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MODULES AND QUIVER REPRESENTATIONS WHOSE ORBIT CLOSURES ARE HYPERSURFACES

BY

NGUYEN QUANG LOC (Hanoi) and GRZEGORZ ZWARA (Toruń)

Abstract. Let A be a finitely generated associative algebra over an algebraically closed field. We characterize the finite-dimensional A-modules whose orbit closures are local hypersurfaces. The result is reduced to an analogous characterization for orbit closures of quiver representations obtained in Section 3.

1. Introduction and the main results. Throughout the paper k denotes a fixed algebraically closed field. By an *algebra* we mean an associative finitely generated k-algebra with identity, and by a *module* a finite-dimensional left module. Let d be a positive integer and denote by $\mathbb{M}_d(k)$ the algebra of $d \times d$-matrices with entries in k. For an algebra A the set $\operatorname{mod}_A(d)$ of A-module structures on the vector space k^d, or equivalently the set of k-algebra homomorphisms from A to $\mathbb{M}_d(k)$, has a natural structure of an affine variety. Indeed, if we fix a k-algebra isomorphism $A \simeq k\langle X_1, \ldots, X_t\rangle/J$, with $t > 0$ and a two-sided ideal J, then $\operatorname{mod}_A(d)$ can be identified with the closed subset of $(\mathbb{M}_d(k))^t$ given by the vanishing of the entries of all the matrices $\rho(X_1, \ldots, X_t)$ for $\rho \in J$. Moreover, the general linear group

$$\mathrm{GL}(d) = \mathrm{GL}(d, k)$$

acts on $\operatorname{mod}_A(d)$ by conjugation, and the $\mathrm{GL}(d)$-orbits in $\operatorname{mod}_A(d)$ correspond bijectively to the isomorphism classes of d-dimensional A-modules. We denote by $\mathcal{O}_M$ the $\mathrm{GL}(d)$-orbit in $\operatorname{mod}_A(d)$ corresponding to (the isomorphism class of) a d-dimensional A-module M.

It is an interesting task to study geometric properties of the Zariski closure $\overline{\mathcal{O}}_M$ of $\mathcal{O}_M$. A general question is how they are related to the representation-theoretic properties of the algebra A and the corresponding A-modules. We notice that, using the geometric equivalence described in [4], the above problem is closely related to an analogous one for k-linear representations of quivers. We refer to [3], [13], [14] and to a survey [15] for results on singularities of orbit closures of modules or quiver representations.

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In [10], a characterization of the A-modules M for which the orbit closure $\overline{\mathcal{O}}_M$ is a non-singular variety is given. More precisely, we have the following two theorems, where the first one follows from the second.

THEOREM 1.1 ([10, Theorem 1.1]). *The orbit closure $\overline{\mathcal{O}}_M$ is a non-singular variety if and only if the algebra $B = A/\mathrm{Ann}(M)$ is hereditary and $\mathrm{Ext}^1_B(M, M) = 0$.*

By definition, the finite-dimensional algebra B is *hereditary* if the functor $\mathrm{Ext}^2_B(-,-)$ vanishes.

THEOREM 1.2 ([10, Theorem 2.1]). *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector. Let N be a representation in $\mathrm{rep}_Q(\mathbf{d})$ such that $\mathrm{Ann}(N)$ is an admissible ideal in kQ. Then $\overline{\mathcal{O}}_N$ is a non-singular variety if and only if $\mathrm{Ann}(N) = \{0\}$ and $\overline{\mathcal{O}}_N = \mathrm{rep}_Q(\mathbf{d})$.*

A (commutative, Noetherian) local ring (R, m) is called a *hypersurface* if it has the form $T/(f)$ for a regular local ring T and a non-unit $f \in T$. We say an algebraic variety $\mathcal{X}$ is a *hypersurface at a point $x \in \mathcal{X}$* if the local ring $\mathcal{O}_{\mathcal{X},x}$ is a hypersurface, and $\mathcal{X}$ a *local hypersurface* if it is a hypersurface at each of its points. Of course, non-singular varieties are local hypersurfaces. Other simple examples of local hypersurfaces are *hypersurfaces in affine spaces*, i.e., the zero sets of a non-constant polynomial.

The assumptions on $\mathrm{Ann}(N)$ in Theorem 1.2 imply that N is a *nilpotent representation*, i.e., $\{0\}$ is the unique closed orbit in $\overline{\mathcal{O}}_N$. Observe that $\overline{\mathcal{O}}_N$ is non-singular if and only if it is non-singular at the point 0, as the singular locus is a closed $\mathrm{GL}(\mathbf{d})$-invariant subset of $\overline{\mathcal{O}}_N$. It is therefore natural to ask when $\overline{\mathcal{O}}_N$ is a singular hypersurface at 0. It turns out that this is the case only when $\overline{\mathcal{O}}_N$ is a singular affine hypersurface.

For a finite-dimensional algebra B, there is a uniquely determined quiver Γ, called the *Gabriel quiver* of B, and an admissible ideal I in the path algebra $k\Gamma$ such that the categories of modules over B and over $k\Gamma/I$ are equivalent.

Our first main result characterizes the orbit closures of modules which are (singular) local hypersurfaces.

THEOREM 1.3. *Assume $\mathrm{char}\, k = 0$. Let M be an A-module, $B = A/\mathrm{Ann}(M)$, and let $k\Gamma \supseteq I$ be as above such that $\mathrm{mod}(B) \simeq \mathrm{mod}(k\Gamma/I)$. The orbit closure $\overline{\mathcal{O}}_M$ is a singular local hypersurface if and only if one of the following conditions holds:*

(1) *The algebra B is hereditary and $\mathrm{Ext}^1_B(M, M) \simeq k$.*
(2) *$I = \langle \gamma^2 \rangle$, where γ is a loop in Γ at a vertex i with $(\mathbf{dim}\, M)_i = 2$, and $\mathrm{Ext}^1_B(M, M) = 0$.*
(3) *$I = \langle \rho \rangle$, where ρ is a relation in Γ from a vertex i to a vertex j with $(\mathbf{dim}\, M)_i = (\mathbf{dim}\, M)_j = 1$, and $\mathrm{Ext}^1_B(M, M) = 0$.*

Using the geometric equivalence described in [4] (see Section 2), Theorem 1.3 will be a consequence of the following result proved in Section 3.

THEOREM 1.4. *Assume* $\operatorname{char} k = 0$. *Let* Q *be a quiver and* $\mathbf{d} = (d_i)_{i\in Q_0}$ *be a dimension vector. Let* N *be a representation in* $\operatorname{rep}_Q(\mathbf{d})$ *such that* $\operatorname{Ann}(N)$ *is an admissible ideal in* kQ. *Then* $\overline{\mathcal{O}}_N$ *is a singular hypersurface if and only if one of the following conditions holds:*

(1) $\operatorname{Ann}(N) = 0$ *and* $\operatorname{Ext}^1_{kQ}(N,N) \simeq k$.
(2) $\operatorname{Ann}(N) = \langle\gamma^2\rangle$, *where* γ *is a loop in* Q *at a vertex* i *with* $d_i = 2$, *and* $\operatorname{Ext}^1_{kQ/\langle\gamma^2\rangle}(N,N) = 0$.
(3) $\operatorname{Ann}(N) = \langle\rho\rangle$, *where* ρ *is a relation in* Q *from a vertex* i *to a vertex* j *with* $d_i = d_j = 1$, *and* $\operatorname{Ext}^1_{kQ/\langle\rho\rangle}(N,N) = 0$.

Moreover, $\overline{\mathcal{O}}_N$ *is a singular hypersurface if and only if it is a singular hypersurface at the point* 0.

In Section 2, we recall some notions on representations of quivers and explain the geometric relation between the orbit closures of modules and of quiver representations. We also deduce that a local hypersurface is preserved by smooth morphisms, which implies that Theorem 1.3 is a consequence of Theorem 1.4, proved in Section 3. For basic background on the representation theory of algebras and quivers we refer to [1]. The results presented in this paper form a part of the first author's doctoral dissertation [9] written under the supervision of the second author.

2. Representations of quivers and geometric relation of orbit closures. Let $Q = (Q_0, Q_1; s,t : Q_1 \to Q_0)$ be a finite quiver, i.e., Q_0 is a finite set of vertices, Q_1 is a finite set of arrows $\alpha : s(\alpha) \to t(\alpha)$, where $s(\alpha)$ and $t(\alpha)$ denote the starting and terminating vertex of α, respectively. By an *oriented path* (*path*, for short) of length $m \geq 1$ in Q we mean a sequence of arrows in Q_1:

$$\omega = \alpha_m \dots \alpha_1,$$

such that $s(\alpha_{l+1}) = t(\alpha_l)$ for $l = 1,\dots,m-1$. In this situation we write $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_m)$, and say that ω is a path from $s(\alpha_1)$ to $t(\alpha_m)$. We agree to associate to each vertex $i \in Q_0$ a path ε_i in Q of length zero with $s(\varepsilon_i) = t(\varepsilon_i) = i$. We call a path ω of positive length with $s(\omega) = t(\omega)$ an *oriented cycle*. By a *primitive cycle* we mean an oriented cycle which does not contain other oriented cycles as proper subpaths. A *loop* is an oriented cycle of length one.

The paths in Q form a k-linear basis of the *path algebra* kQ, in which the product of two paths ω and ρ is the path $\omega\rho$ if $s(\omega) = t(\rho)$, and is zero otherwise. Observe that the algebra kQ is finite-dimensional if and only if Q has no oriented cycles. A *relation* from a vertex i to a vertex j is a k-linear combination of paths from i to j of length at least two. In particular, a relation is an element in the vector space $\varepsilon_j \cdot kQ \cdot \varepsilon_i$. Given ρ in $\varepsilon_j \cdot kQ \cdot \varepsilon_i$, we denote by $\langle \rho \rangle$ the two-sided ideal in kQ generated by ρ.

By a *representation* of Q we mean a collection $V = (V_i, V_\alpha)$ of finite-dimensional k-vector spaces V_i, $i \in Q_0$, together with linear maps $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$, $\alpha \in Q_1$. The *dimension vector* of the representation V is the vector

$$\mathbf{dim}\, V = (\dim_k V_i) \in \mathbb{N}^{Q_0}.$$

A *morphism* $f : V \to W$ between two representations is a collection of linear maps $f_i : V_i \to W_i$, $i \in Q_0$, such that $f_{t(\alpha)} V_\alpha = W_\alpha f_{s(\alpha)}$ for each $\alpha \in Q_1$. The category of representations of Q is an abelian k-linear category, which is naturally equivalent to the category $\mathrm{mod}(kQ)$ of finite-dimensional left kQ-modules. The category $\mathrm{mod}(kQ)$ is hereditary, which means that $\mathrm{Ext}^2_{kQ}(-,-) = 0$.

For a path $\omega = \alpha_m \dots \alpha_1$ and a representation V we define

$$V_\omega = V_{\alpha_m} \circ \cdots \circ V_{\alpha_1} : V_{s(\omega)} \to V_{t(\omega)}$$

and extend easily this definition to $V_\rho : V_i \to V_j$ for any ρ in $\varepsilon_j \cdot kQ \cdot \varepsilon_i$, where $i, j \in Q_0$, as ρ is a linear combination of paths ω with $s(\omega) = i$ and $t(\omega) = j$. We set

$$\mathrm{Ann}(V) = \{\rho \in kQ \mid V_{\varepsilon_j \cdot \rho \cdot \varepsilon_i} = 0 \text{ for all } i, j \in Q_0\},$$

which is a two-sided ideal in kQ. In fact, it is the annihilator of the kQ-module corresponding to V with underlying vector space $\bigoplus_{i \in Q_0} V_i$.

Let $\mathcal{R}_Q$ denote the two-sided ideal in kQ generated by the paths of length one (i.e., arrows) in Q. A two-sided ideal I in kQ is called *admissible* if $(\mathcal{R}_Q)^r \subseteq I \subseteq (\mathcal{R}_Q)^2$ for some integer $r \geq 2$. For such an ideal I, the category $\mathrm{mod}(kQ/I)$ of kQ/I-modules is equivalent to the full subcategory consisting of all the representations V of Q such that $\mathrm{Ann}(V) \supseteq I$. We shall identify these two categories.

Let $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. The representations $V = (V_i, V_\alpha)$ of Q with $V_i = k^{d_i}$, $i \in Q_0$, form an affine space

$$\mathrm{rep}_Q(\mathbf{d}) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)}) = \bigoplus_{\alpha \in Q_1} \mathbb{M}_{d_{t(\alpha)} \times d_{s(\alpha)}}(k),$$

where $\mathbb{M}_{d' \times d''}(k)$ stands for the space of $d' \times d''$-matrices with entries in k. The group

$$\mathrm{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \mathrm{GL}(d_i)$$

acts regularly on $\mathrm{rep}_Q(\mathbf{d})$ via

$$(g_i)_{i \in Q_0} * (V_\alpha)_{\alpha \in Q_1} = (g_{t(\alpha)} \cdot V_\alpha \cdot g^{-1}_{s(\alpha)})_{\alpha \in Q_1}.$$

Given a representation $W = (W_i, W_\alpha)$ of Q with $\mathbf{dim}\, W = \mathbf{d}$, we denote by $\mathcal{O}_W$ the $\mathrm{GL}(\mathbf{d})$-orbit in $\mathrm{rep}_Q(\mathbf{d})$ of representations isomorphic to W.

If I is an admissible ideal in kQ, then the representations V in $\mathrm{rep}_Q(\mathbf{d})$ such that $\mathrm{Ann}(V) \supseteq I$ form a closed $\mathrm{GL}(\mathbf{d})$-stable subset $\mathrm{rep}_{Q,I}(\mathbf{d})$ of $\mathrm{rep}_Q(\mathbf{d})$. This set is the underlying variety of the affine scheme $\mathbf{rep}_{Q,I}(\mathbf{d})$ defined as follows. Let

$$k[\mathrm{rep}_Q(\mathbf{d})] = k[X_{\alpha,p,q} \mid \alpha \in Q_1,\, p \le d_{t(\alpha)},\, q \le d_{s(\alpha)}]$$

denote the algebra of polynomial functions on the affine space $\mathrm{rep}_Q(\mathbf{d})$. Here, $X_{\beta,p,q}$ maps a representation $W = (W_\alpha)$ to the (p,q)-entry of the matrix W_β. Let X_α stand for the $d_{t(\alpha)} \times d_{s(\alpha)}$-matrix whose (p,q)-entry is the variable $X_{\alpha,p,q}$, for any arrow $\alpha \in Q_1$. We define the $d_j \times d_i$-matrix X_ρ for $\rho \in \varepsilon_j \cdot kQ \cdot \varepsilon_i$, with entries in $k[\mathrm{rep}_Q(\mathbf{d})]$, in a similar way to that for representations of Q. Then $\mathbf{rep}_{Q,I}(\mathbf{d})$ is the closed subscheme defined by the ideal in $k[\mathrm{rep}_Q(\mathbf{d})]$ generated by the entries of all the matrices X_ρ for $\rho \in \varepsilon_j \cdot I \cdot \varepsilon_i$, where $i, j \in Q_0$.

We need [6, Corollary 1.2] formulated in terms of representations:

LEMMA 2.1. *Let* $N \in \mathrm{rep}_{Q,I}(\mathbf{d})$. *Then* $\mathrm{Ext}^1_{kQ/I}(N,N) = 0$ *if and only if the orbit* $\mathcal{O}_N$ *is open in the scheme* $\mathbf{rep}_{Q,I}(\mathbf{d})$.

In the case when the scheme $\mathbf{rep}_{Q,I}(\mathbf{d})$ is reduced, $\mathrm{Ext}^1_{kQ/I}(N,N) = 0$ if and only if $\mathcal{O}_N$ is open in $\mathrm{rep}_{Q,I}(\mathbf{d})$.

Now let A be an algebra and let M be an A-module of dimension d. The *annihilator* $\mathrm{Ann}(M)$ of M is the kernel of the algebra homomorphism $A \to \mathbb{M}_d(k)$ induced by M, thus the algebra $B = A/\mathrm{Ann}(M)$ is finite-dimensional. Observe that $\mathrm{mod}_B(d)$ is a closed $\mathrm{GL}(d)$-subvariety of $\mathrm{mod}_A(d)$ containing $\overline{\mathcal{O}}_M$. Moreover, M is faithful as a B-module.

The orbit closures in $\mathrm{mod}_B(d)$ and in $\mathrm{rep}_\Gamma(\mathbf{d})$ are closely related [4], where Γ is the Gabriel quiver of B. Indeed, let $\{e_1, \ldots, e_n\}$ be a complete set of primitive pairwise orthogonal idempotents of B such that $Be_i \not\simeq Be_j$ for $i \ne j$. Then the algebra eBe is the basic algebra associated to B, where $e = e_1 + \cdots + e_n$ (see [1, I.6]). We have an equivalence of module categories

$$\mathcal{F} : \mathrm{mod}(B) \to \mathrm{mod}(k\Gamma/I),$$

where the quiver Γ is defined as follows: the vertices of Γ correspond bijectively to the idempotents $e_1, \ldots, e_n$; for $i, j \in \Gamma_0$, the arrows $\alpha : i \to j$ in Γ_1 correspond bijectively to the vectors in some basis of the vector space $e_j(\mathrm{rad}(B)/\mathrm{rad}^2(B))e_i$. Moreover, I is an admissible ideal in $k\Gamma$ such that $eBe \simeq k\Gamma/I$. The functor $\mathcal{F}$ associates to any faithful B-module M the module eM over the algebra $eBe = k\Gamma/I$, thus a representation N of Γ with $\mathrm{Ann}(N) = I$. In particular, $N \in \mathrm{rep}_{\Gamma,I}(\mathbf{d})$ for the dimension vector $\mathbf{d} = (\dim_k e_iM) \in \mathbb{N}^{\Gamma_0}$. Note that $\mathbf{d}$ is nothing but the dimension vector

dim M of M, viewed as an element of the Grothendieck group $K_0(B)$ of the category $\mathrm{mod}(B)$. By [4], $\overline{\mathcal{O}}_M$ is isomorphic to the associated fibre bundle $\mathrm{GL}(d) \times^{\mathrm{GL}(\mathbf{d})} \overline{\mathcal{O}}_N$. Thus $\overline{\mathcal{O}}_M$ and $\overline{\mathcal{O}}_N$ share all local geometric properties which are preserved under smooth morphisms, including the normality, regularity in some codimension, Cohen–Macaulayness, etc.

We now deduce from a result of Avramov (see [2, Section 7]) that the property of being local hypersurface is also preserved under smooth morphisms. Recall that a local ring R is a hypersurface if it is the quotient of a regular local ring by a principal ideal. If the local ring R is the quotient of a regular local ring by some ideal (for example, the local ring of a point on an algebraic variety), then it is a hypersurface if and only if its second deviation $\varepsilon_2(R)$ is at most 1. Moreover, R is regular if and only if $\varepsilon_2(R) = 0$. We recall that the deviations $\varepsilon_n(R)$, $n \geq 1$, of a local ring R are unique integers such that the Poincaré series of R is equal to

$$\frac{\prod_{i=1}^{\infty}(1+t^{2i-1})^{\varepsilon_{2i-1}(R)}}{\prod_{i=1}^{\infty}(1-t^{2i})^{\varepsilon_{2i}(R)}},$$

where the products converge in the (t)-adic topology of the ring $\mathbb{Z}[[t]]$.

LEMMA 2.2. *Let $\varphi : (R, \mathrm{m}) \to (S, \mathrm{n})$ be a flat, local homomorphism of local rings. If the fibre $S/\mathrm{m}S$ is regular, then $\varepsilon_2(R) = \varepsilon_2(S)$.*

Proof. Using [2, Theorem 7.4.2] we have

$$\varepsilon_2(R) \leq \varepsilon_2(S) = \varepsilon_2(R) + \varepsilon_2(S/\mathrm{m}S) - \delta$$

for some integer $\delta \geq 0$. By assumptions $\varepsilon_2(S/\mathrm{m}S) = 0$, thus $\varepsilon_2(R) = \varepsilon_2(S)$. ■

It follows from Lemma 2.2 that if $\varphi : \mathcal{X} \to \mathcal{Y}$ is a smooth surjective morphism of varieties, then $\mathcal{X}$ is a local hypersurface if and only if so is $\mathcal{Y}$. Now it is clear from our discussion that Theorem 1.4 will imply Theorem 1.3.

3. Proof of Theorem 1.4. If $\overline{\mathcal{O}}_N$ is a singular hypersurface, then it is a singular hypersurface at the point 0. Therefore, let $N = (N_\alpha)$ be a representation in $\mathrm{rep}_Q(\mathbf{d})$ such that $\mathrm{Ann}(N)$ is an admissible ideal in kQ and $\overline{\mathcal{O}}_N$ is a singular hypersurface at 0. Recall that in particular N is a nilpotent representation. This is equivalent to the fact that the endomorphism N_ω is nilpotent for any oriented cycle ω in Q. The assumption $\mathrm{char}\, k = 0$ guarantees that the space $\{f \in \mathrm{End}_k(V) \mid \mathrm{tr}(f) = 0\}$ is a simple $\mathrm{GL}(V)$-submodule of $\mathrm{End}_k(V)$, where V is a finite-dimensional k-vector space and the group $\mathrm{GL}(V)$ acts on $\mathrm{End}_k(V)$ by conjugation. Here, tr stands for the trace of a linear endomorphism or of a square matrix.

We need two auxiliary results, whose proofs are straightforward.

LEMMA 3.1. *Let $\xi = \alpha_m \ldots \alpha_1$ be a path in the quiver Q such that $d_{t(\alpha_l)} \geq 2$ for $l = 1, \ldots, m-1$ and the arrows $\alpha_1, \ldots, \alpha_m$ are pairwise distinct.*

Then the entries of the matrix X_ξ *are irreducible polynomials in* $k[\mathrm{rep}_Q(\mathbf{d})]$. *In particular, if* $d_{s(\xi)} = d_{t(\xi)} = 1$, *then the polynomial* X_ξ *is irreducible in* $k[\mathrm{rep}_Q(\mathbf{d})]$.

LEMMA 3.2. *Let* $\omega = \beta_n \dots \beta_1$ *be a primitive cycle in* Q *such that* $d_{t(\beta_l)} \geq 2$ *for* $l = 1, \dots, n$. *Then the polynomial* $\mathrm{tr}(X_\omega)$ *is irreducible in* $k[\mathrm{rep}_Q(\mathbf{d})]$.

3.1. Tangent spaces of orbit closures. The action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{rep}_Q(\mathbf{d})$ induces an action on $k[\mathrm{rep}_Q(\mathbf{d})]$ by $(g * f)(W) = f(g^{-1} * W)$ for $g \in \mathrm{GL}(\mathbf{d})$, $f \in k[\mathrm{rep}_Q(\mathbf{d})]$ and $W \in \mathrm{rep}_Q(\mathbf{d})$. Clearly, the defining ideal $I(\overline{\mathcal{O}}_N)$ of $\overline{\mathcal{O}}_N$ is invariant under the action of $\mathrm{GL}(\mathbf{d})$ on $k[\mathrm{rep}_Q(\mathbf{d})]$.

Now let $\mathcal{N}_Q(\mathbf{d})$ denote the set of all nilpotent representations in $\mathrm{rep}_Q(\mathbf{d})$. Observe that it is a closed $\mathrm{GL}(\mathbf{d})$-invariant subset of $\mathrm{rep}_Q(\mathbf{d})$ containing $\overline{\mathcal{O}}_N$. We shall identify the tangent space $\mathcal{T}_0(\mathrm{rep}_Q(\mathbf{d}))$ of $\mathrm{rep}_Q(\mathbf{d})$ at the point 0 with $\mathrm{rep}_Q(\mathbf{d})$ itself. Thus the tangent space $\mathcal{T}_0(\overline{\mathcal{O}}_N)$ is a subspace of $\mathrm{rep}_Q(\mathbf{d})$ and is invariant under the action of $\mathrm{GL}(\mathbf{d})$, i.e., it is a $\mathrm{GL}(\mathbf{d})$-submodule of $\mathrm{rep}_Q(\mathbf{d})$.

LEMMA 3.3. *Let* $W = (W_\alpha)$ *be a tangent vector in* $\mathcal{T}_0(\overline{\mathcal{O}}_N)$. *Then* $\mathrm{tr}(W_\gamma) = 0$ *for any loop* $\gamma \in Q_1$.

Proof. The set $\mathcal{N}_Q(\mathbf{d})$ is the zero locus of the (non-leading) coefficients of the characteristic polynomials of all square matrices X_ω, where ω is any oriented cycle in Q. Since $\overline{\mathcal{O}}_N \subseteq \mathcal{N}_Q(\mathbf{d})$, these coefficients belong to $I(\overline{\mathcal{O}}_N)$. In particular, $\mathrm{tr}(X_\gamma) \in I(\overline{\mathcal{O}}_N)$ for any loop $\gamma \in Q_1$. By the definition of tangent spaces, $\mathrm{tr}(W_\gamma) = 0$. ■

We view the set

$$\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d}) = \{W = (W_\alpha) \in \mathrm{rep}_Q(\mathbf{d}) \mid \mathrm{tr}(W_\gamma) = 0 \text{ for any loop } \gamma \in Q_1\}$$

as a vector subspace of $\mathrm{rep}_Q(\mathbf{d})$.

The following result holds for an arbitrary *admissible representation* N (i.e., $\mathrm{Ann}(N)$ is an admissible ideal).

PROPOSITION 3.4. *If* $\mathrm{char}\, k = 0$, *then* $\mathcal{T}_0(\overline{\mathcal{O}}_N) = \mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})$.

Let $V_i = k^{d_i}$ and $R_{i,j}$ be the vector space of formal linear combinations of arrows $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$, for any $i, j \in Q_0$. We identify

$$\mathrm{rep}_Q(\mathbf{d}) = \bigoplus_{i,j \in Q_0} \mathrm{Hom}_k(R_{i,j}, \mathrm{Hom}_k(V_i, V_j)) \quad \text{and} \quad \mathrm{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \mathrm{GL}(V_i).$$

Applying Lemma 3.3, we get

$$\mathcal{T}_0(\overline{\mathcal{O}}_N) \subseteq \bigoplus_{i,j \in Q_0,\, i \neq j} \mathrm{Hom}_k(R_{i,j}, \mathrm{Hom}_k(V_i, V_j)) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}_k(R_{i,i}, \mathrm{End}_k^{\mathrm{tr}}(V_i)),$$

where $\mathrm{End}_k^{\mathrm{tr}}(V_i) = \{f \in \mathrm{End}_k(V_i) \mid \mathrm{tr}(f) = 0\}$.

Since $\operatorname{char} k = 0$, the space $\operatorname{End}_k^{\mathrm{tr}}(V_i)$, $i \in Q_0$, is a simple $\mathrm{GL}(\mathbf{d})$-submodule of $\operatorname{End}_k(V_i)$. Moreover, the $\mathrm{GL}(\mathbf{d})$-modules $\operatorname{Hom}_k(V_i, V_j)$, $i \neq j$, are simple and pairwise non-isomorphic. Thus we have

$$\begin{aligned}\mathcal{T}_0(\overline{\mathcal{O}}_N) = &\bigoplus_{i,j\in Q_0,\, i\neq j} \{\varphi : R_{i,j} \to \operatorname{Hom}_k(V_i, V_j) \mid \varphi(U_{i,j}) = 0\} \\ &\oplus \bigoplus_{i\in Q_0} \{\psi : R_{i,i} \to \operatorname{End}_k^{\mathrm{tr}}(V_i) \mid \psi(U_{i,i}) = 0\}\end{aligned}$$

for some subspaces $U_{i,j}$ of $R_{i,j}$, where $i, j \in Q_0$.

As was shown in [10], we may assume that the spaces $U_{i,j}$ are spanned by arrows in Q_1. Consequently,

$$\mathcal{T}_0(\overline{\mathcal{O}}_N) = \operatorname{rep}_{Q'}^{\mathrm{tr}}(\mathbf{d}) \tag{3.1}$$

for some subquiver Q' of Q such that $Q'_0 = Q_0$.

It is our aim to prove that in fact $Q' = Q$. Note that the proof of [10, Proposition 4.2] does not apply, since the quiver Q may contain oriented cycles. Under the assumption $\operatorname{char} k = 0$ we shall give a different proof, which does not depend on whether Q has oriented cycles or not.

Let G be a linearly reductive group and $\mathcal{X}$ an affine G-variety. Then there exists a unique *Reynolds operator* $\mathcal{R} : k[\mathcal{X}] \to k[\mathcal{X}]^G$, where $k[\mathcal{X}]^G$ denotes the invariant ring of G in $k[\mathcal{X}]$ (for instance, see [5, 2.2]). Recall that a Reynolds operator is a linear map $\mathcal{R} : k[\mathcal{X}] \to k[\mathcal{X}]^G$ such that $\mathcal{R}(f) = f$ for all $f \in k[\mathcal{X}]^G$ and $\mathcal{R}(g * f) = \mathcal{R}(f)$ for all $f \in k[\mathcal{X}]$, $g \in G$. If W is a G-submodule of $k[\mathcal{X}]$, then $\mathcal{R}(W) = W^G$.

Proof of Proposition 3.4. Suppose, to the contrary, that there is an arrow $\beta : b \to a$ in $Q_1 \setminus Q'_1$. Since $\mathcal{T}_0(\overline{\mathcal{O}}_N) \subseteq \operatorname{rep}_{Q'}(\mathbf{d})$, we have $X_{\beta,u,v} \in \mathrm{m}^2 + I(\overline{\mathcal{O}}_N)$ for $u \leq d_a$ and $v \leq d_b$, where m is the maximal ideal in $k[\operatorname{rep}_Q(\mathbf{d})]$ generated by all the variables. Hence there are polynomials $f_{u,v} \in k[\operatorname{rep}_Q(\mathbf{d})]$ of order at least 2, i.e., belonging to m^2, such that $X_{\beta,u,v} - f_{u,v} \in I(\overline{\mathcal{O}}_N)$.

Let Δ be the quiver obtained from Q by appending an arrow $\gamma : a \to b$, i.e., of reverse direction to β. Let $\mathcal{C} = \overline{\mathcal{O}}_N \times \mathbb{M}_{d_b\times d_a}(k) \subseteq \operatorname{rep}_\Delta(\mathbf{d})$. Consider $k[\operatorname{rep}_Q(\mathbf{d})]$ as a subalgebra of $k[\operatorname{rep}_\Delta(\mathbf{d})]$; then $I(\mathcal{C}) = I(\overline{\mathcal{O}}_N) \cdot k[\operatorname{rep}_\Delta(\mathbf{d})]$. In particular, $I(\mathcal{C})$ contains the polynomial

$$w = \sum_{u\leq d_a} \sum_{v\leq d_b} (X_{\beta,u,v} - f_{u,v}) \cdot X_{\gamma,v,u}.$$

Let $\mathcal{R} : k[\operatorname{rep}_\Delta(\mathbf{d})] \to k[\operatorname{rep}_\Delta(\mathbf{d})]^{\mathrm{GL}(\mathbf{d})}$ be the Reynolds operator. For the $\mathrm{GL}(\mathbf{d})$-submodule $I(\mathcal{C})$ of $k[\operatorname{rep}_\Delta(\mathbf{d})]$ and the polynomial $w \in I(\mathcal{C})$, we have

$$\mathcal{R}(w) = \mathcal{R}\Big(\sum_{u,v} X_{\beta,u,v} \cdot X_{\gamma,v,u}\Big) - \mathcal{R}\Big(\sum_{u,v} f_{u,v} \cdot X_{\gamma,v,u}\Big) \in I(\mathcal{C})^{\mathrm{GL}(\mathbf{d})}.$$

The polynomial $\sum_{u,v} X_{\beta,u,v} \cdot X_{\gamma,v,u}$ is $\mathrm{GL}(\mathbf{d})$-invariant, thus

$$\mathcal{R}(w) = \sum_{u,v} X_{\beta,u,v} \cdot X_{\gamma,v,u} - z,$$

where the polynomial $z = \mathcal{R}(\sum_{u,v} f_{u,v} \cdot X_{\gamma,v,u})$ belongs to $k[\mathrm{rep}_\Delta(\mathbf{d})]^{\mathrm{GL}(\mathbf{d})}$.

Consider the natural $\mathbb{N}^{\Delta_1}$-grading on $k[\mathrm{rep}_\Delta(\mathbf{d})]$ and observe that the space of homogeneous polynomials of given degree with respect to this grading, together with 0, is a $\mathrm{GL}(\mathbf{d})$-module. Hence the Reynolds operator maps homogeneous polynomials to homogeneous polynomials of the same degree. The same is true for homogeneous polynomials with respect to the usual $\mathbb{N}$-grading on $k[\mathrm{rep}_\Delta(\mathbf{d})]$. Thus z is a linear polynomial in the variables $X_{\gamma,v,u}$ and is of order at least 3 (in the usual $\mathbb{N}$-grading).

By a result of Le Bruyn and Procesi [8, Theorem 1], the invariant algebra $k[\mathrm{rep}_\Delta(\mathbf{d})]^{\mathrm{GL}(\mathbf{d})}$ is generated by the polynomials $\mathrm{tr}(X_\omega)$, where ω is any oriented cycle in Δ. The usual degree of $\mathrm{tr}(X_\omega)$ equals the length of ω, while the degree with respect to the variables $X_{\gamma,v,u}$ is the multiplicity of the arrow γ in the path ω. It follows that the polynomial z is a linear combination of products

$$\mathrm{tr}(X_{\omega_1}) \cdot \ldots \cdot \mathrm{tr}(X_{\omega_r}),$$

where ω_l are oriented cycles in Δ, the arrow γ appears in only one of these cycles and precisely once, and the sum of their lengths is at least 3. If $r \geq 2$, then the above product belongs to $I(\mathcal{C})$. Indeed, then there exists an oriented cycle ω_l not containing γ, thus being an oriented cycle in Q. Since the representation N is nilpotent, $\mathrm{tr}(X_{\omega_l}) \in I(\overline{\mathcal{O}}_N)$ and consequently $\mathrm{tr}(X_{\omega_1}) \cdot \ldots \cdot \mathrm{tr}(X_{\omega_r}) \in I(\mathcal{C})$.

Let z' be the polynomial obtained from z by deleting all summands of the form $\mathrm{tr}(X_{\omega_1}) \cdot \ldots \cdot \mathrm{tr}(X_{\omega_r})$ for $r \geq 2$. Then z' is a linear combination of polynomials $\mathrm{tr}(X_\omega)$, where ω is an oriented cycle in Δ of length at least 3 passing precisely once through the arrow γ, and

$$\sum_{u,v} X_{\beta,u,v} \cdot X_{\gamma,v,u} - z' \in I(\mathcal{C}).$$

Observe that the polynomial $\mathrm{tr}(X_\omega)$ does not depend on the choice of the starting vertex of ω. Indeed, if $\omega = \omega'\omega''$, then $\omega''\omega'$ is also an oriented cycle and

$$\mathrm{tr}(X_{\omega'\omega''}) = \mathrm{tr}(X_{\omega'} \cdot X_{\omega''}) = \mathrm{tr}(X_{\omega''} \cdot X_{\omega'}) = \mathrm{tr}(X_{\omega''\omega'}).$$

Let Ω be the set of all paths in Q of length at least 2 from b to a. Then

$$z' = \sum_{\omega \in \Omega} \lambda(\omega) \cdot \mathrm{tr}(X_{\omega\gamma}) = \sum_{\omega \in \Omega} \lambda(\omega) \cdot \sum_{u,v} X_{\omega,u,v} X_{\gamma,v,u}, \qquad \lambda(\omega) \in k,$$

and consequently,

$$\sum_{u,v}\Big(X_{\beta,u,v}-\sum_{\omega\in\Omega}\lambda(\omega)\cdot X_{\omega,u,v}\Big)\cdot X_{\gamma,v,u}\in I(\mathcal{C}).$$

Since $I(\mathcal{C})=I(\overline{\mathcal{O}}_N)\cdot k[\mathrm{rep}_\Delta(\mathbf{d})]$, it follows that for any $u\le d_a$, $v\le d_b$, we have

$$X_{\beta,u,v}-\sum_{\omega\in\Omega}\lambda(\omega)\cdot X_{\omega,u,v}\in I(\overline{\mathcal{O}}_N).$$

This means that all the entries of the matrix $X_{\beta-\rho}$ belong to the ideal $I(\overline{\mathcal{O}}_N)$, where $\rho=\sum_{\omega\in\Omega}\lambda(\omega)\cdot\omega$. Therefore $\beta-\rho$ belongs to $\mathrm{Ann}(N)$. Since $\beta-\rho$ does not belong to $(\mathcal{R}_Q)^2$, the ideal $\mathrm{Ann}(N)$ is not admissible, a contradiction. ∎

Corollary 3.5. *$\overline{\mathcal{O}}_N$ is a closed $\mathrm{GL}(\mathbf{d})$-subvariety of codimension 1 in $\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})$.*

Proof. By assumption, $\overline{\mathcal{O}}_N$ is a singular hypersurface at 0. This implies that

$$\dim\overline{\mathcal{O}}_N=\dim_k\mathcal{T}_0(\overline{\mathcal{O}}_N)-1=\dim_k\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})-1.$$

Since $\overline{\mathcal{O}}_N$ is contained in $\mathcal{N}_Q(\mathbf{d})$, it is also contained in $\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})$. Hence the corollary follows. ∎

3.2. The case when Q is acyclic. First, we consider the case when Q is *acyclic*, i.e., there are no oriented cycles (in particular no loops) in Q. Then $\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})=\mathrm{rep}_Q(\mathbf{d})$.

A non-zero polynomial f in $k[\mathrm{rep}_Q(\mathbf{d})]$ is called a $\mathrm{GL}(\mathbf{d})$-*semi-invariant* (*of weight* χ) if $g*f=\chi(g)\cdot f$ for all $g\in\mathrm{GL}(\mathbf{d})$, where $\chi:\mathrm{GL}(\mathbf{d})\to k^*$ is a k-regular character of $\mathrm{GL}(\mathbf{d})$. The following result is a consequence of Corollary 3.5.

Corollary 3.6. *There exists an irreducible $\mathrm{GL}(\mathbf{d})$-semi-invariant F such that $I(\overline{\mathcal{O}}_N)=(F)$.*

Proof. By Corollary 3.5, $\overline{\mathcal{O}}_N$ is an irreducible hypersurface in $\mathrm{rep}_Q(\mathbf{d})$, thus $I(\overline{\mathcal{O}}_N)=(F)$ for some irreducible polynomial F. Since the variety $\overline{\mathcal{O}}_N$ is $\mathrm{GL}(\mathbf{d})$-invariant, we get the equality $(g*F)=(F)$ of ideals, for any $g\in\mathrm{GL}(d)$. Consequently,

$$g*F=\chi(g)\cdot F$$

for some non-zero scalar $\chi(g)$, $g\in\mathrm{GL}(\mathbf{d})$. The map $\chi:\mathrm{GL}(\mathbf{d})\to k^*$ is easily seen to be a k-regular character of $\mathrm{GL}(\mathbf{d})$. ∎

Since $\mathrm{codim}_{\mathrm{rep}_Q(\mathbf{d})}\overline{\mathcal{O}}_N=\dim_k\mathrm{Ext}^1_{kQ}(N,N)$ by the Artin–Voigt formula (see [11]), we also obtain:

Corollary 3.7. *$\mathrm{Ext}^1_{kQ}(N,N)\simeq k$.*

We consider two gradings on the algebra $k[\mathrm{rep}_Q(\mathbf{d})] = k[X_{\alpha,p,q}]$, induced by two torus actions. We choose a standard maximal torus T in $\mathrm{GL}(\mathbf{d})$ consisting of $g = (g_i)$, where all $g_i \in \mathrm{GL}(d_i)$ are diagonal matrices. Let $\widetilde{Q}_0$ denote the set of pairs (i, p) with $i \in Q_0$ and $1 \leq p \leq d_i$. Then the action of T on $\mathrm{rep}_Q(\mathbf{d})$ leads to a $\mathbb{Z}^{\widetilde{Q}_0}$-grading on $k[\mathrm{rep}_Q(\mathbf{d})]$ with

$$\widetilde{\deg}\, X_{\alpha,p,q} = e_{t(\alpha),p} - e_{s(\alpha),q},$$

where $\{e_{i,p}\}_{(i,p)\in\widetilde{Q}_0}$ is the standard basis of $\mathbb{Z}^{\widetilde{Q}_0}$ identified with the group of k-regular characters of T.

The torus $(k^*)^{|Q_0|}$, being the center of $\mathrm{GL}(\mathbf{d})$, is contained in the torus T. Its action on $\mathrm{rep}_Q(\mathbf{d})$ is the restriction of the action of T. Thus there is a $\mathbb{Z}^{Q_0}$-grading on $k[\mathrm{rep}_Q(\mathbf{d})]$ with

$$\deg X_{\alpha,p,q} = e_{t(\alpha)} - e_{s(\alpha)},$$

where $\{e_i\}_{i\in Q_0}$ is the standard basis of $\mathbb{Z}^{Q_0}$. Observe that any $\mathrm{GL}(\mathbf{d})$-semi-invariant is homogeneous with respect to both gradings.

The following lemma is obvious.

Lemma 3.8. *Assume that Q is acyclic and let h be a monomial in $k[\mathrm{rep}_Q(\mathbf{d})]$.*

(1) *If $\deg h = 0$, then $h = 1$.*
(2) *If $\deg h = e_j - e_i \in \mathbb{Z}^{Q_0}$ for vertices $i \neq j$, then*

$$h = X_{\alpha_m,p_m,q_m} \cdot X_{\alpha_{m-1},p_{m-1},q_{m-1}} \cdot \ldots \cdot X_{\alpha_1,p_1,q_1}$$

for some path $\omega = \alpha_m \ldots \alpha_1$ of length $m \geq 1$ from i to j in Q and indices $1 \leq p_l \leq d_{t(\alpha_l)}$, $1 \leq q_l \leq d_{s(\alpha_l)}$, $l = 1, \ldots, m$.
(3) *If $\widetilde{\deg}\, h = e_{j,1} - e_{i,1} \in \mathbb{Z}^{\widetilde{Q}_0}$ for vertices $i \neq j$, then additionally $p_m = q_1 = 1$ and $q_m = p_{m-1}, q_{m-1} = p_{m-2}, \ldots, q_2 = p_1$.*

Lemma 3.9. *Let f be a $\mathrm{GL}(\mathbf{d})$-semi-invariant in $k[\mathrm{rep}_Q(\mathbf{d})]$ such that $\deg f = e_j - e_i$ for vertices $i \neq j$ and $d_i = d_j = 1$. Then $f = X_\rho$ for some ρ in $\varepsilon_j \cdot kQ \cdot \varepsilon_i$.*

Proof. Clearly $\widetilde{\deg}\, f = e_{j,1} - e_{i,1}$. By Lemma 3.8, f is a linear combination

$$f = \sum \lambda(\alpha_m, p_{m-1}, \alpha_{m-1}, \ldots, p_1, \alpha_1) \cdot X_{\alpha_m,1,p_{m-1}} \cdot X_{\alpha_{m-1},p_{m-1},p_{m-2}} \cdot \ldots \cdot X_{\alpha_2,p_2,p_1} \cdot X_{\alpha_1,p_1,1},$$

where the sum runs over all paths $\omega = \alpha_m \ldots \alpha_1$ in Q from i to j and integers $1 \leq p_l \leq d_{t(\alpha_l)}$ for $l = 1, \ldots, m-1$.

We claim that the scalars $\lambda(\alpha_m, p_{m-1}, \alpha_{m-1}, \ldots, \alpha_1, p_1)$ depend only on the path $\alpha_m \ldots \alpha_1$. Indeed, let $p'_l \leq d_{t(\alpha_l)}$ and $p'_l \neq p_l$ for some $1 \leq l \leq m$. In particular, $d_{t(\alpha_l)} \geq 2$. We choose $g = (g_a) \in \mathrm{GL}(\mathbf{d})$ such that g_a for $a \neq t(\alpha_l)$

is the identity matrix and $g_{t(\alpha_l)} = I_{d_{t(\alpha_l)}} + E_{p_l,p'_l}$, where E_{p_l,p'_l} is the matrix whose (p_l, p'_l)-entry is 1 while the other entries are 0. Then the monomial

$$X_{\alpha_m,1,p_{m-1}} \cdot \ldots \cdot X_{\alpha_{l+1},p_{l+1},p_l} \cdot X_{\alpha_l,p'_l,p_{l-1}} \cdot \ldots \cdot X_{\alpha_1,p_1,1}$$

appears in $g * f$ with the coefficient

$$\lambda(\alpha_m, p_{m-1}, \ldots, \alpha_{l+1}, p'_l, \ldots, p_1, \alpha_1) - \lambda(\alpha_m, p_{m-1}, \ldots, \alpha_{l+1}, p_l, \ldots, p_1, \alpha_1).$$

Since $g * f$ is a homogeneous polynomial of degree $e_{j,1} - e_{i,1}$, this coefficient must be 0, which proves the claim.

Let Ω denote the set of all paths ω in Q from i to j. Then there are scalars $\lambda(\omega)$, $\omega \in \Omega$, such that

$$f = \sum_{\omega=\alpha_m\ldots\alpha_1\in\Omega} \lambda(\omega) \cdot \sum_{p_1 \le d_{t(\alpha_1)}} \cdots \sum_{p_{m-1} \le d_{t(\alpha_{m-1})}} X_{\alpha_m,1,p_{m-1}} \cdot \ldots \cdot X_{\alpha_1,p_1,1}.$$

Hence $f = X_\rho$ for $\rho = \sum_{\omega\in\Omega} \lambda(\omega) \cdot \omega \in \varepsilon_j \cdot kQ \cdot \varepsilon_i$. ∎

Let $i_1, \ldots, i_r, j_1, \ldots, j_s$ be vertices (not necessarily distinct) in Q_0 such that

$$\sum_{l=1}^{r} d_{i_l} = \sum_{m=1}^{s} d_{j_m}. \tag{3.2}$$

For any $1 \le l \le r$, $1 \le m \le s$, let $\rho_{l,m} \in \varepsilon_{j_m} \cdot kQ \cdot \varepsilon_{i_l}$ be a linear combination of paths in Q from i_l to j_m. We form an $s \times r$-block matrix whose (m, l)-block is the $d_{j_m} \times d_{i_l}$-matrix $X_{\rho_{l,m}}$. By (3.2), this is a square matrix with entries in $k[\mathrm{rep}_Q(\mathbf{d})]$. Its determinant is a $\mathrm{GL}(\mathbf{d})$-semi-invariant in $k[\mathrm{rep}_Q(\mathbf{d})]$, called a *determinantal semi-invariant*. By [12, Theorem 2.3], the algebra of $\mathrm{GL}(\mathbf{d})$-semi-invariants in $k[\mathrm{rep}_Q(\mathbf{d})]$ is spanned, as a vector space, by the determinantal semi-invariants. In particular, the semi-invariant F in Corollary 3.6 is a linear combination of such determinantal semi-invariants.

Observe that if $i_l = j_m$ for some $l \le r$ and $m \le s$, and $\rho_{l,m} = \lambda \cdot \varepsilon_{i_l}$, $\lambda \in k^*$, so that $X_{\rho_{l,m}} = \lambda \cdot I_{d_{i_l}}$, then the determinant of the $s \times r$-block matrix above is equal to that of a suitable $(s-1) \times (r-1)$-block matrix associated to the other vertices, without i_l and j_m. Thus we can assume that the elements $\rho_{l,m} \in \varepsilon_{j_m} \cdot kQ \cdot \varepsilon_{i_l}$ are linear combinations of paths of positive length.

PROPOSITION 3.10. *If* $\mathrm{Ann}(N) \neq 0$, *then* $\mathrm{Ann}(N) = \langle\rho\rangle$, *where* ρ *is a relation from a vertex* i *to a vertex* j *with* $d_i = d_j = 1$, *and* $\mathrm{Ext}^1_{kQ/\langle\rho\rangle}(N, N) = 0$.

Proof. We use the $\mathbb{Z}^{Q_0}$-grading on $k[\mathrm{rep}_Q(\mathbf{d})]$ with

$$\deg X_{\alpha,p,q} = e_{t(\alpha)} - e_{s(\alpha)} =: e_\alpha, \qquad \alpha \in Q_1.$$

The semi-invariant F is homogeneous with respect to this grading, thus it is a linear combination of determinantal semi-invariants of the same degree. We consider such a non-zero determinantal semi-invariant and assume that it is given by the vertices $i_1, \dots, i_r, j_1, \dots, j_s$ with the equality (3.2) satisfied. We denote the entries of the corresponding square matrix by $x_{p,q}$. There is a permutation σ of the columns of the matrix such that

$$x_{1,\sigma(1)} \cdot \dots \cdot x_{n,\sigma(n)} \neq 0,$$

where $n = \sum d_{i_l}$. Assume $x_{1,\sigma(1)}$ belongs to the $(1,l)$-block $X_{\rho_{l,1}}$ of the square matrix for some $1 \le l \le r$. Then $x_{1,\sigma(1)} \neq 0$ implies that there is a non-zero combination of paths from i_l to j_1 and

$$\deg x_{1,\sigma(1)} = \sum_{\alpha \in Q_1} c_{1,\alpha} e_\alpha = e_{j_1} - e_{i_l},$$

where $c_{1,\alpha} \in \{0,1\}$ and the arrows α for which $c_{1,\alpha} = 1$ form a path from i_l to j_1. Of course, similar arguments can be applied for $x_{2,\sigma(2)}, \dots, x_{n,\sigma(n)}$ with

$$\deg x_{p,\sigma(p)} = \sum_{\alpha \in Q_1} c_{p,\alpha} e_\alpha, \qquad c_{p,\alpha} \in \{0,1\},\ 2 \le p \le n.$$

Thus

$$\begin{aligned}\deg F = \deg(x_{1,\sigma(1)} \cdot \dots \cdot x_{n,\sigma(n)}) &= \sum_{\alpha \in Q_1} (c_{1,\alpha} + \dots + c_{n,\alpha}) e_\alpha \\ &= \sum_{\alpha \in Q_1} a_\alpha e_\alpha \quad \text{for } a_\alpha = c_{1,\alpha} + \dots + c_{n,\alpha}.\end{aligned}$$

By assumption, there is a non-zero element ω in $\varepsilon_{j'} \cdot \mathrm{Ann}(N) \cdot \varepsilon_{i'}$ for some vertices i' and j'. Observe that the common zero set of the polynomials $X_{\omega,u,v}$ for $u \le d_{j'}$, $v \le d_{i'}$ is a closed $\mathrm{GL}(\mathbf{d})$-invariant subset in $\mathrm{rep}_Q(\mathbf{d})$ containing $\overline{\mathcal{O}}_N$. Thus $X_{\omega,1,1} \in (F)$, so $X_{\omega,1,1} = Fh$ for a homogeneous polynomial h in $k[\mathrm{rep}_Q(\mathbf{d})]$. It follows that $\deg F + \deg h = e_{j'} - e_{i'}$. Clearly $\deg h = \sum b_\alpha e_\alpha$ for some non-negative integers b_α, $\alpha \in Q_1$. Hence we conclude that $a_\alpha + b_\alpha \in \{0,1\}$ and the arrows α for which $a_\alpha + b_\alpha = 1$ form a path ξ from i' to j' (see Lemma 3.8). Taking into account the information about $c_{p,\alpha}$, this implies that the vertices $i_1, \dots, i_r, j_1, \dots, j_s$ lie on ξ. Furthermore, by renumbering the vertices we may assume that

$$\deg F = (e_{j_1} - e_{i_1}) + \dots + (e_{j_{l'}} - e_{i_{l'}})$$

for some $l' \le r, s$ and distinct vertices $i_1, \dots, i_{l'}, j_1, \dots, j_{l'}$ such that the subpaths of ξ from i_1 to j_1, …, from $i_{l'}$ to $j_{l'}$ have no arrow in common. Then $d_{i_1} = \dots = d_{i_{l'}} = d_{j_1} = \dots = d_{j_{l'}} = 1$, as $\deg F = \sum_{m=1}^{s} d_{j_m} e_{j_m} - \sum_{l=1}^{r} d_{i_l} e_{i_l}$. It follows from Lemma 3.9 that, up to a scalar, $F = X_{\rho_1} \cdot \ldots \cdot X_{\rho_{l'}}$ for some $\rho_p \in \varepsilon_{j_p} \cdot kQ \cdot \varepsilon_{i_p}$, $1 \le p \le l'$. Since F is irreducible, $l' = 1$ and

$F = X_{\rho_1}$. In particular ρ_1 belongs to the admissible ideal $\mathrm{Ann}(N)$, so it is a relation in Q from i_1 to j_1.

We show next that $\mathrm{Ann}(N) = \langle \rho_1 \rangle$. Let $\omega \in \varepsilon_{j'} \cdot \mathrm{Ann}(N) \cdot \varepsilon_{i'}$ be a non-zero linear combination of paths ω_l, for $i', j' \in Q_0$. Then $X_{\omega,1,1}$ is a multiple of X_{ρ_1}. This implies that each arrow on the path ρ_1 appears also on ω_l for all l, as $X_{\omega,1,1}$ is a linear combination of $X_{\omega_l,1,1}$. The quiver Q has no oriented cycles, thus ρ_1 must be a subpath of ω_l. Therefore $\omega_l \in \langle \rho_1 \rangle$ and consequently $\omega \in \langle \rho_1 \rangle$.

By Lemma 3.1, the scheme $\mathbf{rep}_{Q,\langle \rho_1 \rangle}(\mathbf{d}) = \mathrm{Spec}(k[\mathrm{rep}_Q(\mathbf{d})]/(X_{\rho_1}))$ is reduced. Moreover, $\overline{\mathcal{O}}_N = \mathrm{rep}_{Q,\langle \rho_1 \rangle}(\mathbf{d})$. Hence $\mathrm{Ext}^1_{kQ/\langle \rho_1 \rangle}(N, N) = 0$, by Lemma 2.1. ∎

3.3. The case when Q contains a loop. Next, we consider the case when the quiver Q contains a loop. Let $\gamma : i \to i$ be a loop in Q and denote by Q'' the subquiver of Q consisting of the vertex i and the loop γ. The obvious $\mathrm{GL}(\mathbf{d})$-equivariant linear projection $\pi : \mathrm{rep}^{\mathrm{tr}}_Q(\mathbf{d}) \to \mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i)$ induces a dominant morphism $\overline{\mathcal{O}}_N \to \overline{\mathcal{O}}_{N''}$ with $N'' = \pi(N)$. It follows that

$$I_{\mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i)}(\overline{\mathcal{O}}_{N''}) = I_{\mathrm{rep}^{\mathrm{tr}}_Q(\mathbf{d})}(\overline{\mathcal{O}}_N) \cap k[\mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i)].$$

By Corollary 3.5, we have $I_{\mathrm{rep}^{\mathrm{tr}}_Q(\mathbf{d})}(\overline{\mathcal{O}}_N) = (f)$ for some f in the polynomial ring

$$k[\mathrm{rep}^{\mathrm{tr}}_Q(\mathbf{d})] = k[\mathrm{rep}_Q(\mathbf{d})]/(\{\mathrm{tr}(X_\rho) \mid \rho \text{ is a loop in } Q\}).$$

Therefore $I_{\mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i)}(\overline{\mathcal{O}}_{N''}) = (f)$ if $f \in k[\mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i)]$, and $I_{\mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i)}(\overline{\mathcal{O}}_{N''}) = 0$ otherwise. Consequently,

$$\dim \mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i) - \dim \overline{\mathcal{O}}_{N''} \leq 1.$$

On the other hand, we have:

Lemma 3.11. $\dim \mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i) - \dim \overline{\mathcal{O}}_{N''} \geq d_i - 1$.

Proof. The representation N'' is easily seen to be nilpotent, thus $\overline{\mathcal{O}}_{N''}$ is contained in the closed set of nilpotent representations $\mathcal{N}_{Q''}(d_i)$ in $\mathrm{rep}_{Q''}(d_i)$. It is known that $\mathcal{N}_{Q''}(d_i)$ is a complete intersection of codimension d_i, where its defining ideal is generated by the (non-leading) coefficients of the characteristic polynomial of X_γ (see [7]). Therefore

$$\dim \overline{\mathcal{O}}_{N''} \leq \dim \mathcal{N}_{Q''}(d_i) = {d_i}^2 - d_i.$$

Moreover

$$\dim \mathrm{rep}^{\mathrm{tr}}_{Q''}(d_i) = \dim \mathrm{rep}_{Q''}(d_i) - 1 = {d_i}^2 - 1,$$

thus the claim follows. ∎

Recall that a *primitive cycle* is an oriented cycle which does not contain other oriented cycles as proper subpaths.

COROLLARY 3.12. *The loop* $\gamma : i \to i$ *is the only primitive cycle in* Q, $d_i = 2$ *and* $I(\overline{\mathcal{O}}_N) = (X_{\gamma,1,1} + X_{\gamma,2,2}, X_{\gamma,1,1}X_{\gamma,2,2} - X_{\gamma,1,2}X_{\gamma,2,1})$.

Proof. Let ω be a primitive cycle in Q. The coefficients of the characteristic polynomial of X_ω belong to $I(\overline{\mathcal{O}}_N)$, as $\overline{\mathcal{O}}_N \subseteq \mathcal{N}_Q(\mathbf{d})$. Consequently, their images in $k[\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})]$ belong to (f), i.e., these images are polynomial multiples of f. On the other hand, they are polynomials of variables $X_{\alpha,p,q}$ only, where α is an arbitrary arrow in ω.

Now the above inequalities show that $d_i = 1$ or $d_i = 2$. If $d_i = 1$, then $N_\gamma = 0$, thus $\gamma \in \mathrm{Ann}(N)$ and $\mathrm{Ann}(N)$ is not admissible, a contradiction. Hence $d_i = 2$. Since the irreducible polynomial

$$\det X_\gamma = -X_{\gamma,1,1}^2 - X_{\gamma,1,2}X_{\gamma,2,1}$$

in $k[\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d})]$ belongs to (f), we see that, up to a scalar, $f = \det X_\gamma$. The expression of f involves the variables $X_{\gamma,1,1}$, $X_{\gamma,1,2}$ and $X_{\gamma,2,1}$, thus there cannot exist a second primitive cycle in Q. ∎

PROPOSITION 3.13. $\mathrm{Ann}(N) = \langle\gamma^2\rangle$ *and* $\mathrm{Ext}^1_{kQ/\langle\gamma^2\rangle}(N,N) = 0$.

Proof. Clearly $\langle\gamma^2\rangle \subseteq \mathrm{Ann}(N)$. Let $\xi \in \varepsilon_b \cdot \mathrm{Ann}(N) \cdot \varepsilon_a$ be a non-zero linear combination of paths ξ_l, for $a, b \in Q_0$. The zero set of the polynomials $X_{\xi,u,v}$ for $u \le d_b$, $v \le d_a$ contains $\overline{\mathcal{O}}_N$, thus in particular $X_{\xi,1,1} \in I(\overline{\mathcal{O}}_N) = (\mathrm{tr}\, X_\gamma, \det X_\gamma)$. Since $X_{\xi,1,1}$ is a linear combination of the polynomials $X_{\xi_l,1,1}$, this implies that γ appears on each path ξ_l. Then from the formula for $X_{\xi_l,1,1}$ we deduce that the variable $X_{\gamma,1,2}$ or $X_{\gamma,2,1}$ appears in some term p_l of $X_{\xi_l,1,1}$, which is also a term of $X_{\xi,1,1}$. For instance, in the extreme case $\xi_l = \alpha_m \ldots \alpha_1\gamma$, the variable $X_{\gamma,2,1}$ appears in such a term of $X_{\xi_l,1,1}$.

Since

$$X_{\xi,1,1} = (X_{\gamma,1,1} + X_{\gamma,2,2})h + (X_{\gamma,1,1}X_{\gamma,2,2} - X_{\gamma,1,2}X_{\gamma,2,1})h'$$

for some $h, h' \in k[\mathrm{rep}_Q(\mathbf{d})]$, each term p_l must be divided by another variable $X_{\gamma,p,q}$. This can happen only when γ appears on each path ξ_l at least twice. Hence $\xi_l \in \langle\gamma^2\rangle$ for all l (notice that γ is the only primitive cycle in Q), and so is ξ.

It is easy to see that the ideal $(\mathrm{tr}\, X_\gamma, \det X_\gamma)$ in $k[\mathrm{rep}_Q(\mathbf{d})]$ is prime. Thus the scheme $\mathbf{rep}_{Q,\langle\gamma^2\rangle}(\mathbf{d})$ is reduced. Hence $\mathrm{Ext}^1_{kQ/\langle\gamma^2\rangle}(N,N) = 0$, by Lemma 2.1. ∎

3.4. The case when Q contains an oriented cycle and no loop. Finally, we consider the case when the quiver Q contains an oriented cycle of length at least 2, which is not a power of a loop. In view of Corollary 3.12, this implies that Q does not contain loops. Hence $\mathrm{rep}_Q^{\mathrm{tr}}(\mathbf{d}) = \mathrm{rep}_Q(\mathbf{d})$ and Corollary 3.6 applies.

Let $\omega = \beta_n \dots \beta_1$ ($n \geq 2$) be a primitive cycle in Q.

LEMMA 3.14. $\min\{d_{t(\beta_l)} \mid 1 \leq l \leq n\} = 1$.

Proof. Suppose that $d_{t(\beta_l)} \geq 2$ for all l. Since $\overline{\mathcal{O}}_N \subseteq \mathcal{N}_Q(\mathbf{d})$, the coefficients of the characteristic polynomial of the square matrix X_ω belong to (F). By Lemma 3.2, the polynomial

$$\operatorname{tr}(X_\omega) = X_{\omega,1,1} + \dots + X_{\omega,r,r}$$

is irreducible, where $r = d_{s(\beta_1)}$. Thus up to a scalar, $F = \operatorname{tr}(X_\omega)$. Consequently, the sum of the principal 2×2 minors of X_ω is a multiple of $\operatorname{tr}(X_\omega)$. Observe that this sum has the term

$$h = -(X_{\beta_n,1,2} X_{\beta_{n-1},2,2} \dots X_{\beta_2,2,2} X_{\beta_1,2,2})(X_{\beta_n,2,1} X_{\beta_{n-1},1,1} \dots X_{\beta_2,1,1} X_{\beta_1,1,1})$$

(which is a term of $X_{\omega,1,1}X_{\omega,2,2} - X_{\omega,1,2}X_{\omega,2,1}$). Since it is clear that h cannot be a term of a multiple of $\operatorname{tr}(X_\omega)$, we get a contradiction. ∎

Replacing ω by one of its cyclic permutations of the form $\beta_{l-1} \dots \beta_1 \beta_n \dots \beta_{l+1}\beta_l$, we may assume that $d_{s(\beta_1)} = 1$, so that X_ω is a polynomial in $k[\operatorname{rep}_Q(\mathbf{d})]$. The oriented cycle ω can be decomposed as a product $\omega_r \dots \omega_1$, where ω_l's are subpaths of ω satisfying the assumptions of Lemma 3.1, for $1 \leq l \leq r$. Since $X_\omega = X_{\omega_r} \dots X_{\omega_1}$ belongs to the prime ideal $I(\overline{\mathcal{O}}_N)$ and the polynomials $X_{\omega_1}, \dots, X_{\omega_r}$ are irreducible, it follows that up to a scalar, $F = X_{\omega_l}$ for some l. Letting $\rho = \omega_l$, we obtain $I(\overline{\mathcal{O}}_N) = (X_\rho)$.

If ω' is another primitive cycle in Q which differs from ω and its cyclic permutations, then ω' must contain ρ as a subpath. Up to a cyclic permutation we have $\omega = \eta\rho$ and $\omega' = \eta'\rho$ for subpaths η and η'; thus $d_{s(\eta)} = d_{t(\eta)} = 1$ and $d_{s(\eta')} = d_{t(\eta')} = 1$. Then there exist scalars λ, μ, not both zero, such that $\lambda \cdot N_\eta + \mu \cdot N_{\eta'} = 0$. Equivalently, we have $\lambda \cdot X_\eta + \mu \cdot X_{\eta'} \in I(\overline{\mathcal{O}}_N)$. However, this is impossible, since the polynomials X_ρ and X_η, and X_ρ and $X_{\eta'}$ contain no variables in common. Hence ω is the only (up to cyclic permutations) primitive cycle in Q.

COROLLARY 3.15. *$I(\overline{\mathcal{O}}_N) = (X_\rho)$, where $\rho : i \to j$ is a subpath of the only primitive cycle* (*not being a loop*) *in Q and $d_i = d_j = 1$.*

PROPOSITION 3.16. *The subpath ρ is a relation,* $\operatorname{Ann}(N) = \langle\rho\rangle$ *and* $\operatorname{Ext}^1_{kQ/\langle\rho\rangle}(N, N) = 0$.

Proof. Apply the final part of the proof of Proposition 3.10. ∎

Proof of Theorem 1.4. We have shown that one of the conditions (1), (2), (3) is necessary in Corollary 3.7 (for the condition (1)), in Corollary 3.12 and Proposition 3.13 (for the condition (2)) and in Proposition 3.10, Corollary 3.15 and Proposition 3.16 (for the condition (3)).

Conversely, if the condition (1) is satisfied, then $\operatorname{codim}_{\mathrm{rep}_Q(\mathbf{d})} \overline{\mathcal{O}}_N = 1$. If the condition (2) holds, then the orbit $\mathcal{O}_N$ is open in the scheme $\mathbf{rep}_{Q,\langle\gamma^2\rangle}(\mathbf{d})$, by Lemma 2.1. Hence $\mathcal{O}_N$ is open in the variety $\mathrm{rep}_{Q,\langle\gamma^2\rangle}(\mathbf{d})$, which is isomorphic to the zero set of the polynomial $X^2_{\gamma,1,1} + X_{\gamma,1,2}X_{\gamma,2,1}$ in the affine space $\{W \in \mathrm{rep}_Q(\mathbf{d}) \mid \mathrm{tr}(W_\gamma) = 0\}$ and is irreducible. Thus $\overline{\mathcal{O}}_N = \mathrm{rep}_{Q,\langle\gamma^2\rangle}(\mathbf{d})$. If the condition (3) holds, the orbit $\mathcal{O}_N$ is open in the variety $\mathrm{rep}_{Q,\langle\rho\rangle}(\mathbf{d})$. Hence $\overline{\mathcal{O}}_N$ is an irreducible component of $\mathrm{rep}_{Q,\langle\rho\rangle}(\mathbf{d})$, and this component is of codimension one in $\mathrm{rep}_Q(\mathbf{d})$.

We see that in all three cases, $\overline{\mathcal{O}}_N$ is a hypersurface, and it is singular, by Theorem 1.2. This finishes the proof of Theorem 1.4. ∎

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Nguyen Quang Loc
Faculty of Mathematics and Computer Science
Hanoi National University of Education
136 Xuan Thuy
Hanoi, Vietnam
E-mail: nqloc2007@gmail.com

Grzegorz Zwara
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: gzwara@mat.umk.pl

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