]. For a positive integer n, we denote by $\pi(n)$ the set of all primes dividing n. Obviously the order of the group G can be expressed as the product of $m_1, \ldots, m_{t(G)}$, where m_i is a positive integer with $\pi(m_i) = \pi_i(G)$; such m_i are called the *order components* of G in [3].

In 1987, J. G. Thompson proposed the following conjecture (Problem 12.39 in [28]). Let M be a non-abelian simple group. If G is a finite group satisfying $Z(G) = 1$ and $N(G) = N(M)$, where $N(G) = \{n \in N \mid G$ has a conjugacy class C such that $|C| = n\}$, then $G \cong M$.

Let $\pi_e(G)$ denote the set of orders of elements of G. W. J. Shi [29] proposed the following conjecture in 1990: if M is a finite non-abelian simple group such that $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$ then $G \cong M$. In [5], it was proved that if M is a simple group with disconnected prime graph and G is a finite group satisfying the conditions of J. G. Thompson's conjecture then $\mathrm{OC}(G) = \mathrm{OC}(M)$, where $\mathrm{OC}(G) = \{m_1, \ldots, m_{t(G)}\}$ is the set of order components of G. Obviously, if G is a finite group satisfying the conditions of W. J. Shi's conjecture, then $\mathrm{OC}(G) = \mathrm{OC}(M)$ too. As a consequence, these two conjectures hold for a simple group M which can be characterized

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by its order components consisting of at least of two elements. Hence, it is an important problem to find out those simple groups satisfying the above mentioned properties.

It is known that the following simple groups with disconnected prime graphs can be characterized by their order components: a finite simple group with at least three prime graph components [5], sporadic simple groups [3], Suzuki–Ree groups [6], $G_2(q)$ where $q \equiv 0 \pmod 3$ [4], $E_8(q)$ [1], $\mathrm{PSL}_2(q)$ [7], ${}^3D_4(q)$ [8], ${}^2D_n(3)$, $9 \le n = 2^m + 1 \ne p$ [9], ${}^2D_{p+1}(2)$, $5 \le p \ne 2^m - 1$ [30], A_p where p and $p-2$ are primes [12], $\mathrm{PSL}(5,q)$ [13], $\mathrm{PSL}(3,q)$ where q is an odd prime power [14], $\mathrm{PSL}(3,q)$ for $q = 2^n$ [15], $F_4(q)$ where q is even [16], $C_2(q)$ where $q > 5$ [17], $\mathrm{PSU}_5(q)$ [18], $\mathrm{PSU}(3,q)$ for $q > 5$ [19], ${}^2D_4(q)$ [22], ${}^2E_6(q)$ [24], $E_6(q)$ [23], $\mathrm{PSL}(p,q)$ [20], $\mathrm{PSU}(p,q)$ [25], $\mathrm{PSL}(p+1,q)$ [26], $\mathrm{PSU}(p+1,q)$ [21], $C_p(2)$ [31].

In this paper, we continue this work and prove the following theorem:

Theorem. *Let $M = D_p(3)$ ($p \ge 5$ is a prime). If a finite group G has the same order components as M, then $G \cong M$.*

2. Preliminary results

Lemma 1. *Suppose that $t(G) \ge 2$. If H is a π_i-subgroup of G and $H \trianglelefteq G$, then $\prod_{j=1,\, j\ne i}^{t(G)} m_i \mid |H| - 1$.*

Proof. For any $x \in H$, $x \ne 1$, by the definition of order components, $C_G(x)$ is a π_i-subgroup, and $\prod_{j=1,\, j\ne i}^{t(G)} m_i$ is a divisor of $|G : C_G(x)|$, i.e., $\prod_{j=1,\, j\ne i}^{t(G)} m_i$ divides the length of the conjugacy class of x. Since $H \trianglelefteq G$, H contains the complete G-conjugacy class of x. Hence $\prod_{j=1,\, j\ne i}^{t(G)} m_i \mid |H| - 1$ by the class equation.

In fact, Lemma 1 and its proof is a generalization of Lemma 6 of [3].

Lemma 2 ([2, Theorem 2]). *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = m_2$, $|H| \cdot |G/K| = m_1$, $|G/K| \mid |K/H| - 1$, and H is nilpotent.*

Lemma 3 ([32, Lemma 3]). *If M is a simple group with $t(M) = 2$, G is a finite group and $\mathrm{OC}(G) = \mathrm{OC}(M)$, then one of the following holds:*

(1) *G is a Frobenius group or 2-Frobenius group.*
(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H is a nilpotent π_1-group, K/H is a non-abelian simple group, the odd order component of M is equal to one of those of K/H, G/K is a cyclic π_1-group, and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 4 ([33]). *Let p be a prime and $n \ge 2$ be a natural number. Then there exists a prime divisor r of $p^n - 1$ which does not divide $p^m - 1$ for any* *natural number $m \leq n$, except the cases $n = 6$, $p = 2$ or $n = 2$, $p + 1$ a power of 2. Such an r is called a* primitive prime divisor *of $p^n - 1$.*

Of course a primitive prime divisor of $p^n - 1$ cannot divide $p^n + 1$ or $p^m - 1$ for $n \nmid m$.

Lemma 5 ([11, Remark]). *The only solution of the equation $p^m - q^n = 1$ is $3^2 - 2^3 = 1$, where p, q are primes and $m, n > 1$.*

Lemma 6 ([27, Lemma 1]). *If $n \geq 6$ is a natural number, then there exist at least $s(n)$ primes p_i such that $(n+1)/2 < p_i < n$, where*

$$s(n) = \begin{cases} 6 & \text{for } n \geq 49, \\ 5 & \text{for } 42 \leq n \leq 47, \\ 4 & \text{for } 38 \leq n \leq 41, \\ 3 & \text{for } 18 \leq n \leq 37, \\ 2 & \text{for } 14 \leq n \leq 17, \\ 1 & \text{for } 6 \leq n \leq 13. \end{cases}$$

Let p be a prime number and a a natural number. We denote the power of p in the prime factorization of a by a_p, and if $a_p = p^n$, then we write $p^n \parallel a$. Of course $p^n \parallel a$ means that $p^n \mid a$ and $p^{n+1} \nmid a$.

Lemma 7 ([31, Lemmas 7–9]). *Suppose that p is a prime, $q > 1$ and set $e = \min\{d > 0 : p \mid q^d - 1\}$. Assume that $q^e = 1 + p^r k$, $p \nmid k$, $s = \prod_{i=1}^{n}(q^i - 1)$, t is a natural number and $p^u \parallel t$. If $p > 2$ or $r > 2$, then $p^{r+u} \parallel q^{et} - 1$, $s_p < q^{np/(p-1)}$ and $s_p < q^{1.5n}$ if $p = 2$.*

Definition 1. Suppose that a and f are two integers. If $f \mid a$ and $(f, a/f) = 1$, then f is called a *Hall factor* of a.

Lemma 8 ([10, Theorem 1]). *If q is a power of a prime number, and $c = \prod_{i=1}^{n}(q^{2i} - 1)$ or $(q^n \pm 1)\prod_{i=1}^{n-1}(q^{2i} - 1)$, then there exists a Hall factor f of c satisfying:*

(1) *If $n \geq 23$, then $f > q^{8n}$.*
(2) *If $n = 22$, then $f > q^{7n}$.*
(3) *If $18 \leq n \leq 21$, then $f > q^{6n}$.*
(4) *If $16 \leq n \leq 17$, then $f > q^{5n}$.*
(5) *If $14 \leq n \leq 15$, then $f > q^{4n}$.*

Moreover, if the prime factorization of f is $\prod_{k=1}^{t} r_k^{\delta_k}$, then

$$r_k^{\delta_k} \leq \frac{q^{n-1} - 1}{q - 1}.$$

3. Proof of the theorem. Because $M = D_p(3)$ ($p \geq 5$ is a prime), and G has the same order components with M, the even order component of G is $m_1 = 2 \cdot 3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$, and the odd order component is $m_2 = (3^p - 1)/2$.

We divide the proof into several cases based on Lemma 3 and Tables 1–4 which were obtained in [3].

Table 1. Order components of Lie type simple groups G with $t(G) \geq 3$ (except $E_8(q)$)

Group	orcmp 1	orcmp 2	orcmp 3	orcmp 4
A_p, p, $p-2$ primes	$3 \cdot 4 \cdots (p-3)(p-1)$	$p-2$	p	
$A_1(q)$, $4 \mid q+1$	$q+1$	q	$(q-1)/2$	
$A_1(q)$, $4 \mid q-1$	$q-1$	q	$(q+1)/2$	
$A_1(q)$, $4 \mid q$	q	$q+1$	$q-1$	
$G_2(q)$, $3 \mid q$	$q^6(q^2-1)^2$	q^2+q+1	q^2-q+1	
${}^2G_2(q)$, $q = 3^{2m+1}$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$	
${}^2D_p(3)$ $p = 2^n+1$, $n \geq 2$	$3^{p(p-1)}\ (3^{p-1}-1)$	$(3^{p-1}+1)/2$	$(3^p+1)/4$	
${}^2D_{p+1}(2)$, $p = 2^n-1$, $n \geq 2$	$2^{p(p+1)}\ (2^p-1)$ $\cdot \prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$	
$F_4(q)$, $2 \mid q$, $q > 2$	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1	
${}^2B_2(q)$	q^2	$q+\sqrt{2q}+1$	$q-\sqrt{2q}+1$	$q-1$
$E_7(2)$	$2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3$ $\cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	73	127	
$E_7(3)$	$2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13^3$ $\cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547$	757	1093	
$A_2(4)$	2^6	5	7	9
$A_2(2)$	8	3	7	
${}^2A_5(2)$	$2^{15} \cdot 3^6 \cdot 5$	7	11	
${}^2E_6(2)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11$	13	17	19
${}^2F_4(q)$ $q = 2^{2m+1}$, $m \geq 1$	$q^{12}(q^4-1)(q^3+1)$ $\cdot(q^2+1)(q-1)$	$q^2-\sqrt{2q^3}+$ $q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+$ $q+\sqrt{2q}+1$	

Table 2. Order components of $E_8(q)$

Group	$E_8(q)$, $q \equiv 0, 1, 4 \pmod 5$
orcmp 1	$q^{120}(q^{18}-1)(q^{14}-1)(q^{12}-1)^2(q^{10}-1)^2(q^8-1)^2(q^4+q^2+1)$
orcmp 2	$q^8+q^7-q^5-q^4-q^3+q+1$
orcmp 3	$q^8-q^7+q^5-q^4+q^3-q+1$
orcmp 4	q^8-q^4+1
orcmp 5	$q^8-q^6+q^4-q^2+1$

Table 2 (cont.)

Group	$E_8(q)$, $q \equiv 2, 3 \pmod 5$
orcmp 1	$q^{120}(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)$ $\cdot(q^8-1)(q^4+1)(q^4+q^2+1)$
orcmp 2	$q^8+q^7-q^5-q^4-q^3+q+1$
orcmp 3	$q^8-q^7+q^5-q^4+q^3-q+1$
orcmp 4	q^8-q^4+1

Table 3. Order components of Lie type simple groups G with $t(G) = 2$ (p an odd prime)

Group	orcmp 1	orcmp 2
$\mathrm{Alt}(n)$, $n \neq 5, 6$, $n = p$, $p+1$ or $p+2$ not both n, $n-2$ primes	$3 \cdot 4 \cdots (p-1)(p-2) \cdots n$	p
$A_{p-1}(q)$	$q^{p(p-1)/2} \prod_{i=2}^{p-1}(q^i-1)$	$(q^p-1)/(q-1)(p, q-1)$
$A_p(q)$, $q-1 \mid p-1$	$q^{p(p+1)/2} \prod_{i=2}^{p-1}(q^i-1)(q^{p+1}-1)$	$q^p-1/q-1$
$B_n(q)$ or $C_n(q)$ $n = 2^m$, $m \geq 2$	$q^{n^2}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$(q^n-1)/(2, q-1)$
$B_p(q)$ or $C_p(q)$, p an odd prime, $q = 2, 3$	$3^{p^2}(3^p-1) \prod_{i=1}^{n-1}(3^{2i}-1)$	$(3^p-1)/2$
$D_p(q)$, $p \geq 5$, $q = 3, 5$	$q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i}-1)$	$(q^p-1)/(q-1, 4)$
$D_{p+1}(3)$, $p \geq 3$	$3^{p(p+1)}(3^{p+1}-1) \prod_{i=1}^{p-1}(3^{2i}-1)(3^p+1)$	$(3^p-1)/2$
$E_6(q)$	$q^{36}(q^{12}-1)(q^3-1)(q^8-1)$ $\cdot(q^6-1)(q^5-1)(q^2-1)$	$(q^6+q^3+1)/(3, q-1)$
$F_4(q)$, q odd	$q^{24}(q^6-1)(q^8-1)(q^4-1)(q^2-1)$ $\cdot(q^2-1)(q^4+q^2+1)$	q^4-q^2+1
$G_2(q)$, $3 \mid q-1$	$q^3(q^6-1)^2(q^2+q+1)$	q^2-q+1
$G_2(q)$, $3 \mid q+1$	$q^3(q^6-1)^2(q^2-q+1)$	q^2+q+1
${}^2A_{p-1}(q)$	$q^{p(p-1)/2} \prod_{i=2}^{p-1}(q^i-(-1)^i)$	$(q^p+1)/(q+1)(p, q+1)$
${}^2A_p(q)$, $q+1 \mid p+1$	$q^{p(p+1)/2} \prod_{i=2}^{p-1}(q^i-(-1)^i)(q^{p+1}-1)$	$(q^p+1)/(q+1)$
${}^2D_n(q)$, $n = 2^m$, $m \geq 2$ $(n, q) \neq (p+1, 2)$	$q^{n(n-1)} \prod_{i=1}^{n-1}(q^{2i}-1)$	$(q^n+1)/(2, q-1)$
${}^2D_p(3)$, $p \geq 5$, $p \neq 2^n+1$	$3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i}-1)$	$(3^p+1)/4$
${}^2D_{p+1}(2)$, $p \neq 2^m-1$	$2^{p(p+1)} \prod_{i=1}^{p-1}(2^{2i}-1)(2^p+1)(2^{p+1}+1)$	2^p-1
${}^2D_n(3)$, $n = 2^m+1$, n not a prime	$3^{n(n+1)} \prod_{i=1}^{n-2}(3^{2i}-1)(3^n+1)(3^{n-1}-1)$	$(3^{n-1}+1)/2$
${}^3D_4(q)$	$q^{12}(q^4+q^2+1)(q^6-1)(q^5+1)$	q^4-q^2+1
${}^2E_6(q)$	$q^{36}(q^{12}-1)(q^8+1)(q^6-1)(q^5+1)$ $\cdot(q^3+1)(q^2-1)$	$(q^6-q^3+1)/(3, q+1)$
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2$	13

Table 4. Order components of sporadic simple groups G

Group	orcmp 1	orcmp 2	orcmp 3	orcmp 4	orcmp 5	orcmp 6
M_{11}	$2^4 \cdot 3^2$	5	11			
M_{12}	$2^6 \cdot 3^3 \cdot 5$	11				
M_{22}	$2^7 \cdot 3^2$	5	7	11		
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	11	23			
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	11	23			
J_1	$2^3 \cdot 3 \cdot 5$	7	11	19		
J_2	$2^7 \cdot 3^3 \cdot 5^2$	7				
J_3	$2^7 \cdot 3^5 \cdot 5$	17	19			
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$	23	29	31	37	43
HS	$2^9 \cdot 3^2 \cdot 5^3$	7	11			
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	29				
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$	11	13			
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3$	17				
ON	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3$	11	19	31		
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7$	11				
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11$	31	37	61		
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$	23				
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7$	11	23			
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	23				
F_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11$	13				
F_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	17	23			
F'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13$	17	23	29		
M	$2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2$ $\cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47$	41	59	71		
B	$2^{41} \cdot 3^{31} \cdot 5^6 \cdot 7^2$ $\cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	31	47			
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13$	19	31			
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11$	19				

Case 1. We prove that G cannot be a Frobenius group or a 2-Frobenius group.

Subcase 1.1. If G is a Frobenius group with Frobenius kernel H and complement K, then $|H| = m_1$ and $|K| = m_2$. There exists a primitive prime divisor r of $3^{2(p-1)}-1$ by Lemma 4. Let $S_r \in \mathrm{Syl}_r(H)$. Then $|S_r| \mid (3^{p-1}+1)/2$ and $S_r \trianglelefteq G$. Thus $|S_r| \equiv 1 \pmod{m_2}$ by Lemma 1, which is impossible.

Subcase 1.2. If G is a 2-Frobenius group, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H is a nilpotent π_1-group, $|K/H| = m_2$, and

$|G/K| \mid |K/H| - 1 = 3(3^{p-1} - 1)/2$. Hence $3^{p-1} + 1 \mid |H|$. Again there exists a primitive prime divisor r of $3^{2(p-1)} - 1$. Considering the order of a Sylow r-subgroup of H, we come to a contradiction by the same reasoning as in Subcase 1.1.

From Subcases 1.1 and 1.2 and Lemma 3 we deduce that G has the following properties:

(1) There is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group, H and G/K are π_1-groups and H is nilpotent.
(2) The odd order component of G is one of those of K/H, consequently $t(K/H) \geq 2$. Hence K/H may be one of the simple groups listed in Tables 1–4.

Case 2. We prove that K/H is not isomorphic to $E_7(2)$, $E_7(3)$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$ or one of the sporadic simple groups.

This follows from the fact that no group mentioned above has an order component of the form $(3^p - 1)/2$, for any prime $p \geq 5$, which is one of the odd order components of M.

Case 3. We prove that $K/H \ncong A_n$.

Indeed, otherwise A_n has an odd component equal to $(3^p - 1)/2$. Thus $|A_{(3^p-1)/2}| \mid |A_n| \mid |D_p(3)|$. By Lemma 6, there exist at least six primes p_i satisfying $(3^p + 1)/4 < p_i < (3^p - 1)/2$ since $p \geq 5$. But there exists at most one prime divisor p' of $|D_{p+1}(3)|$ satisfying $(3^p + 1)/4 < p' < (3^p - 1)/2$, a contradiction to $|A_{(3^p-1)/2}| \mid |D_p(3)|$.

Case 4. We prove that $K/H \ncong A_n(q)$, ${}^2A_n(q)$, $E_6(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(q)$, ${}^2E_6(q)$.

Subcase 4.1. If $K/H \cong A_1(q)$, then $(3^p - 1)/2$ is equal to q, $q \pm 1$ or $(q \pm 1)/2$. In each of these cases, $q \leq 3^p$, hence $|K/H| < 3^{3p}$. Write $q = r^f$. Then $|G/K| < 3^p$ for $2^{3^p/2} > 3^p$ and $|G/K| \mid |\mathrm{Out}(K/H)| = 2f$ since $|\mathrm{Out}(A_1(r^f))| = 2f$.

If $p \geq 14$ then there exists a Hall factor g of $|G| = 3^{p(p+1)}(3^p - 1) \cdot \prod_{i=1}^{p-1}(3^{2i} - 1)$ satisfying $g > 3^{4p}$. Moreover, for any prime $r' \mid g$ we have $g_{r'} < (3^p - 1)/2$ by Lemma 8. Clearly $(g, |H|) \neq 1$. Now let p' be a prime such that $p' \mid (g, |H|)$ and let $S_{p'} \in \mathrm{Syl}_{p'}(G)$. Then $S_{p'}$ is a normal π_1-subgroup of G and $|S_{p'}| < (3^p - 1)/2$, which contradicts Lemma 1.

By a trivial calculation we can show that p' cannot be 5, 7, 11 or 13.

Subcase 4.2. If $K/H \cong A_{p'}(q)$ $(q - 1 \mid p' - 1)$, then we have $(3^p - 1)/2 = (q^{p'} - 1)/(q - 1)$. Obviously $q \neq 2$, and hence $q^{p'} \geq 3^p$.

If $p' \geq 5$, one has $q^{p'(p'+1)/2} \geq 3^{3p}$; then Lemma 7 implies that q is a power of 3. But $3^p - 3 = 2(q^{p'-1} + q^{p'-2} + \cdots + q)$. Thus $q = 3$ and $p = p'$. Therefore $|K/H| = 3^{p(p+1)/2} \prod_{i=1}^{p}(3^{i+1} - 1)$. On the other hand, since $|G/K| \mid |\mathrm{Out}(K/H)| = |\mathrm{Out}(A_p(3))| = 4$, we have $3^{p-1}+1 \mid |H|$. Take a primitive prime r of $3^{2(p-1)}-1$, and let $S_r \in \mathrm{Syl}_r(H)$. By Lemma 2, we have $S_r \trianglelefteq G$. Once again Lemma 1 yields $|S_r| \equiv 1 \pmod{(3^p-1)/2}$. But $|S_r| \mid 3^{p-1}+1$, a contradiction.

By a trivial calculation we can show that p' cannot be 2 or 3.

SUBCASE 4.3. If $K/H \cong A_{p'-1}(q)$, then $(q^{p'}-1)/((q-1)(p',q-1)) = (3^p-1)/2$. Here $q \neq 2$, and so $q^{p'} \geq 3^p$.

If $p' \geq 7$, then $q^{p'(p'-1)/2} \geq 3^{3p}$, hence q is a power of 3 by Lemma 7. Since $(q^{p'}-1)/((q-1)(p',q-1)) = (3^p-1)/2$, one has $q^3 \mid 3^p$. Thus $(q-1)(p',q-1) \equiv 2 \pmod{q^3}$. Therefore $q = 3$ and $p = p'$, which means that $3^{p-1}+1 \mid |H|$. Take a primitive prime r of $3^{2(p-1)}-1$, and let $S_r \in \mathrm{Syl}_r(H)$. By Lemma 2 we find that $S_r \trianglelefteq G$, and $|S_r| \equiv 1 \pmod{(3^p-1)/2}$ by Lemma 1. But $|S_r| \mid 3^{p-1}+1$, a contradiction.

By a trivial calculation we can show that p' cannot be 2, 3 or 5.

SUBCASE 4.4. If $K/H \cong {}^2A_{p'-1}(q)$, then we have $(3^p-1)/2 = (q^{p'}+1)/((q+1)(p',q+1))$, and $q^{p'} > 3^p$.

If $p' \geq 7$, we have $q^{p'(p'-1)/2} > 3^{3p}$. Hence q is a power of 3 by Lemma 7. Since $(q^{p'}+1)/((q+1)(p',q+1)) = (3^p-1)/2$, one has $q^3 \mid 3^p$, and hence $(q+1)(p',q+1)+2 \equiv 0 \pmod{q^3}$, a contradiction.

By a trivial calculation we can show that p' cannot 2, 3 or 5.

SUBCASE 4.5. If $K/H \cong {}^2A_{p'}(q)$, then $(3^p-1)/2 = (q^{p'}+1)/(q+1)$, and $q^{p'} > 3^p$.

If $p' \geq 5$, then $q^{p'(p'+1)/2} > 3^{3p}$. So q is a power of 3 by Lemma 7. Since $(q^{p'}-1)/((q-1)(p',q-1)) = (3^p-1)/2$, we have $q^3 \mid 3^p$, and hence $q+3 \equiv 0 \pmod{q^3}$, a contradiction.

If p' is 2, 3 or 5, we get a contradiction by a trivial calculation.

SUBCASE 4.6. If $K/H \cong E_6(q)$, then $(q^6+q^3+1)/(3,q-1) = (3^p-1)/2$. Hence $q^9 > 3^p$ and $q^{36} > 3^{4(p-1)}$. Therefore q is a power of 3 by Lemma 7, and so $2(q^6+q^3) = 3^p-3$, a contradiction.

If $K/H \cong {}^2E_6(q)$, then $(q^6-q^3+1)/(3,q-1) = (3^p-1)/2$, hence $q^9 > 3^p$, so $q^{36} > 3^{4(p-1)}$. Therefore q is a power of 3 by Lemma 7, and so $2(q^6-q^3) = 3^p-3$, a contradiction.

If $K/H \cong F_4(q)$, then $q^4-q^2+1 = (3^p-1)/2$. Hence $q^4 > 3^{p-1}$ and $q^{24} > 3^{3(p-1)}$. Therefore q is a power of 3 by Lemma 7 and so $2(q^4-q^2) = 3^p-3$, a contradiction.

If $K/H \cong {}^2F_4(q)$, $q = 2^{2m+1}$ $(m \geq 1)$, then $q^2 \pm \sqrt{2q^2}+q \pm \sqrt{2q}+1 = (3^p-1)/2$. Hence $2q^2 > (3^p-1)/2$, as $q^2 > 3^{p-2}$. Therefore $q^{12} > 3^{4(p-1)}$ since $p \geq 5$, and so q is a power of 3 by Lemma 7, a contradiction.

If $K/H \cong E_8(q)$, then $(3^p-1)/2$ equals $q^8+q^7-q^5-q^4-q^3+q+1$, or $q^8-q^7+q^5-q^4+q^3-q+1$, or q^8-q^4+1, or $q^8-q^6+q^4-q^2+1$.

Hence $q^{10} > 3^p$, so $q^{120} > 3^{4p}$. Therefore q is a power of 3 by Lemma 7, a contradiction.

Case 5. We prove that $K/H \ncong B_n(q), C_n(q)$.

Subcase 5.1. If $K/H \cong C_{p'}(2)$, then $2^{p'} - 1 = (3^p - 1)/2$, so $2^{p'+1} - 1 = 3^p$, which contradicts Lemma 5.

Subcase 5.2. If $K/H \cong B_{p'}(3)$ or $C_{p'}(3)$, then $(3^p - 1)/2 = (3^{p'} - 1)/2$, thus $p = p'$, hence $|B_p(3)|$ and $|C_p(3)|$ are divisors of $|D_p(3)|$, which is impossible.

Subcase 5.3. If $K/H \cong B_n(q)$ or $C_n(q)$ (q odd, $4 \leq n = 2^m$), then $(q^n + 1)/2 = (3^p - 1)/2$, and furthermore $q^{n+1} > 3^p$. Noticing that $n \geq 4$, one sees that $q^{n^2} > 3^{3p}$. By Lemma 7, q is a power of 3, which contradicts $q^n + 1 = 3^p - 1$.

Case 6. We prove that $K/H \ncong {}^2D_n(q)$.

Subcase 6.1. If $K/H \cong {}^2D_{p'}(3)$ $(5 \leq p' \neq 2^k + 1)$, then $(3^p - 1)/2 = (3^{p'} + 1)/4$, so that $2 \cdot 3^p - 3^{p'} = 3$, a contradiction.

If $K/H \cong {}^2D_n(3)$, where $9 \leq n = 2^k + 1$ is not a prime, then $(3^p - 1)/2 = (3^{n-1} + 1)/2$, and we have $2 \cdot (3^p - 3^{n-1}) = 3$, which is impossible.

If $K/H \cong {}^2D_{p'}(3)$, where $5 \leq p' = 2^k + 1$, then $(3^p - 1)/2 = (3^{p'} + 1)/4$, hence $4 \cdot 3^p - 2 \cdot 3^{p'} = 6$, which is impossible.

Subcase 6.2. If $K/H \cong {}^2D_{p'+1}(2)$ $(p' \neq 2^m - 1)$, then $(3^p - 1)/2 = 2^{p'} - 1$, $3^p = 2^{p'+1} - 1$, which contradicts Lemma 5.

If $K/H \cong {}^2D_{p'+1}(2)$ $(3 \leq p' = 2^k - 1)$, then $(3^p - 1)/2 = 2^{p'} - 1$, hence $2^{p'+1} - 3^p = 1$, a contradiction to Lemma 5.

Subcase 6.3. If $K/H \cong {}^2D_n(q)$ $(2 \leq n = 2^k)$, then we have $(3^p - 1)/2 = (q^n + 1)(2, q - 1)$, hence $q^{n+1} > 3^p$.

If $n \geq 8$, we have $q^{n(n-1)} > 3^{3p}$. By Lemma 7, it follows that q is a power of 3, which contradicts $(3^p - 1)/2 = (q^n + 1)(2, q - 1)$.

Case 7. We prove that $K/H \ncong G_2(q)$, ${}^2G_2(q)$ $(q = 3^{2k+1})$, ${}^3D_4(q)$.

Subcase 7.1. If $K/H \cong G_2(q)$, then $(3^p - 1)/2 = q^2 \pm q + 1$.

If $3 \mid q$, then $q = 3$ by the above equations, but this is impossible.

If $q \not\equiv 1 \pmod 3$, then $q^6 > 3^{3(p-1)}$. Because q^6 divides $|G_2(q)|$, by Lemma 8, we have $3 \mid q$. For the same reason as above, we get a contradiction.

If $K/H \cong {}^2G_2(q)$ $(q = 3^{2k+1})$, then $(3^p - 1)/2 = q \pm \sqrt{3q} + 1$. It follows that $3^p - 3 = 2 \cdot (q \pm \sqrt{3q})$, a contradiction obviously.

Subcase 7.2. If $K/H \cong {}^3D_4(q)$, then $q^4 - q^2 + 1 = (3^p - 1)/2$, hence $q^4 > 3^{p-1}$, so $q^{12} > 3^{3(p-1)}$. Therefore q is a power of 3 by Lemma 7. Further $2(q^4 - q^2) = 3^p - 3$, a contradiction.

Case 8. We prove that $K/H \ncong {}^2B_2(q)$ $(4 \leq q = 2^{2k+1})$.

Indeed, otherwise $(3^p-1)/2 = q \pm \sqrt{2q}+1$ or $q-1$. Clearly $(3^p-1)/2 \neq q-1$.

If $(3^p-1)/2 = q \pm \sqrt{2q}+1$, then $3(3^{p-1}-1) = 2^{k+2}(2^k \pm 1)$. Hence $2^k \mid p-1$ by Lemma 7. Furthermore, for $k \geq 4$, $2^{k+2}(2^k \pm 1) = 3(3^{p-1}-1) > 3^{p-1} > 3^{2^k} > 2^{2^k} > 2^{2k+3} > 2^{k+2}(2^k \pm 1)$, a contradiction. By calculation we can prove that k cannot be 1, 2 or 3.

Case 9. Concluding the proof.

From Cases 1–8 and Lemma 3 we deduce that K/H is isomorphic to one of $D_n(q)$.

If $K/H \cong D_{p'+1}(3)$ $(p' \geq 3)$, then $(3^p-1)/2 = (3^{p'}-1)/2$, and hence $p = p'$. Therefore $|D_{p+1}(3)| \mid |D_p(3)|$, a contradiction.

If $K/H \cong D_{p'}(5)$ $(p' \geq 5)$, then $(5^{p'}-1)/4 = (3^p-1)/2$, and $5^{p'} > 3^p$. Therefore $5^{p'(p'-1)} > 3^{3(p+1)}$, which contradicts Lemma 7.

So $K/H \cong D_{p'}(3)$ and $(3^{p'}-1)/2 = (3^p-1)/2$. It is easy to see that $p = p'$, and hence $K/H \cong D_p(3)$. Therefore $H = 1$ and thus $G \cong M$. ∎

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