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# STABILITY RESULTS FOR ROTATIONALLY INVARIANT CONSTANT MEAN CURVATURE SURFACES IN HYPERBOLIC SPACE

BY

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**Abstract.** We prove the existence of many constant mean curvature surfaces of revolution with two ends which are immersed or embedded in hyperbolic space. We also study their stability.

**1. Introduction.** All constant mean curvature surfaces of revolution in $\mathbb{R}^3$ were classified in [D]. In particular, Delaunay discovered a beautiful one-parameter family of complete noncompact surfaces of constant mean curvature one, now called the *Delaunay surfaces*. The elements of this family which are embedded are called *unduloids*; all other elements, which correspond to negative Delaunay parameters, are immersed and are called *nodoids*.

The unduloids are stable in the sense that their global constant mean curvature deformations all lead to other elements of this Delaunay family. The same property is shared by nodoids only when the Delaunay parameter is sufficiently close to zero. On the other hand, in [MP1] it is shown that as the Delaunay parameter decreases to $-\infty$, infinitely many new families of complete, cylindrically bounded constant mean curvature surfaces bifurcate from this Delaunay family. The surfaces in these branches have only a discrete symmetry group.

In 1981, Hsiang and Yu [HY] generalized Delaunay's classification to the hypersurfaces of revolution in higher dimensions which have constant mean curvature equal to 1 and some symmetries. Specifically in $\mathbb{R}^{n+1}$, for $n > 3$, there exist two families of complete constant mean curvature hypersurfaces of revolution. The elements of the first family are embedded in $\mathbb{R}^{n+1}$ while the second family constitutes a one-parameter family of immersed hypersurfaces.

Most of the results known for Euclidean spaces can be generalized to other space forms. In particular, a representation formula for constant mean

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curvature surfaces in the hyperbolic 3-space $\mathbb{H}^3$ has been discovered by Bryant [B]. This shows that, to some extent, constant mean curvature surfaces in $\mathbb{H}^3$ behave like minimal surfaces in $\mathbb{R}^3$. This is not surprising in view of the local isometry between minimal surfaces in $\mathbb{R}^3$ and constant mean curvature surfaces in $\mathbb{H}^3$ discovered by Lawson [L]. Beside the well-known horospheres, which have one end, there exists a one-parameter family of constant mean curvature surfaces of revolution with two ends which are commonly known as "catenoid cousins" [B], referring to the fact that they are related to catenoids through Lawson's correspondence.

In this paper, we initiate the study of constant mean curvature surfaces with two ends in the hyperbolic space $\mathbb{H}^3$. As in the Euclidean case, using an argument based on solving a second order ordinary differential equation one easily shows that there exists a one-parameter family of embedded constant mean curvature surfaces of revolution in $\mathbb{H}^3$. However, the other one-parameter family of immersed not embedded constant mean curvature surfaces of revolution has to be constructed. Specifically, we prove

MAIN THEOREM 1.1. *Let $H > 1$. There exists $\tau_H > 0$ and a one-parameter family of surfaces of revolution in the hyperbolic 3-space $\mathbb{H}^3$ denoted by $\mathcal{D}_\tau$ for $\tau \in (-\infty, 0) \cup (0, \tau_H]$ such that for $\tau \in (0, \tau_H]$, the surface $\mathcal{D}_\tau$ is embedded and has mean curvature equal to $H$. For $\tau \in (-\infty, 0)$, $\mathcal{D}_\tau$ also has mean curvature $H$ but is only immersed rather than embedded.*

The second part of the paper is devoted to proving a maximum principle for the Jacobi operator associated with a Delaunay surface. However, for the result to hold, we need to impose a lower bound on the Delaunay parameter ($\tau \in [\tau^H, 0) \cup (0, \tau_H]$), where $\tau^H$ depends only on the curvature $H$. More precisely, we prove

MAIN THEOREM 1.2. *There exists $\tau^H < 0$ such that for all $\tau \in (\tau^H, 0) \cup (0, \tau_H]$, the Jacobi operator associated with the surface $\mathcal{D}_\tau$ satisfies the maximum principle.*

**2. Constant mean curvature surfaces of revolution.** We still consider the upper-half-space model

$$\mathbb{H}^3 := \{(x_1, x_2, y) \in \mathbb{R}^3 : y > 0\},$$

endowed with the metric

$$g_{\mathrm{hyp}} := \frac{1}{y^2}(dx_1^2 + dx_2^2 + dy^2) = \frac{1}{y^2} g_{\mathrm{Eucl}},$$

and we fix the geodesic

$$\zeta(t) = (0, 0, e^t).$$

The profile curve of the surface of revolution with constant mean curvature equal to $H > 1$ in hyperbolic space is described, say in the vertical 2-

dimensional plane $\{x_1, y\}$, as a geodesic graph. The point $\xi(t)$ in the profile curve is at geodesic distance $\rho(t)$ from the point $\zeta(t)$. Let $\psi(t)$ be the angle $(\overrightarrow{O\zeta(t)}, \overrightarrow{O\xi(t)})$. Then

$$\sinh(\rho(t)) = \tan(\psi(t)).$$

Using this notation, the constant mean curvature surfaces of revolution (say around the $y$-axis) can be parameterized by

$$\begin{aligned} &X : \mathbb{R} \times S^1 \to \mathbb{H}^3, \\ &\qquad (t, \theta) \mapsto \big(e^t \sin(\psi(t)) \cos(\theta), e^t \sin(\psi(t)) \sin(\theta), e^t \cos(\psi(t))\big). \end{aligned}$$

Let us assume that the orientation of this surface is chosen so that the unit inward normal vector field is given by

$$N_{\mathrm{hyp}}(t, \theta) := \frac{\cos(\psi)}{\sqrt{1 + (\partial_t \psi)^2}} \big(-\partial_t g(t) \cos(\theta), -\partial_t g(t) \sin(\theta), \partial_t f(t)\big),$$

where

$$f(t) := e^t \sin(\psi(t)) \quad \text{and} \quad g(t) := e^t \cos(\psi(t)).$$

In the upper-half-space model, the mean curvature $H_{\mathrm{hyp}}$ of the surface parameterized by $X$, endowed with the metric induced by $g_{\mathrm{hyp}}$, can be compared with the mean curvature $H_{\mathrm{Eucl}}$ of the same surface, this time considered in $\mathbb{R}^3$, and hence endowed with the metric induced by $g_{\mathrm{Eucl}}$. This is the content of the following classical result whose proof can be found in [BE] and [PP].

THEOREM 2.1. *Let $\Sigma$ be a surface contained in the upper half-space. We denote by $z$ the height function and, if $N_{\mathrm{Eucl}}$ denotes the normal to the surface in $(\mathbb{R}^3, g_{\mathrm{Eucl}})$, we denote by $N^z_{\mathrm{Eucl}}$ the vertical coordinate of $N_{\mathrm{Eucl}}$ (the projection of $N_{\mathrm{Eucl}}$ onto the $z$-axis). Then the mean curvatures of $\Sigma$ in $(\mathbb{H}^3, g_{\mathrm{hyp}})$ and in $(\mathbb{R}^3, g_{\mathrm{Eucl}})$ are related by*

$$H_{\mathrm{hyp}} = z H_{\mathrm{Eucl}} + N^z_{\mathrm{Eucl}}.$$

It is easy to see that the first fundamental form, the normal vector and the second fundamental form of the surface parameterized by $X$ (endowed with the metric induced by $g_{\mathrm{Eucl}}$) are respectively given by:

$$\begin{aligned} I &= e^{2t}\big((1 + (\partial_t \psi)^2) dt \otimes dt + \sin^2(\psi) d\theta \otimes d\theta\big), \\ N_{\mathrm{Eucl}} &= \frac{e^{-t}}{\sqrt{1 + (\partial_t \psi)^2}} \big(-\partial_t g(t) \cos(\theta), -\partial_t g(t) \sin(\theta), \partial_t f(t)\big), \\ II &= \frac{e^{t}}{\sqrt{1 + (\partial_t \psi)^2}} \big((\partial_t^2 \psi + (\partial_t \psi)^3 + \partial_t \psi) dt \otimes dt \\ &\qquad\qquad - \sin(\psi)(\cos(\psi) - \partial_t \psi \sin(\psi)) d\theta \otimes d\theta\big). \end{aligned}$$

It follows at once from the above expressions that the mean curvature $H_{\mathrm{Eucl}}$ of the surface parameterized by $X$ satisfies

$$2H_{\mathrm{Eucl}} = \frac{-e^{-t}}{(1+(\partial_t\psi)^2)^{3/2}}(\partial_t^2\psi + (\partial_t\psi)^3 + \partial_t\psi) + \frac{e^{-t}(\cos(\psi) - \partial_t\psi\sin(\psi))}{\sin(\psi)(1+(\partial_t\psi)^2)^{1/2}}.$$

Finally, due to the last equation and Theorem 2.1, the mean curvature of the surface parameterized by $X$ is equal to the constant $H_{\mathrm{hyp}} > 1$ if and only if the scalar function $\psi$ solves

$$\partial_t^2\psi - \frac{1+\sin^2(\psi)}{\sin(\psi)\cos(\psi)}(1+(\partial_t\psi)^2) + \frac{2H_{\mathrm{hyp}}}{\cos(\psi)}(1+(\partial_t\psi)^2)^{3/2} = 0, \tag{2.1}$$

which is equivalent to the following Hamiltonian being constant:

$$\mathcal{H}(\psi, \partial_t\psi) := \frac{\tan(\psi)}{\cos(\psi)\sqrt{1+(\partial_t\psi)^2}} - H_{\mathrm{hyp}}\tan^2(\psi). \tag{2.2}$$

In the rest of paper we write $H$ for $H_{\mathrm{hyp}}$.

**2.1. Embedded constant mean curvature surfaces.** It is easy to show that there exists a one-parameter family of embedded constant mean curvature surfaces of revolution in hyperbolic space. Indeed, there are two special solutions of (2.1) which can be immediately determined.

For $H > 1$, we denote by $c_H$ the unique scalar in $(0, \pi/2)$ such that

$$\sin(c_H) = H - \sqrt{H^2 - 1}.$$

Then, the first solution is the constant solution $\psi \equiv c_H$ which corresponds to the cone. The other explicit solution, corresponding to the horosphere, is given by

$$\psi(t) = \arccos\left(\frac{\sqrt{H^2-1}}{H}\sinh(t)\right) \quad \text{for} \quad t \in \left(0, \sinh^{-1}\left(\frac{\pi H}{2\sqrt{H^2-1}}\right)\right).$$

These two solutions are the two end-points of a one-parameter family of solutions of (2.2) which produce embedded constant mean curvature surfaces.

To describe this family it will be convenient to define a positive real $\tau_H$ such that

$$2\tau_H^2 = \frac{1}{H + \sqrt{H^2-1}}.$$

Given $\tau \in (0, \tau_H]$, we define $\varepsilon \in (0, c_H]$ by the identity

$$\varepsilon\sqrt{1+\varepsilon^2} - H\varepsilon^2 = \tau^2$$

and we define the function $\psi_\tau$ to be the unique solution of (2.1), with $H > 1$, such that

$$\tan(\psi_\tau(0)) = \varepsilon \quad \text{and} \quad \partial_t\psi_\tau(0) = 0.$$

Then

$$\mathcal{H}(\psi_\tau, \partial_t\psi_\tau) = \tau^2$$

is constant along solutions of (2.1). Since the curve $(\psi_\tau, \partial_t \psi_\tau)$ is closed, it is easy to show that the function $t \mapsto \psi_\tau(t)$ is periodic with period $T_\tau$ given by

$$T_\tau := 2 \int_{\eta_1}^{\eta_2} \frac{\tau^2 + H\eta^2}{1+\eta^2} \frac{d\eta}{f(\eta)},$$

where $\eta_1$ and $\eta_2$ are the positive roots of

$$f(\eta) := \left((1-H^2)\eta^4 + (1-2H\tau^2)\eta^2 - \tau^4\right)^{1/2}.$$

We will denote by $\mathcal{D}_\tau$ the surface of revolution parameterized by

$$X_\tau(t,\theta) = \left(e^t \sin(\psi_\tau(t)) \cos(\theta), e^t \sin(\psi_\tau(t)) \sin(\theta), e^t \cos(\psi_\tau(t))\right).$$

By construction, $\mathcal{D}_\tau$ is an embedded constant mean curvature surface of revolution, which will be called an *unduloid*.

**2.2. Immersed constant mean curvature hypersurfaces.** Given $\tau \in (-\infty, 0)$, we define $\varepsilon > 0$ by the identity

$$\varepsilon\sqrt{1+\varepsilon^2} + H\varepsilon^2 = \tau^2$$

and we define the function $\bar{\psi}_\tau$ to be the unique solution of (2.1), with $H$ replaced by $-H$, such that $\bar{\psi}_\tau(0) = \tan^{-1}(\varepsilon)$ and $\partial_t \bar{\psi}_\tau(0) = 0$. Using the fact that

$$\mathcal{H}(\bar{\psi}_\tau, \partial_t \bar{\psi}_\tau) = \tau^2$$

is constant along solutions of (2.1), it is a simple exercise to show that $\bar{\psi}_\tau$ is defined over the maximal interval $(-\bar{T}_\tau, \bar{T}_\tau)$, where

$$\bar{T}_\tau := 2 \int_{\eta_1}^{\eta_2} \frac{\tau^2 - H\eta^2}{1+\eta^2} \frac{d\eta}{g(\eta)},$$

with $\eta_1$ and $\eta_2$ denoting the positive roots of

$$g(\eta) := \left((1-H^2)\eta^4 + (1+2H\tau^2)\eta^2 - \tau^4\right)^{1/2}.$$

Furthermore, we have

$$\lim_{t \to \pm \bar{T}_\tau} \bar{\psi}_\tau = -\tan^{-1}\left(\frac{\tau}{\sqrt{H}}\right) \quad \text{and} \quad \lim_{t \to \pm \bar{T}_\tau} \partial_t \bar{\psi}_\tau = \pm\infty.$$

Now, for the same value of $\tau$, we define $\tilde{\varepsilon} \in (1, +\infty)$ by the identity

$$\tilde{\varepsilon}\sqrt{1+\tilde{\varepsilon}^2} - H\tilde{\varepsilon}^2 = -\tau^2.$$

This being done, we define the function $\tilde{\psi}_\tau$ to be the unique solution of (2.1), with $H > 1$ such that $\tilde{\psi}_\tau(0) = \tan^{-1}(\tilde{\varepsilon})$ and $\partial_t \tilde{\psi}_\tau(0) = 0$. Using the fact that

$$\mathcal{H}(\tilde{\psi}_\tau, \partial_t \tilde{\psi}_\tau) = -\tau^2$$

is constant along solutions of (2.1), as above there exists $\tilde{\psi}_\tau$ which is defined in $(-\tilde{T}_\tau, \tilde{T}_\tau)$, where

$$\tilde{T}_\tau := 2\int_{\eta_1}^{\eta_2} \frac{H\eta^2 - \tau^2}{1+\eta^2}\,\frac{d\eta}{h(\eta)},$$

with $\eta_1$ and $\eta_2$ denoting the positive roots of

$$h(\eta) := \big((1-H^2)\eta^4 + (1+2H\tau^2)\eta^2 - \tau^4\big)^{1/2}.$$

Furthermore, we have

$$\lim_{t\to\pm\tilde{T}_\tau} \tilde{\psi}_\tau = -\tan^{-1}\left(\frac{\tau}{\sqrt{H}}\right) \quad \text{and} \quad \lim_{t\to\pm\tilde{T}_\tau} \partial_t\tilde{\psi}_\tau = \mp\infty.$$

Finally, the graph of the function $\bar{\psi}_\tau$ and the graph of the function $\tilde{\psi}_\tau$ defined above (once translated by $\tilde{T}_\tau - \bar{T}_\tau$) can be glued together to produce a piece of constant mean curvature surface of revolution. Now we can extend this piece of surface by periodicity to produce a complete immersed constant mean curvature surface in $\mathbb{H}^3$. These surfaces will be referred to as *nodoids*.

**3. Isothermal parameterization.** The previous parameterization can probably be used to investigate the geometric properties of the surfaces. However, in our analysis, it will be more interesting to consider an isothermal type parameterization which obscures the geometric properties of the surfaces, but is more convenient for analytical purposes.

Thus, we are now looking for surfaces of revolution which can be parameterized by

$$X(s,\theta) = \big(\varphi(s)\kappa(s)\cos(\theta), \varphi(s)\kappa(s)\sin(\theta), \kappa(s)\big)$$

for $(s,\theta) \in \mathbb{R}\times S^1$. The constant $\tau$ being fixed, the functions $\varphi$ and $\kappa$ are determined by requiring that the surface parameterized by $X$ have constant mean curvature equal to $H$ and the metric associated with the parameterization be conformal to the product metric on $\mathbb{R}\times S^1$, namely

$$(\partial_s(\varphi\kappa))^2 + (\partial_s\kappa)^2 = \varphi^2\kappa^2. \tag{3.1}$$

This time, the first fundamental form $g$ of the surface parameterized by $X$ in the hyperbolic space is given by

$$g_{\text{hyp}} = \varphi^2(ds\otimes ds + d\theta\otimes d\theta).$$

Since

$$\varphi(s) = \tan(\psi(t)) \quad \text{and} \quad e^{2t} = \kappa^2(s)(1+\varphi^2(s)), \tag{3.2}$$

it is easy to deduce that

$$(1+(\partial_t\psi)^2)(\varphi\partial_s\varphi + (1+\varphi^2)\kappa^{-1}\partial_s\kappa)^2 = (1+\varphi^2)\varphi^2,$$

and

$$\partial_t^2\psi(\varphi\partial_s\varphi+(1+\varphi^2)\kappa^{-1}\partial_s\kappa)^4=(1+\varphi^2)(\varphi^2\partial_s^2\varphi\,(1+\varphi^2)-\varphi(\partial_s\varphi)^2(1+2\varphi^2)).$$

Due to (3.1), (3.2) and the last two equations, (2.1) can be rewritten as

$$\partial_s^2\varphi-(1+2\varphi^2)\varphi+2H(\varphi^2\partial_s\varphi+(1+\varphi^2)\varphi\kappa^{-1}\partial_s\kappa)=0. \tag{3.3}$$

Since the function $\varphi$ is positive, we have

$$\varphi(s):=|\tau|e^{\sigma(s)}$$

for some scalar function $s\mapsto\sigma(s)$. Then (3.1) becomes

$$(1+\tau^2e^{2\sigma})(\kappa^{-1}\partial_s\kappa)^2+2\tau^2e^{2\sigma}\partial_s\sigma(\kappa^{-1}\partial_s\kappa)+\tau^2e^{2\sigma}((\partial_s\sigma)^2-1)=0, \tag{3.4}$$

and (3.3) becomes

$$\partial_s^2\sigma-(1+2\tau^2e^{2\sigma}-(\partial_s\sigma)^2)+2H(\tau^2e^{2\sigma}\partial_s\sigma+(1+\tau^2e^{2\sigma})\kappa^{-1}\partial_s\kappa)=0. \tag{3.5}$$

Thus, in order to find constant mean curvature surfaces of revolution, we have to solve (3.4) together with (3.5). This is the content of the next subsection.

**3.1. The unduloids.** Recall that we have defined the positive real number $\tau_H$ by

$$2\tau_H^2:=\frac{1}{H+\sqrt{H^2-1}}.$$

For all $\tau\in(0,\tau_H]$, we define $\sigma_\tau$ to be the unique smooth nonconstant solution of

$$(\partial_s\sigma)^2+\tau^2((He^{\sigma}+e^{-\sigma})^2-e^{2\sigma})=1, \tag{3.6}$$

with the initial conditions $\partial_s\sigma(0)=0$ and $\sigma(0)<0$. Next we define the function $\kappa_\tau$ to be the unique solution of

$$\kappa^{-1}\partial_s\kappa=(H-\partial_s\sigma+e^{-2\sigma})\frac{\tau^2e^{2\sigma}}{1+\tau^2e^{2\sigma}}. \tag{3.7}$$

It is easy to check that $\sigma_\tau$ and $\kappa_\tau$ satisfy (3.4) and (3.5). Moreover, the function $\kappa_\tau$ is increasing since $\partial_s\kappa_\tau>0$.

In particular, the surface parameterized by

$$X_\tau(s,\theta):=\big(\tau e^{\sigma_\tau(s)}\kappa_\tau(s)\cos(\theta),\tau e^{\sigma_\tau(s)}\kappa_\tau(s)\sin(\theta),\kappa_\tau(s)\big),$$

for $(s,\theta)\in\mathbb{R}\times S^1$, is an embedded constant mean curvature surface of revolution. It coincides with the surface defined in §2.1 and, as already mentioned, this surface will be referred to as the unduloid of parameter $\tau$.

Observe that the extreme element in this family which corresponds to $\tau=\tau_H$ is a cone, while, as $\tau$ tends to 0, the family of unduloids converges to a sequence of spheres arranged along the $y$-axis.

**3.2. The nodoids.** For all $\tau < 0$, we define $\sigma_\tau$ to be the unique smooth nonconstant solution of

$$(\partial_s \sigma)^2 + \tau^2((He^{\sigma} - e^{-\sigma})^2 - e^{2\sigma}) = 1, \tag{3.8}$$

with the initial conditions $\partial_s \sigma(0) = 0$ and $\sigma(0) < 0$. Next, we define the function $\kappa_\tau$ to be the unique solution of

$$\kappa^{-1}\partial_s \kappa = (H - \partial_s \sigma - e^{-2\sigma})\frac{\tau^2 e^{2\sigma}}{1 + \tau^2 e^{2\sigma}}. \tag{3.9}$$

Again, it is easy to check that $\sigma_\tau$ and $\kappa_\tau$ satisfy (3.4) and (3.5). However, this time the function $\kappa_\tau$ is not monotone anymore since $\partial\kappa_\tau$ changes sign.

Hence, the surface parameterized by

$$X_\tau(s,\theta) := \big(-\tau e^{\sigma_\tau(s)}\kappa_\tau(s)\cos(\theta), -\tau e^{\sigma_\tau(s)}\kappa_\tau(s)\sin(\theta), \kappa_\tau(s)\big)$$

for $(s,\theta) \in \mathbb{R} \times S^1$, is an immersed constant mean curvature surface of revolution. This surface coincides with the surface defined in §2.2 and will be referred to as the nodoid of parameter $\tau$.

**4. The Jacobi operator.** In this section, we define and study the Jacobi operator associated with a Delaunay hypersurface in the hyperbolic space. Recall that this surface can be parameterized as

$$\mathbb{R} \times S^1 \ni (s,\theta) \mapsto X(s,\theta) := \big(\varphi(s)\kappa(s)\cos(\theta), \varphi(s)\kappa(s)\sin(\theta), \kappa(s)\big).$$

Assume that the orientation of this surface is chosen so that the unit normal vector field is given by

$$N_\tau := \varphi^{-1}\big(-\partial_s\kappa\cos(\theta), -\partial_s\kappa\sin(\theta), \partial_s(\varphi\kappa)\big). \tag{4.1}$$

Any surface close enough to $\mathcal{D}_\tau$ can be parameterized (at least locally) as a normal graph over $\mathcal{D}_\tau$, that is, by

$$X_\omega = X_\tau + \omega N_\tau$$

for some (small) smooth function $\omega$. The surface parameterized by $X_\omega$ will be denoted by $\mathcal{D}_\tau(\omega)$ and we define the mean curvature operator $H(\omega)$ to be the mean curvature of $\mathcal{D}_\tau(\omega)$.

It is well known that the linearized mean curvature operator associated with $\mathcal{D}_\tau$, which is usually referred to as the *Jacobi operator*, is given by

$$\mathcal{L}_\tau := \Delta_\tau + |A_\tau|^2 + \mathrm{Ric}_{\mathbb{H}^3}(N_\tau, N_\tau).$$

where $\Delta_\tau$ is the Laplace–Beltrami operator, $|A_\tau|^2$ is the square of the norm of the second fundamental form of $\mathcal{D}_\tau$, and $\mathrm{Ric}_{\mathbb{H}^3}$ is the Ricci tensor of $(\mathbb{H}^3, g_{\mathrm{hyp}})$.

A simple computation (see [BLR] and [KKMS]) shows that

$$\mathcal{L}_\tau := A^{-1}B^{-1}\partial_t(A^{-1}B\partial_t) + B^{-2}\partial_\theta^2 + 2(H^2 - 1) + 2\tau^4 B^{-4}, \tag{4.2}$$

with

$$A := (1 + (\partial_t \psi)^2)^{1/2}(1 + \tan^2(\psi))^{1/2} \quad \text{and} \quad B := \tan(\psi).$$

Using the isothermal parameterization, we express the Jacobi operator explicitly in terms of the function $\sigma$ by

$$\tau^2 e^{2\sigma} \mathcal{L}_\tau = \partial_s^2 + \partial_\theta^2 + 2\tau^2((H^2 - 1)e^{2\sigma} + e^{-2\sigma}). \tag{4.3}$$

**4.1. Geometric Jacobi fields.** Some *Jacobi fields*, i.e., solutions of the homogeneous problem

$$\mathcal{L}_\tau \omega = 0,$$

can be explicitly computed when they correspond to a smooth one-parameter family of constant mean curvature surfaces $\mathcal{C}_\lambda$, for $\lambda \in (-1, 1)$ to which $\mathcal{D}_\tau$ belongs, for example $\mathcal{C}_0 = \mathcal{D}_\tau$. For $\lambda$ small enough, the surface $\mathcal{C}_\lambda$ can be written (at least locally) as a normal graph over $\mathcal{D}_\tau$, for some function $\omega_\lambda$. Differentiation with respect to $\lambda$, at $\lambda = 0$, yields a Jacobi field.

In the special case where the one-parameter family of constant mean curvature surfaces is given by the action of a one-parameter family of rigid motions, the corresponding Jacobi field can be obtained by projecting, onto the normal bundle of $\mathcal{D}_\tau$, the Killing vector field associated to the one-parameter family of rigid motions under consideration. These Killing vector fields arise from the isometries of $(\mathbb{H}^3, g_{\mathrm{hyp}})$ and Liouville's theorem shows that these isometries are the restrictions to $\mathbb{H}^3$ of conformal transformations of $\mathbb{R}^3$ that take $\mathbb{H}^3$ onto itself. More details are given in [PP]. We now describe these Jacobi fields as well as another independent Jacobi field which arises by varying the Delaunay parameter $\tau$.

- *Translation along the axis of revolution:* To begin, for $\tau \in (-\infty, 0) \cup (0, \tau_H)$, we define $\Phi_\tau^{0,+}$ to be the Jacobi field corresponding to the translation of $\mathcal{D}_\tau$ along its axis of revolution. As explained above, this Jacobi field can be obtained by projecting, onto the normal bundle, the Killing field

$$\chi_1 := (\varphi\kappa \cos(\theta), \varphi\kappa \sin(\theta), \kappa).$$

Then

$$\Phi_\tau^{0,+} := \langle N_\tau, \chi_1 \rangle_{\mathbb{H}^3} = \kappa^{-2} \langle N_\tau, \chi_1 \rangle_{\mathbb{R}^3} = \varphi^{-1} \partial_s \varphi \tag{4.4}$$

is a solution of $\mathcal{L}_\tau \omega = 0$. It is easy to check that $\Phi_\tau^{0,+}$ only depends on $s$ and is periodic. In particular, this implies that this Jacobi field is bounded.

- *Translations orthogonal to the axis of revolution:* Let

$$\chi_2 := (a, 0) \in \mathbb{R}^2 \times \mathbb{R}$$

denote the constant Killing field which corresponds to the translations in the $a$ direction in $\mathbb{R}^2$. Then we have

$$\langle N_\tau, \chi_2 \rangle_{\mathbb{H}^3} = -\varphi^{-1} \kappa^{-2} \partial_s \kappa a \cdot (\cos(\theta), \sin(\theta)).$$

Taking $a$ to be any vector of the canonical basis of $\mathbb{R}^2$, we get

$$\Phi_\tau^{\pm 1,+} := \varphi^{-1}\kappa^{-2}\partial_s\kappa e^{\pm i\theta} \tag{4.5}$$

which are solutions of $\mathcal{L}_\tau\omega = 0$.

- *Composition of translation and two inversions:* Let

$$\chi_3 := (1+\varphi^2)\kappa^2(a,0) - 2\varphi\kappa a\cdot(\cos(\theta),\sin(\theta))\chi_1$$

denote the Killing field which corresponds to the composition of translation in the $a$ direction in $\mathbb{R}^2$ and two central inversions of the Delaunay surface at the origin. Then we have

$$\langle N_\tau,\chi_3\rangle_{\mathbb{H}^3} = -(\varphi^{-1}(1+\varphi^2)\partial_s\kappa + 2\kappa\partial_s\varphi)a\cdot(\cos(\theta),\sin(\theta)).$$

Again taking $a$ to be any vector of the canonical basis of $\mathbb{R}^2$, we get

$$\Phi_\tau^{\pm 1,-} := (\varphi^{-1}(1+\varphi^2)\partial_s\kappa + 2\kappa\partial_s\varphi)e^{\pm i\theta} \tag{4.6}$$

which are solutions of $\mathcal{L}_\tau\omega = 0$.

- *Variation of the Delaunay parameter:* Finally, the Jacobi field corresponding to varying the parameter $\tau\in(-\infty,0)\cup(0,\tau_H)$ will be denoted by $\Phi_\tau^{0,-}$. It can be obtained by writing, for $\eta$ small enough, the constant mean curvature hypersurface $\mathcal{D}_{\tau+\eta}$ as a normal graph over $\mathcal{D}_\tau$ for some function $\omega_\eta$ and differentiating $\omega_\eta$ with respect to $\eta$ at $\eta=0$. More precisely, for $\tau'$ close to $\tau$ there exists a local diffeomorphism $\Phi_{\tau'}$ such that

$$X_{\tau'}\circ\Phi_{\tau'} = X_\tau + \omega_{\tau'}N_\tau.$$

In particular, we get

$$\omega_{\tau'} = \langle X_{\tau'}\circ\Phi_{\tau'} - X_\tau, N_\tau\rangle_{\mathbb{H}^3}$$

and differentiating $\omega_{\tau'}$ with respect to $\tau'$ at $\tau'=\tau$ we get

$$\partial_{\tau'}\omega_{\tau'}|_{\tau'=\tau} = \langle\partial_\tau X_\tau + \ DX_\tau(\partial_\tau\Phi_\tau), N_\tau\rangle_{\mathbb{H}^3}.$$

Since $\langle DX_\tau(\partial_\tau\Phi_\tau),N_\tau\rangle_{\mathbb{H}^3}\equiv 0$, the corresponding Jacobi field takes the form

$$\Phi_\tau^{0,-} = \kappa^{-2}(\partial_\tau\varphi\partial_s(\varphi\kappa) - \varphi^{-1}\partial_s\kappa\partial_\tau(\varphi\kappa)). \tag{4.7}$$

**4.2. Maximum principle for the Delaunay surface.** We prove a maximum principle for $\mathcal{L}_\tau$, the Jacobi operator associated with a Delaunay surface.

PROPOSITION 4.1. *There exists $\tau^H<0$ such that for all $\tau\in(\tau^H,0)\cup(0,\tau_H]$, if $v$ is a bounded solution of $\mathcal{L}_\tau v=0$ in $(s_1,s_2)\times S^1$, with boundary data $v=0$ on $\{s_1,s_2\}\times S^1$ and if, for all $s\in(s_1,s_2)$, the function $v(s,\cdot)$ is $L^2$-orthogonal to* Span$\{1,e^{\pm i\theta}\}$ *on $S^1$, then $v=0$.*

*Proof.* We consider the eigenfunction decomposition of $v$

$$v=\sum_{j\geq 2} v_j e^{ij\theta}.$$

Multiplying the equation $\mathcal{L}_\tau v = 0$ by $\varphi_\tau^2 v_j e^{ij\theta}$ and integrating by parts over $(s_1, s_2)\times S^1$, we obtain

$$\int(\partial_s v_j)^2 + j^2\int v_j^2 = 2(H^2-1)\int\varphi_\tau^2 v_j^2 + 2\tau^4\int\varphi_\tau^{-2}v_j^2 \tag{4.8}$$

where all integrals are over $(s_1, s_2)$.

Due to (3.6), (3.8) and the fact that $\varphi_\tau = |\tau|e^{\sigma_\tau}$, we have

$$(\partial_s\varphi_\tau)^2 = \varphi_\tau^2 + \varphi_\tau^4 - (H\varphi_\tau^2 + i\tau^2)^2,$$

Multiplying this equality by $\varphi_\tau^{-2}v_j^2$ gives

$$\int\varphi_\tau^{-2}(\partial_s\varphi_\tau)^2 v_j^2 = (1-2iH\tau^2)\int v_j^2 + (1-H^2)\int\varphi_\tau^2 v_j^2 - \tau^4\int\varphi_\tau^{-2}v_j^2.$$

Adding the last equation multiplied by 2 and (4.8), we obtain

$$(j^2 - 2 + 4i\tau^2 H)\int v_j^2 + \int\varphi_\tau^{-2}(\partial_s\varphi_\tau)^2 v_j^2 + \int(\partial_s v_j)^2 = 0.$$

It is easy to see that if $\tau\in[-1/\sqrt{2H}, 0)\cup(0,\tau_H]$ and $j\ge 2$, then $v_j\equiv 0$. ■

Similar results to the last proposition can be obtained if $s_2 = +\infty$ and $v$ decays exponentially at $+\infty$.

As in [MP1] and [J1], the last stability result proves that the set of constant mean curvature surfaces in hyperbolic space is "singular". In a forthcoming paper we will use the Crandall–Rabinowitz Theorem [CR] to study the existence of new constant mean curvature surfaces which bifurcate from the family of the immersed surfaces. In particular, we will try to generalize to $\mathbb{H}^3$ the results obtained in [MP2] and [J2].

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