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REAL HYPERSURFACES WITH PSEUDO-$\mathbb{D}$-PARALLEL STRUCTURE JACOBI OPERATOR IN COMPLEX HYPERBOLIC SPACES

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Abstract. The aim of the present paper is to classify real hypersurfaces with pseudo-$\mathbb{D}$-parallel structure Jacobi operator, in non-flat complex space forms.

1. Introduction. An n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a *complex space form* and is denoted by $M_n(c)$. A complete and simply connected complex space form is complex analytically isometric to a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g). The vector field ξ is defined by $\xi = -JN$, where J is the complex structure of $M_n(c)$ and N is a unit normal vector field.

Real hypersurfaces have been studied by many authors and under several conditions ([B], [BD], [IR1], [IR2], [KR], [T], [T1]).

An important class of hypersurfaces consists of *Hopf hypersurfaces*, that is, real hypersurfaces satisfying $A\xi = \alpha\xi$, where A is the shape operator and $\alpha = g(A\xi, \xi)$.

Several authors have studied real hypersurfaces under conditions which involve the structure Jacobi operator l given by $lX = R_\xi X = R(X, \xi)\xi$ ([OPS], [PS], [PSS]).

In [LPS], H. Lee, J. D. Pérez and Y. J. Suh introduced the notion of *pseudo-$\mathbb{D}$-parallel* structure Jacobi operator, that is, the case where l satisfies the condition

$$(\nabla_X l)Y = \kappa\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\} \tag{1.1}$$

where κ is a non-zero constant, $X \in \mathbb{D}$ and $Y \in TM$. They classified the real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, satisfying (1.1).

However, the problem remains open for the case of $\mathbb{C}H^n$. In the present paper the condition (1.1) is treated in an even more generalized form for

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both $\mathbb{C}P^n$ and $\mathbb{C}H^n$: the constant κ is replaced by a C^2 function satisfying $\kappa + c/4 \neq 0$. Namely we prove the following:

MAIN THEOREM. *Let M be a real hypersurface of a complex space form $M_n(c)$, $n \geq 3$, whose structure Jacobi operator satisfies condition* (1.1) *for some non-vanishing C^2 function κ. Then M is a Hopf hypersurface and κ is a negative constant. Furthermore:*

- *If $M_n(c) = \mathbb{C}P^n$, then M is a geodesic hypersurface of radius r, satisfying* $\cot^2 r = -\kappa$.
- *If $M_n(c) = \mathbb{C}H^n$, then M is:*
  - (i) *a horosphere in $\mathbb{C}H^n$, where $c = 4\kappa$, or*
  - (ii) *a geodesic sphere of radius $r = \frac{1}{\sqrt{|c|}} \ln\big(\frac{2\sqrt{\kappa/c}+1}{2\sqrt{\kappa/c}-1}\big)$, where $4\kappa > c$, or*
  - (iii) *a tube of radius $r = \frac{1}{\sqrt{|c|}} \ln\big(\frac{1+2\sqrt{\kappa/c}}{1-2\sqrt{\kappa/c}}\big)$, where $4\kappa > c$, around a totally geodesic $\mathbb{C}H^{n-1}$.*

2. Preliminaries. Let M_n be a Kählerian manifold of real dimension $2n$, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold: $J^2X = -X$, $G(JX, JY) = G(X, Y)$, $\widetilde{\nabla} J = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection of G.

Let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and $\xi = -JN$, $\eta(X) = g(X, \xi)$, $g = G|_M$.

By properties of the almost complex structure J and the definitions of η and g, the following relations hold [BL]:

$$\text{(2.1)} \quad \text{(i) } \phi^2 = -I + \eta \otimes \xi, \quad \text{(ii) } \eta \circ \phi = 0, \quad \text{(iii) } \phi\xi = 0, \quad \text{(iv) } \eta(\xi) = 1,$$

$$\text{(2.2)} \quad \text{(i) } g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \text{(ii) } g(X, \phi Y) = -g(\phi X, Y).$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). When an almost contact metric structure is defined on M, we can define a local orthonormal basis $\{e_1, \ldots, e_{n-1}, \phi e_1, \ldots, \phi e_{n-1}, \xi\}$, called a *$\phi$-basis*. Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then A is symmetric and

$$\text{(2.3)} \qquad \text{(i) } \nabla_X \xi = \phi AX, \quad \text{(ii) } (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$\begin{aligned}R(X,Y)Z = \frac{c}{4}\big[&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\\ &- 2g(\phi X,Y)\phi Z\big] + g(AY,Z)AX - g(AX,Z)AY,\end{aligned} \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi]. \tag{2.5}$$

The tangent space T_pM, for every point $p \in M$, is decomposed as follows: $T_pM = \mathbb{D}^\perp \oplus \mathbb{D}$, where $\mathbb{D} = \ker(\eta) = \{X \in T_pM : \eta(X) = 0\}$.

Based on the above decomposition, by virtue of (2.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{2.6}$$

where $\beta = |\phi\nabla_\xi\xi|$, α is a smooth function on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in \ker(\eta)$, provided that $\beta \neq 0$. If $A\xi = \alpha\xi$, then ξ is called a *principal vector field.*

Finally, the differentiation of a function f along a vector field X will be denoted by (Xf). All manifolds, vector fields, etc., in this paper are assumed to be connected and of class C^∞.

3. Auxiliary lemmas. Let $\mathcal{N} = \{p \in M : \beta \neq 0$ in a neighborhood of $p\}$. We define the open subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{N}$ by setting

$$\begin{aligned}\mathcal{N}_1 &= \{p \in \mathcal{N} : \alpha \neq 0 \text{ in a neighborhood of } p\},\\ \mathcal{N}_2 &= \{p \in \mathcal{N} : \alpha = 0 \text{ in a neighborhood of } p\}.\end{aligned}$$

Then $\mathcal{N}_1 \cup \mathcal{N}_2$ is open and dense in the closure of $\mathcal{N}$.

Lemma 3.1. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1)*. Then the following relations hold on $\mathcal{N}_1$:*

$$AU = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)U + \beta\xi + \lambda W, \quad A\phi U = \Big(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\Big)\phi U + \mu Z, \tag{3.1}$$

$$\begin{aligned}&\nabla_\xi\xi = \beta\phi U, \quad \nabla_U\xi = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\phi U + \lambda\phi W,\\ &\nabla_{\phi U}\xi = -\Big(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\Big)U + \mu\phi Z,\end{aligned} \tag{3.2}$$

$$\nabla_\xi U = W_1, \quad \nabla_U U = W_2, \quad \nabla_{\phi U}U = W_3 + \Big(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\Big)\xi, \tag{3.3}$$

$$\begin{aligned}&\nabla_\xi(\phi U) = \phi W_1 - \beta\xi, \quad \nabla_U(\phi U) = \phi W_2 - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\xi,\\ &\nabla_{\phi U}(\phi U) = \phi W_3,\end{aligned} \tag{3.4}$$

where W_1, W_2, W_3 are vector fields on $\mathbb{D}$ satisfying $W_1, W_2, W_3 \perp U$, W_1, W_2 $\perp \xi$ and W, Z are vector fields in $\mathrm{span}^\perp\{U, \phi U, \xi\}$.

Proof. From (2.4) we get

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - g(AX, \xi)A\xi, \tag{3.5}$$

which for $X = U$ yields

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi. \tag{3.6}$$

The scalar product of (3.6) with U yields

$$g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \tag{3.7}$$

where $\gamma = g(lU, U)$.

In addition, from (1.1), we have $(\nabla_{\phi U} l)\xi = \kappa\phi A\phi U$, which is expanded by virtue of (2.3) and (3.5), giving

$$\left(\frac{c}{4} + \kappa\right)\phi A\phi U + \alpha A\phi A\phi U + \beta g(A\phi U, \phi U)\xi = 0.$$

The inner product of the above relation with ϕU, because of the symmetry of the shape operator A and (2.2)(ii), implies

$$g(AU, \phi U) = 0. \tag{3.8}$$

The symmetry of A and (2.6) imply

$$g(AU, \xi) = \beta. \tag{3.9}$$

From relations (3.7)–(3.9), we obtain

$$AU = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U + \beta\xi + \lambda W, \tag{3.10}$$

where $W \in \text{span}^{\perp}\{U, \phi U, \xi\}$. Combining the last equation with (3.6) we obtain

$$lU = \gamma U + \lambda\alpha W. \tag{3.11}$$

Equation (3.5), for $X = \phi U$, gives $l\phi U = (c/4)\phi U + \alpha A\phi U$, whose inner product with ϕU (due to (3.8)) leads to

$$g(A\phi U, \phi U) = \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right), \tag{3.12}$$

where $\epsilon = g(l\phi U, \phi U)$. Furthermore, the symmetry of A and (2.6) give

$$g(A\phi U, \xi) = 0. \tag{3.13}$$

Therefore, from (3.8), (3.12) and (3.13) we conclude the second relation of (3.1). Using (2.3)(i), for $X = \xi$, $X = U$, $X = \phi U$ and by virtue of (2.6), (3.1), we obtain (3.2).

It is well known that

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{3.14}$$

Let us set $\nabla_\xi U = W_1$ and $\nabla_U U = W_2$. If we use (3.1), (3.2) and (3.14), it is easy to verify that $g(\nabla_\xi U, U) = 0 = \eta(\nabla_\xi U)$ and $g(\nabla_U U, U) = 0 = \eta(\nabla_U U)$, which means $W_1 \perp \{\xi, U\}$ and $W_2 \perp \{\xi, U\}$.

On the other hand, using (3.14) and (3.2) we find $\eta(\nabla_{\phi U} U) = \epsilon/\alpha - c/4\alpha$ and $g(\nabla_{\phi U} U, U) = 0$ which means that $\nabla_{\phi U} U$ is decomposed as $\nabla_{\phi U} U = W_3 + (\epsilon/\alpha - c/4\alpha)\xi$, $W_3 \perp U$. We also observe that

$$\begin{aligned} g(W_3, \xi) &= g\left(\nabla_{\phi U} U + \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\xi, \xi\right) = g(\nabla_{\phi U} U, \xi) + \frac{\epsilon}{\alpha} - \frac{c}{4\alpha} \\ &= (\phi U g(U, \xi)) - g(U, \nabla_{\phi U}\xi) + \frac{\epsilon}{\alpha} - \frac{c}{4\alpha}, \end{aligned}$$

which by virtue of (3.2) yields $g(W_3, \xi) = 0$. So (3.3) has been proved too.

In order to prove (3.4), we use (2.3)(ii) with (i) $X = \xi$, $Y = U$, (ii) $X = Y = U$, (iii) $X = \phi U$, $Y = U$, combined with (3.1), (3.3). ∎

Let $X \in \operatorname{span}^\perp\{U, \phi U, \xi\}$. Then (1.1) implies that $\nabla_X l\phi X - l\nabla_X \phi X = \kappa g(AX, X)\xi$. Taking the inner product of the last relation with ξ and using (3.5) and (2.3)(i) we obtain

$$\left(\frac{c}{4} + \kappa\right) g(AX, X) = -\alpha g(A\phi X, \phi AX). \tag{3.15}$$

Similarly, (1.1) yields $\nabla_{\phi X} lX - l\nabla_{\phi X} X = -\kappa g(A\phi X, \phi X)\xi$, whose inner product with ξ has the form

$$\left(\frac{c}{4} + \kappa\right) g(A\phi X, \phi X) = -\alpha g(A\phi X, \phi AX).$$

The above equation and (3.15) lead to

$$g(AX, X) = g(A\phi X, \phi X), \qquad \forall X \in \operatorname{span}^\perp\{U, \phi U, \xi\}. \tag{3.16}$$

Lemma 3.2. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying (1.1). Then $\lambda = \mu = 0$ on $\mathcal{N}_1$.*

Proof. Condition (1.1) yields $(\nabla_{\phi U} l)\xi = \kappa\phi A\phi U$, which is further expanded with the aid of Lemma 3.1, giving

$$\left(\frac{c}{4} + \kappa\right)\phi A\phi U + \alpha A\phi A\phi U + \beta\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)A\xi = 0.$$

The inner products of the above equation with U, ϕZ and W (with the aid of Lemma 3.1 and (3.16)) yield, respectively,

$$\begin{aligned} &\text{(i) } (\kappa + \gamma)\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) + \alpha\lambda\mu g(Z, \phi W) = 0, \\ &\text{(ii) } \left(\frac{c}{4} + \kappa\right)\mu - \left(\epsilon - \frac{c}{4}\right)\lambda g(W, \phi Z) + \alpha\mu g(AZ, Z) = 0, \\ &\text{(iii) } \left(\frac{c}{4} + \kappa\right)\mu g(W, \phi Z) - \left(\epsilon - \frac{c}{4}\right)\lambda + \alpha\mu g(A\phi Z, W) = 0. \end{aligned} \tag{3.17}$$

Similarly, condition (1.1) yields $(\nabla_U l)\xi = \kappa\phi AU$ which is further expanded with the aid of Lemma 3.1, giving

$$\left(\frac{c}{4}+\kappa\right)\phi AU + \alpha A\phi AU = 0.$$

The inner products of the above equation with ϕU and Z yield, respectively,

$$\text{(3.18)}\quad \begin{aligned} &\text{(i)}\ (\kappa+\epsilon)\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\right)+\lambda\mu g(Z,\phi W)=0,\\ &\text{(ii)}\ \left(\frac{c}{4}+\kappa\right)\lambda g(Z,\phi W)+\left(\gamma-\frac{c}{4}+\beta^2\right)\mu+\alpha\lambda g(A\phi W,Z)=0. \end{aligned}$$

Again from (1.1) we have $(\nabla_{\phi Z} l)\xi = \kappa\phi A\phi Z$ which is further expanded with the aid of Lemma 3.1, (2.2)(ii), (2.3)(i) and (3.5), giving

$$\left(\frac{c}{4}+\kappa\right)\phi A\phi Z + \alpha A\phi A\phi Z = 0.$$

The inner product of the last equation with ϕU, because of (2.2), (3.16) and the symmetry of A, yields

$$(\kappa+\epsilon)\lambda g(W,\phi Z) - \alpha\mu g(AZ,Z) = 0.$$

Combining the above relation with (3.17)(ii) we obtain

$$\text{(3.19)}\qquad \lambda g(W,\phi Z) = \mu.$$

From (1.1), (2.2)(ii) and the symmetry of A we further get $\nabla_Z lU - l\nabla_Z U = -\kappa g(A\phi U, Z)\xi$. Taking the inner product of the last relation with ξ, and using (2.2)(ii), (2.3)(i), (3.14) and the symmetry of A, we find that $\mu = 0$. Therefore (3.17)(iii) yields $(\epsilon - c/4)\lambda = 0$.

Let us assume there exists a point $p_1 \in \mathcal{N}_1$ at which $\lambda \neq 0$. Then there exists a neighborhood V_1 of p_1 such that $\lambda \neq 0$ in V_1. This means $\epsilon = c/4$, and so Lemma 3.1 implies $A\phi U = 0$. Because of the last relation, (1.1) and (2.3)(i), we have

$$(\nabla_W l)\xi = \kappa\phi AW \ \Rightarrow\ l\phi AW = -\kappa\phi AW \ \Rightarrow\ \left(\frac{c}{4}+\kappa\right)\phi AW + \alpha A\phi AW = 0.$$

The inner product of the above equation with ϕU, combined with $A\phi U = 0$, (2.2)(i) and the symmetry of A, gives $\lambda = 0$, which is a contradiction. Therefore there do no not exist points in $\mathcal{N}_1$ at which $\lambda \neq 0$ and so $\lambda = 0$ on $\mathcal{N}_1$. ∎

We define the functions $\kappa_1 = g(\phi U, W_1)$, $\kappa_2 = g(\phi U, W_2)$, $\kappa_3 = g(\phi U, W_3)$, which will be needed very often in what follows. Since $\lambda = \mu = 0$, (3.17)(i) and (3.18) are rewritten as

$$\text{(3.20)}\quad \text{(i)}\ (\kappa+\gamma)\left(\frac{\epsilon}{\alpha}-\frac{c}{4\alpha}\right)=0, \qquad \text{(ii)}\ (\kappa+\epsilon)\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\right)=0.$$

Condition (1.1) yields $(\nabla_U l)U = \kappa g(\phi AU, U)$, which is expanded from Lemmas 3.1, 3.2 giving $(U\gamma)U + \gamma\nabla_U U - lW_2 = 0$. This relation is multiplied by U and ϕU (using also the symmetry of l and (3.5)) giving, respectively,

$$(U\gamma) = 0, \qquad (\gamma - \epsilon)\kappa_2 = 0. \tag{3.21}$$

In a similar way, from (1.1) we have $(\nabla_{\phi U} l)U = \kappa g(\phi A\phi U, U)$, which is expanded to give $(\phi U\gamma)U + \gamma W_3 - lW_3 = 0$. The inner products of this equation with U and ϕU (using also the symmetry of l and (3.5)) yield

$$(\phi U\gamma) = 0, \qquad (\gamma - \epsilon)\kappa_3 = 0. \tag{3.22}$$

Finally, condition (1.1) for $X = Y = \phi U$ gives $(\phi U\epsilon)\phi U + \epsilon\phi W_3 - l\phi W_3 = 0$. The inner product of this equation by ϕU (using also the symmetry of l and (3.5)) yields

$$(\phi U\epsilon) = 0. \tag{3.23}$$

Lemma 3.3. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1)*. Then $\kappa = -\gamma$ on $\mathcal{N}_1$.*

Proof. Let us assume there exists a point $p_2 \in \mathcal{N}_1$ at which $\kappa \neq -\gamma$. Then there exists a neighborhood V_2 of p_2 such that $\kappa \neq -\gamma$ in V_2. This means $\epsilon = c/4$ (due to (3.20)(i)). Since $\epsilon = c/4$, (3.20)(ii) yields $\gamma - \epsilon + \beta^2 = 0$ (also because $\kappa + c/4 \neq 0$). So we have proved that

$$\gamma - \epsilon = -\beta^2 \neq 0. \tag{3.24}$$

By making use of (3.21)–(3.24) we obtain $(\phi U\beta) = 0$ and $\kappa_2 = 0$. We are going to combine the last two equations with $\epsilon = c/4$, Lemmas 3.1, 3.2, (3.24) and (2.5):

$$\begin{aligned}(\nabla_U A)\phi U - (\nabla_{\phi U} A) = -\frac{c}{2}\xi \ &\Rightarrow -A(\nabla_U \phi U) - \nabla_{\phi U} AU + A(\nabla_{\phi U} U) = -\frac{c}{2}\xi\\ &\Rightarrow -A\phi W_2 + AW_3 = -\frac{c}{2}\xi.\end{aligned}$$

The inner product of the above equation with ξ, because of the symmetry of A, (2.2)(ii), (2.6), Lemmas 3.1, 3.2 and $\kappa_2 = 0$, $A\phi U = 0$ yields $c = 0$, which is a contradiction. Therefore, there do not exist points in $\mathcal{N}_1$ at which $\kappa \neq -\gamma$ and so $\kappa = -\gamma$ on $\mathcal{N}_1$. ∎

Lemma 3.4. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1)*. Then $\gamma = \epsilon$ on $\mathcal{N}_1$.*

Proof. Let us assume there exists a point $p_3 \in \mathcal{N}_1$ at which $\gamma \neq \epsilon$. Then there exists a neighborhood V_3 of p_3 such that $\gamma \neq \epsilon$ in V_3. So, from Lemma 3.3 and equations (3.20)–(3.22) we obtain $\kappa_2 = \kappa_3 = 0$ and $\gamma - c/4 + \beta^2 = 0$. In addition the differentiation of the last relation along U and ϕU, by virtue of (3.21), (3.22), yields $(U\beta) = (\phi U\beta) = 0$. Summarizing the relations that hold on V_3, we have

$$\text{(3.25)}\quad \text{(i) } \kappa_2 = \kappa_3 = 0, \quad \text{(ii) } \gamma - \frac{c}{4} + \beta^2 = 0, \quad \text{(iii) } (U\beta) = (\phi U\beta) = 0.$$

From (2.5) we deduce $(\nabla_U A)\xi - (\nabla_\xi A)U = -(c/4)\phi U$, which is expanded with the aid of Lemmas 3.1–3.3 and relation (3.25), to give $[(U\alpha) - (\xi\beta)]\xi + \beta W_2 - \beta^2\phi U + AW_1 = -(c/4)\phi U$. The inner product of this equation with ϕU (because of (3.25), the symmetry of A and Lemmas 3.1, 3.2) leads to

$$-\beta^2 + \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\kappa_1 = -\frac{c}{4}. \tag{3.26}$$

Again from (2.5) we have $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = (c/4)U$, which is expanded in a similar way to give

$$\left[(\phi U\alpha) + 3\beta\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) - \alpha\beta\right]\xi - [\epsilon + \beta^2]U - \xi\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\phi U$$
$$+ \beta W_3 - \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\phi W_1 + A\phi W_1 = 0.$$

The inner product of the last equation by U yields

$$-\beta^2 + \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\kappa_1 = \epsilon.$$

Comparing the above relation with (3.26) we are led to $\epsilon = -c/4$, which by virtue of Lemmas 3.1, 3.2 implies

$$A\phi U = -\frac{c}{2\alpha}\phi U.$$

We make use of the last two equations and (3.25), to write $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -(c/2)\xi$ and obtain

$$\frac{c}{2\alpha^2}\phi U - \frac{c}{2\alpha}\phi W_2 - A\phi W_2 - \frac{\beta c}{\alpha}U,$$

whose inner product with U gives $c\beta = 0$, which is a contradiction. Therefore, there do not exist points in $\mathcal{N}_1$ at which $\gamma \neq \epsilon$ and we have $\gamma = \epsilon$ on $\mathcal{N}_1$. ∎

Next we make use of (2.5) with the following substitutions: (i) $X = U$, $Y = \xi$, (ii) $X = \phi U$, $Y = \xi$, (iii) $X = U$, $Y = \phi U$, with the aid of Lemmas 3.1–3.4.

Case (i).

$$[(U\alpha) - (\xi\beta)]\xi + \left[(U\beta) - \xi\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\right]U$$
$$+\left[\gamma - \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right)\right]\phi U + \beta W_2 - \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)W_1 + AW_1 = 0.$$

The inner products of the above equation with ξ, U and ϕU yield, respectively,

$$\text{(3.27)}\qquad \begin{aligned} &\text{(i) } (U\alpha) = (\xi\beta), \qquad \text{(ii) } (U\beta) = \xi\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big),\\ &\text{(iii) } \gamma + \kappa_2\beta - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) - \kappa_1\frac{\beta^2}{\alpha} = 0. \end{aligned}$$

Case (ii).

$$\begin{aligned} &\Big[(\phi U\alpha) + 3\beta\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) - \alpha\beta\Big]\xi\\ &\quad + \Big[(\phi U\beta) + \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) - \beta^2\Big]U\\ &\quad - \xi\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)\phi U + \beta W_3 - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)\phi W_1 + A\phi W_1 = 0. \end{aligned}$$

The inner products of the above equation with $\xi, \phi U$ and U yield, respectively,

$$\text{(3.28)}\qquad \begin{aligned} &\text{(i) } (\phi U\alpha) + 3\beta\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) - \alpha\beta - \kappa_1\beta = 0, \text{ (ii) } \beta\kappa_3 = \xi\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big),\\ &\text{(iii) } \gamma - (\phi U\beta) - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) + \kappa_1\frac{\beta^2}{\alpha} + \beta^2 = 0. \end{aligned}$$

Case (iii).

$$\begin{aligned} &\Big[-2\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) + \gamma + \beta^2 - (\phi U\beta)\Big]\xi\\ &\quad + \Big[3\beta\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) + \frac{\beta^3}{\alpha} - \phi U\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\Big]U + U\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)\phi U\\ &\quad + \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)\phi W_2 - A\phi W_2 - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)W_3 + AW_3 = 0. \end{aligned}$$

The inner products of the above equation with ϕU and U yield, respectively,

$$\text{(3.29)}\qquad \begin{aligned} &\text{(i) } \kappa_2\frac{\beta^2}{\alpha} + 3\beta\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) + \frac{\beta^3}{\alpha} - \phi U\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big) = 0,\\ &\text{(ii) } U\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) = \kappa_3\frac{\beta^2}{\alpha}. \end{aligned}$$

We analyze (3.29)(i) by replacing the terms $(\phi U\alpha)$, $(\phi U\beta)$ from (3.28)(i) and (3.28)(iii), to obtain

$$\phi U\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) = \frac{3\beta}{\alpha}\Big[\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)^2 - \frac{c}{4}\Big].$$

The last relation, because of (3.22) and (3.28)(i), yields

$$\text{(3.30)}\qquad \kappa_1\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big) = c - \gamma.$$

Finally, putting $X = Y = W_1$ in (1.1) we get $\nabla_{W_1} lW_1 - l\nabla_{W_1} W_1 = (c/4)g(\phi AW_1, W_1)\xi$. The inner product of the last equality with ξ, combined with (2.2)(ii), (3.5), (3.14) and the restriction $c/4 + \kappa \neq 0$, gives $g(AW_1, \phi W_1) = 0$. So, taking the inner product of the equation in Case (i) with ϕW_1, due to $g(AW_1, \phi W_1) = 0$ and (2.2)(ii), we get $g(\phi W_1, W_2) = 0$. Furthermore, the inner product of the equation in Case (ii) with W_1, because of $g(AW_1, \phi W_1) = 0$, (2.2)(ii) and (3.28), yields $g(W_1, W_3) = \kappa_1\kappa_3$. Summarizing the relations we have proved in this last paragraph, we have

$$g(\phi W_1, W_2) = 0, \qquad g(W_1, W_3) = \kappa_1\kappa_3. \tag{3.31}$$

4. The hypersurface M is Hopf

LEMMA 4.1. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1). *Then γ is constant on $\mathcal{N}_1$.*

Proof. From (3.21) and (3.22) we have $[\phi U, U]\gamma = 0$. However the same Lie bracket is calculated from Lemma 3.1 as

$$[\phi U, U]\gamma = (W_3\gamma) - (\phi W_2\gamma) + \left[2\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \frac{\beta^2}{\alpha}\right](\xi\gamma).$$

Therefore the two expressions for $[\phi U, U]\gamma$ yield

$$(W_3\gamma) - (\phi W_2\gamma) + \left[2\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \frac{\beta^2}{\alpha}\right](\xi\gamma) = 0. \tag{4.1}$$

Moreover, condition (1.1) yields $(\nabla_{W_3} l)U = \kappa g(\phi AW_3, U)\xi$, which is expanded by Lemma 3.2 and (3.11) to give $(W_3\gamma)U + \gamma\nabla_{W_3}U - l\nabla_{W_3}U = \kappa g(\phi AW_3, U)\xi$. The inner product of this equation with U, due to Lemma 3.2, the symmetry of l, (3.11) and (3.14), yields

$$(W_3\gamma) = 0. \tag{4.2}$$

In a similar way from (1.1) we have $(\phi W_2\gamma)U + \gamma\nabla_{\phi W_2}U - l\nabla_{\phi W_2}U = \kappa g(\phi A\phi W_2, U)\xi$, whose inner product with U yields

$$(\phi W_2\gamma) = 0.$$

The above equation combined with (4.1), (4.2) leads to $[2(\gamma/\alpha - c/4\alpha) + \beta^2/\alpha](\xi\gamma) = 0$.

Let us assume there exists a point $p_4 \in \mathcal{N}_1$ at which $(\xi\gamma) \neq 0$. Then there exists a neighborhood V_4 of p_4 such that $(\xi\gamma) \neq 0$ in V_4. Therefore, from the last inequality and $[2(\gamma/\alpha - c/(4\alpha)) + \beta^2/\alpha](\xi\gamma) = 0$ we get

$$2\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \frac{\beta^2}{\alpha} = 0,$$

which is rewritten as $\gamma/\alpha - c/(4\alpha) + \beta^2/\alpha = -(\gamma/\alpha - c/(4\alpha))$. Differentiating the last equation along ξ and using (3.27), (3.28) we obtain

$$(U\beta) = \kappa_3\beta. \tag{4.3}$$

But since $2(\gamma/\alpha - c/(4\alpha)) + \beta^2/\alpha = 0$ in V_4, we get $2(\gamma - c/4) + \beta^2 = 0$, which is differentiated along U (also with the help of (3.21)), giving $(U\beta) = 0$. The last equality is combined with (4.3) leading to $\kappa_3 = 0$. From $\kappa_3 = 0$, (3.21), (3.29) and (3.27)(i) we have $(\xi\beta) = 0$. Since $(\xi\beta) = 0$, the differentiation of $2(\gamma - c/4) + \beta^2 = 0$ along ξ gives $(\xi\gamma) = 0$, which is a contradiction on V_4.

Therefore there do not exist points on $\mathcal{N}_1$ at which $(\xi\gamma) \neq 0$ and so $(\xi\gamma) = 0$ on $\mathcal{N}_1$.

Now, for every vector field $X \in \operatorname{span}^\perp\{U, \phi U, \xi\}$, condition (1.1) yields $\nabla_X lU - l\nabla_X U = \kappa g(\phi AX, U)\xi$ and hence $(X\gamma)U + \gamma\nabla_X U - l\nabla_X U = \kappa g(\phi AX, U)\xi$, the inner product of which with U, in view of (3.11), (3.14) and Lemma 3.2, yields $(X\gamma) = 0$.

From the last equation, $(\xi\gamma) = 0$ and (3.21), (3.22) the lemma follows. ∎

LEMMA 4.2. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1). *Then $\kappa_3 = 0$ on $\mathcal{N}_1$.*

Proof. From Lemma 4.1, (3.27), (3.28)(ii) and (3.29)(ii) we obtain

$$\begin{aligned} &(U\alpha) = (\xi\beta) = -\frac{\alpha\beta^2}{\gamma - c/4}\kappa_3, \qquad (\xi\alpha) = -\frac{\alpha^2\beta}{\gamma - c/4}\kappa_3, \\ &(U\beta) = \beta\left[1 - \frac{\beta^2}{\gamma - c/4}\right]\kappa_3. \end{aligned} \tag{4.4}$$

By using (4.4), we differentiate (3.30) along U and ξ, respectively, to get

$$(U\kappa_1) = -\frac{\kappa_1\beta^2}{\gamma - c/4}\kappa_3, \qquad (\xi\kappa_1) = -\frac{\kappa_1\alpha\beta}{\gamma - c/4}\kappa_3. \tag{4.5}$$

From (2.5) we have $\nabla_{W_3}A\xi - A\nabla_{W_3}\xi - \nabla_\xi AW_3 + A\nabla_\xi W_3 = -(c/4)\phi W_3$, which is expanded using (2.3)(i) and (2.6) to give $(W_3\alpha)\xi + \alpha\phi AW_3 + (W_3\beta)U + \beta\nabla_{W_3}U - A\phi AW_3 - \nabla_\xi AW_3 + \nabla_\xi AW_3 = -(c/4)\phi W_3$. Taking the inner product of the last relation with ξ and applying (2.2)(ii), (2.3)(i), (2.6), (3.14), the symmetry of A and Lemmas 3.1, 3.2, we see that

$$(W_3\alpha) = \left[-3\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \alpha + 1\right]\beta\kappa_3. \tag{4.6}$$

In a similar way, from (2.5) we have $(\nabla_{\phi W_2}A)\xi - (\nabla_\xi A)\phi W_2 = (c/4)W_2$, which is rewritten as $(\phi W_2\alpha)\xi + \alpha\phi A\phi W_2 + (\phi W_2\beta)U + \beta\nabla_{\phi W_2}U - A\phi A\phi W_2 - \nabla_\xi A\phi W_2 + A\nabla_\xi\phi W_2 = (c/4)W_2$. Taking the inner product of the last equation with ξ and making similar calculations to those in the proof of (4.6) we are led to the equality

$$(\phi W_2\alpha) = \frac{\alpha\beta^2}{\gamma - c/4}\kappa_2\kappa_3. \tag{4.7}$$

Finally, (2.5) for $X = \phi W_1$, $Y = \xi$ gives $(\phi W_1\alpha)\xi + \alpha\phi A\phi W_1 + (\phi W_1\beta)U + \beta\nabla_{\phi W_1}U - A\phi A\phi W_1 - \nabla_\xi A\phi W_1 + A\nabla_\xi\phi W_2 = (c/4)W_1$, the inner product of which with U yields (in a similar way to (4.6) and (4.7))

$$(\phi W_1\beta) = -\kappa_1\beta\left(1 - \frac{\beta^2}{\gamma - c/4}\right)\kappa_3. \tag{4.8}$$

By virtue of (3.28)(i), (3.29)(ii), (4.4) and (4.5), the Lie bracket $[\phi U, U]\alpha = (\phi U(U\alpha)) - (U(\phi U\alpha))$ is calculated as follows:

$$[\phi U, U]\alpha = (\phi U(U\alpha)) + \beta\left[3\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) - \kappa_1 - \alpha + \frac{2\kappa_1\beta^2}{\gamma - c/4} + \frac{2\alpha\beta^2}{\gamma - c/4}\right]\kappa_3.$$

However the same Lie bracket is calculated from $[\phi U, U]\alpha = (\nabla_{\phi U}U - \nabla_U\phi U)\alpha$, Lemmas 3.1, 3.4, and (4.6), (4.7), giving

$$[\phi U, U]\alpha = \beta\left[-3\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \kappa_1 - \alpha - \frac{\kappa_2\alpha\beta}{\gamma - c/4} - \frac{\alpha\beta^2}{\gamma - c/4}\right]\kappa_3.$$

Comparing the two expressions for $[\phi U, U]\alpha$ we end up with

$$(\phi U(U\alpha)) = \beta\left[-6\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + 2\kappa_1 - \frac{2\kappa_1\beta^2}{\gamma - c/4} - \frac{\kappa_2\alpha\beta}{\gamma - c/4} - \frac{3\alpha\beta^2}{\gamma - c/4}\right]\kappa_3. \tag{4.9}$$

The Lie bracket $[\phi U, \xi]\beta = (\phi U(\xi\beta)) - (\xi(\phi U\beta))$ is obtained from (3.27)(i), (3.28)(ii), (3.28)(iii), (4.4) and Lemma 4.1:

$$[\phi U, \xi]\beta = (\phi U(\xi\beta)) + \beta\left[2\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \frac{2\kappa_1\beta^2}{\gamma - c/4} + \frac{2\alpha\beta^2}{\gamma - c/4}\right]\kappa_3.$$

In addition we have $[\phi U, \xi]\beta = (\nabla_{\phi U}\xi - \nabla_\xi\phi U)\beta$, which is further expanded with the aid of Lemmas 3.1, 3.2, 3.4 and (4.4), (4.8) as

$$[\phi U, \xi]\beta = \beta\left[-\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \frac{\beta^2}{\alpha} + \kappa_1 - \frac{\kappa_1\beta^2}{\gamma - c/4} - \frac{\alpha\beta^2}{\gamma - c/4}\right]\kappa_3.$$

The two expressions for $[\phi U, \xi]\beta$ yield

$$(\phi U(\xi\beta)) = \beta\left[-3\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \kappa_1 + \frac{\beta^2}{\alpha} - \frac{3\kappa_1\beta^2}{\gamma - c/4} - \frac{3\alpha\beta^2}{\gamma - c/4}\right]\kappa_3. \tag{4.10}$$

We equate (4.9) with (4.10) (since (3.27) holds) and replace the terms κ_1, κ_2 using (3.30) and (3.27)(iii), which leads to

$$\left[4\beta\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) - \frac{c}{\gamma/\alpha - c/(4\alpha)} + \frac{2\beta^3}{\alpha}\right]\kappa_3 = 0.$$

Let us assume there exists a point $p_5 \in \mathcal{N}_1$ at which $\kappa_3 \neq 0$. So there exists a neighborhood V_5 of p_5 such that $\kappa_3 \neq 0$ in V_5. Then from the above equation we have

$$4\beta\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\right)-\frac{c}{\gamma/\alpha-c/(4\alpha)}+\frac{2\beta^3}{\alpha}=0, \tag{4.11}$$

which is rewritten as

$$\frac{4\beta}{\alpha^2}\left(\gamma-\frac{c}{4}\right)^2-c+\frac{2\beta^3}{\alpha^2}\left(\gamma-\frac{c}{4}\right)=0.$$

The differentiation of the last equation along ξ, due to Lemma 4.1 and (4.4), yields $[4\beta(\gamma/\alpha-c/4\alpha)-\beta^3/\alpha-2(\gamma-c/4)]\kappa_3$, which implies

$$4\beta\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\right)-\frac{\beta^3}{\alpha}=2\left(\gamma-\frac{c}{4}\right),$$

since $\kappa_3\neq 0$ on V_5. Combining the above relation with (4.11) we get

$$\frac{3\beta^3}{\alpha}+2\left(\gamma-\frac{c}{4}\right)=-\frac{\alpha c}{\gamma-c/4}. \tag{4.12}$$

Equation (4.12) is differentiated along ξ and, because of (4.4), $\kappa_3\neq 0$, so that we obtain

$$-\frac{6\beta^3}{\alpha}=\frac{\alpha c}{\gamma-c/4}.$$

The last equation and (4.12) give

$$\frac{3\beta^3}{\alpha}=2\left(\gamma-\frac{c}{4}\right). \tag{4.13}$$

From (4.12) and (4.13) we get

$$4\left(\gamma-\frac{c}{4}\right)=-\frac{\alpha c}{\gamma-c/4}.$$

Differentiating this equation along ξ and using (4.4), we have $\kappa_3=0$, which is a contradiction on V_5. Hence we conclude that $V_5=\emptyset$ and $\kappa_3=0$ on $\mathcal{N}_1$. ∎

LEMMA 4.3. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1). *Then $\mathcal{N}_1=\emptyset$.*

Proof. From Lemma 4.2 and (4.4) we have $[U,\xi]\alpha=0$. In addition, from Lemmas 3.1, 3.2, 3.4 we have

$$[U,\xi]\alpha=(\nabla_U\xi-\nabla_\xi U)\alpha=\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\right)(\phi U\alpha)-(W_1\alpha).$$

So we conclude that

$$\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\right)(\phi U\alpha)-(W_1\alpha)=0. \tag{4.14}$$

In order to obtain the term $(W_1\alpha)$ we make use of (2.5) for $X = W_1$, $Y = \xi$, which results in

$$(W_1\alpha)\xi + \alpha\phi AW_1 + (W_1\beta)U + \beta\nabla_{W_1}U - A\phi AW_1 - \nabla_\xi AW_1 + A\nabla_\xi W_1 = -\frac{c}{4}\phi W_1.$$

We take the inner product of the above equation with ξ and make use of (2.2)(ii), (2.3)(i), (2.6), (3.14) and Lemmas 3.1, 3.2, to get

$$(W_1\alpha) = -3\beta\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right)\kappa_1 + \alpha\beta\kappa_1 + \beta|W_1|^2.$$

The combination of the above relation with (4.14), (3.28)(i) and (3.30) leads eventually to

$$-\frac{c-\gamma}{\gamma - c/4}\alpha^4 + \left[3c - 3\left(\gamma - \frac{c}{4}\right) + \beta^2\frac{c-\gamma}{\gamma - c/4} + \beta^2\right]\alpha^2 - 3\left(\gamma - \frac{c}{4}\right)^2 - 3\beta^2\left(\gamma - \frac{c}{4}\right)^2 = \alpha^2|W_1|^2.$$

Putting $\gamma - c/4 = C_1 = \text{const} \neq 0$ (due to Lemmas 3.3, 4.1), we may rewrite the above relation as

$$-\frac{3c/4 - C_1}{C_1}\alpha^4 + \left[3c - 3C_1 + \frac{\beta^2}{C_1}\left(\frac{3c}{4} - C_1\right) + \beta^2\right]\alpha^2 - [3{C_1}^2 + 3C_1\beta^2] = \alpha^2|W_1|^2. \tag{4.15}$$

Because of (4.15), the quadratic function

$$f(\alpha) = -\frac{3c/4 - C_1}{C_1}\alpha^4 + \left[3c - 3C_1 + \frac{\beta^2}{C_1}\left(\frac{3c}{4} - C_1\right) + \beta^2\right]\alpha^2 - [3{C_1}^2 + 3C_1\beta^2]$$

is non-negative for every α. We are going to prove that $f(\alpha)$ is strictly positive.

If instead we had $f(\alpha) = 0$, then $W_1 = 0$ and so $\kappa_1 = g(\phi U, W_1) = 0$. In addition, from (3.30) we would have $\gamma = c$. Using $W_1 = \kappa_1 = 0$, $\gamma = c$, (3.28)(i) and (4.14), we would obtain

$$\left(\frac{3c}{4} + \beta^2\right)\left(\alpha - \frac{9c}{4\alpha}\right) = 0. \tag{4.16}$$

If we had $3c/4 + \beta^2 = 0$, then (3.28) combined with $\kappa_1 = 0$, $\gamma = c$ would give $c = 0$, which is a contradiction. Therefore $3c/4 + \beta^2 \neq 0$, and (4.16) would yield

$$\alpha^2 = \frac{9c}{4} > 0. \tag{4.17}$$

Moreover, from (4.4) and Lemma 4.2, we would get $[U, \xi]\beta = 0$, which by virtue of $[U, \xi]\beta = (\nabla_U\xi - \nabla_\xi U)\beta$, Lemmas 3.1, 3.2, $\gamma = c$, $W_1 = 0$ and $3c/4 + \beta^2 \neq 0$ would give $(\phi U\beta) = 0$. The last equation, together with

(3.28)(iii), (4.17), $\gamma = c$, $\kappa_1 = 0$, would eventually lead to $\beta^2 = -9c/8$, contradicting (4.17).

Since in the last paragraph we showed that $f(\alpha) \neq 0$, by virtue of (4.15) we have $f(\alpha) > 0$. This can happen only if the discriminant D_f of $f(\alpha)$ is negative. But D_f is calculated to be

$$\begin{aligned} D_f = {}& \frac{9c^2}{16{C_1}^2}\beta^4 + \left[-\frac{9c}{2} + \frac{9c^2}{2C_1} - 12\left(\frac{3c}{4} - C_1\right)\right]\beta^2 9{C_1}^2 \\ &+ 9c^2 - 2cC_1 - 12\left(\frac{3c}{4} - C_1\right). \end{aligned}$$

Thus, D_f cannot always be negative, since it is a quadratic function of β^4 and the coefficient of β^4 is positive. Therefore we have a contradiction and $\mathcal{N}_1 = \emptyset$. ∎

Lemma 4.4. *Let M be a real hypersurface of a complex space form $M_n(c)$ satisfying* (1.1)*. Then the real hypersurface M is Hopf.*

Proof. From Lemma 4.3, we have $\alpha = 0$ on $\mathcal{N}$. So, by virtue of (2.4) and (2.6) we get

$$\begin{aligned} &lX = \frac{c}{4}[X - \eta(X)\xi] - \beta^2 g(X, U)U, \\ &lU = \left(\frac{c}{4} - \beta^2\right)U, \qquad l\phi U = \frac{c}{4}\phi U. \end{aligned} \tag{4.18}$$

Condition (1.1) yields $(\nabla_U l)\xi = \kappa\phi AU$, which is expanded with the help of (4.18), (2.2)(ii) and (2.3)(i), giving

$$-\left(\frac{c}{4} + \kappa\right)\phi AU = g(AU, \phi U)\beta^2 U. \tag{4.19}$$

From (1.1) we have $(\nabla_{\phi U} l)\phi U = \kappa g(\phi A\phi U, \phi U)\xi$. Rewriting this relation with the aid of (4.18), (3.14), (2.3)(i) and (2.2)(i) we obtain $\beta^2 g(\nabla_{\phi U}\phi U, U)U = (c/4 + \kappa)g(AU, \phi U)\xi$. The last equation, with $c/4 + \kappa \neq 0$ and the linear independence of U, ξ, yields $g(AU, \phi U) = 0$. Combining $g(AU, \phi U) = 0$ and (4.19) we obtain $\phi AU = 0$, hence $\phi^2 AU = 0$, so $-AU + g(AU, \xi)\xi = 0$ and therefore

$$AU = \beta\xi. \tag{4.20}$$

Putting $X = \phi U$, $Y = U$ in (1.1) and making use of (2.2)(ii), (2.3)(i), (3.14), (4.18), we have

$$2\beta(\phi U\beta)U + \beta^2\nabla_{\phi U}U = -\left(\frac{c}{4} + \kappa\right)g(A\phi U, \phi U)\xi.$$

Taking the inner product of the above relation with U and ϕU we obtain, respectively,

$$(\phi U\beta) = 0, \quad g(A\phi U, \phi U) = 0. \tag{4.21}$$

Next we make use of (4.20) and (4.21) in order to expand $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -(c/2)\xi$ (which holds due to (2.5)); this leads to

$$\nabla_U A\phi U - A\nabla_U \phi U - \beta\nabla_{\phi U}\xi + A\nabla_{\phi U} U = -\frac{c}{2}\xi.$$

The inner product of the above relation with ξ, combined with (3.14), (4.20), (4.21) and (2.3)(i), gives

$$c = 2\beta g(\nabla_U U, \phi U). \tag{4.22}$$

But from (1.1) and (4.20) we have $\nabla_U lU - l\nabla_U U = 0$, which is expanded, using (3.14), (4.18), (4.20), to give $2(U\beta)U + \beta\nabla_U U = 0$. The inner product of the last equation with ϕU gives $g(\nabla_U U, \phi U) = 0$, which shows, due to (4.22), that $c = 0$. We have arrived at a contradiction, which means that $\mathcal{N}_2 = \emptyset$. From Lemma 4.3 and since $\mathcal{N}_1 \cup \mathcal{N}_2$ is open and dense in the closure of $\mathcal{N}$, we have $\mathcal{N} = \emptyset$. So, the real hypersurface M consists only of points where $\beta = 0$, i.e. M is a Hopf hypersurface. ■

5. The classification. Let $\{e_i, \phi e_i, \xi\}$, $i = 1, \ldots, n-1$, be a local ϕ-basis. If we had $\alpha = 0$ then from (2.4) it would follow that

$$lX = \frac{c}{4}[X - \eta(X)\xi], \quad le_i = \frac{c}{4}e_i, \quad l\phi e_i = \frac{c}{4}\phi e_i. \tag{5.1}$$

Therefore, putting $X = e_i$, $Y = \xi$ in (1.1), and using (2.3), (5.1), $c/4+\kappa \neq 0$ we get $Ae_i = 0$. In a similar way putting $X = \phi e_i$, $Y = \xi$ in (1.1) we obtain $A\phi e_i = 0$. So we have shown that $A = 0$. Applying (2.5) to $X = e_i$, $Y = \phi e_i$ we have $c = 0$, which is a contradiction. Thus, the function α must be non-zero. According to [NR] the function α must be constant.

Due to symmetry of A, the vector fields $Ae_i, A\phi e_i$ are decomposed as follows:

$$Ae_i = \sum_j \lambda_{ij}e_j + \sum_j \mu_{ij}\phi e_j, \quad A\phi e_i = \sum_j \mu_{ji}e_j + \sum_j \nu_{ij}\phi e_j, \tag{5.2}$$

where $\lambda_{ij} = g(Ae_i, e_j) = g(Ae_j, e_i) = \lambda_{ji}$ $(i \neq j)$. In addition, from (2.4) we have

$$\begin{aligned} lX &= \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \alpha^2\eta(X)\xi, \\ le_i &= \frac{c}{4}e_i + \alpha Ae_i, \quad l\phi e_i = \frac{c}{4}\phi e_i + \alpha A\phi e_i. \end{aligned} \tag{5.3}$$

Condition (1.1) for $X = e_i$, $Y = \xi$, combined with (5.2), (5.3) and (2.3)(i), yields

$$\left(\frac{c}{4} + \kappa\right)\phi Ae_i = -\alpha A\phi Ae_i. \tag{5.4}$$

The inner product of (5.4) with e_i yields

$$\mu_{ii} = 0. \tag{5.5}$$

From (1.1) we have $\nabla_{e_i} l e_j - l\nabla_{e_i} e_j = -\kappa\mu_{ij}\xi$ $(i \neq j)$. The inner product of this relation with ξ, due to (5.1) and (2.3)(i), leads to

$$\Big(\frac{c}{4} + \kappa\Big)\mu_{ij} = \alpha\Big(\sum_k \mu_{ik}\lambda_{jk} - \sum_k \lambda_{ik}\mu_{jk}\Big). \tag{5.6}$$

In a similar way, from (1.1) for $X = e_j$, $Y = e_i$ $(i \neq j)$ we eventually get

$$\Big(\frac{c}{4} + \kappa\Big)\mu_{ji} = \alpha\Big(\sum_k \mu_{ik}\lambda_{jk} - \sum_k \lambda_{ik}\mu_{jk}\Big).$$

So from the above equation and (5.6) we have

$$\mu_{ij} = \mu_{ji}. \tag{5.7}$$

Furthermore, the inner product of (5.4) with e_j $(i \neq j)$, with the aid of (5.2), leads to

$$\Big(\frac{c}{4} + \kappa\Big)\mu_{ij} = \alpha\Big(\sum_k \lambda_{ik}\mu_{jk} - \sum_k \lambda_{jk}\mu_{ik}\Big). \tag{5.8}$$

Equation (5.4) is rewritten as

$$\Big(\frac{c}{4} + \kappa\Big)\phi A e_j = -\alpha A\phi A e_j,$$

whose the inner product with e_i $(i \neq j)$, due to (5.7) and by similar calculations, gives

$$-\Big(\frac{c}{4} + \kappa\Big)\mu_{ij} = \alpha\Big(\sum_k \lambda_{ik}\mu_{jk} - \sum_k \lambda_{jk}\mu_{ik}\Big).$$

The last equation and (5.8) imply that

$$\mu_{ij} = 0. \tag{5.9}$$

From (1.1) we get $\nabla_{e_i} l\phi e_j - l\nabla_{e_i}\phi e_j = \kappa\lambda_{ij}\xi$. The inner product of this relation with ξ, due to (5.1), (5.2), (5.7), (5.9) and (2.3)(i), leads to

$$\Big(\frac{c}{4} + \kappa\Big)\lambda_{ij} = -\alpha\sum_k \lambda_{ik}\nu_{jk}. \tag{5.10}$$

In a similar way we have $\nabla_{\phi e_i} l e_j - l\nabla_{\phi e_i} e_j = -\kappa\nu_{ij}\xi$, the inner product of which with ξ yields

$$\Big(\frac{c}{4} + \kappa\Big)\nu_{ij} = -\alpha\sum_k \lambda_{ik}\nu_{jk}.$$

The above relation and (5.10) lead to

$$\lambda_{ij} = \nu_{ij} \tag{5.11}$$

for all $i, j = 1, \ldots, n-1$. Next we expand $\nabla_{e_i} l\phi e_j - l\nabla_{e_i}\phi e_j = \kappa\lambda_{ij}\xi$ $(i \neq j)$, which holds due to (1.1), with the aid of (5.1), (5.2), (5.5), (5.9), (5.10), getting

$$\alpha(\nabla_{e_i} A)\phi e_j = \left(\frac{c}{4} + \kappa + \alpha^2\right)\lambda_{ij}\xi. \tag{5.12}$$

Similarly, by expanding of $\nabla_{\phi e_j} le_i - l\nabla_{\phi e_j} e_i = -\kappa\lambda_{ij}\xi$ $(i \neq j)$ we obtain

$$\alpha(\nabla_{\phi e_j} A)e_i = -\left(\frac{c}{4} + \kappa + \alpha^2\right)\lambda_{ij}\xi. \tag{5.13}$$

Also from (2.5) we have $(\nabla_{e_i} A)\phi e_j = (\nabla_{\phi e_j} A)e_i$ $(i \neq j)$. Therefore, the last equation, (5.12) and (5.13) give

$$\left(\frac{c}{4} + \kappa + \alpha^2\right)\lambda_{ij} = 0, \qquad i \neq j. \tag{5.14}$$

Similarly, from $\nabla_{e_i} l\phi e_i - l\nabla_{e_i}\phi e_i = \kappa\lambda_{ii}\xi$ and $\nabla_{\phi e_i} le_i - l\nabla_{\phi e_i} e_i = \kappa\lambda_{ii}\xi$ we obtain, respectively, $\alpha(\nabla_{e_i} A)\phi e_i = (c/4 + \kappa + \alpha^2)\lambda_{ii}\xi$ and $\alpha(\nabla_{\phi e_i} A)e_i = -(c/4 + \kappa + \alpha^2)\lambda_{ii}\xi$. The last two equations are combined with $(\nabla_{e_i} A)\phi e_i - (\nabla_{\phi e_i} A)e_i = -(c/2)\xi$ (which holds because of (2.5)) to show

$$\left(\frac{c}{4} + \kappa + \alpha^2\right)\lambda_{ii} = -\frac{\alpha c}{4}. \tag{5.15}$$

Evidently, $c/4 + \kappa + \alpha^2 \neq 0$, otherwise from (5.15) we would have $c = 0$, which is a contradiction. So from (5.2), (5.7), (5.9), (5.11), (5.14), (5.15) we deduce $Ae_i = \lambda_{ii}e_i$, $A\phi e_i = \lambda_{ii}\phi e_i$, where

$$\lambda_{ii} = \frac{-\alpha c}{c + 4\kappa + 4\alpha^2}. \tag{5.16}$$

However, the term λ_{ii} is also calculated from (1.1), for $X = e_i$, $Y = \phi e_i$, giving $\nabla_{e_i} l\phi e_i - l\nabla_{e_i\phi e_i} = \kappa\lambda_{ii}\xi$. The inner product of this equation with ξ yields $\lambda_{ii} = -(c/4\alpha + \kappa/\alpha)$. Therefore, from (5.15), (5.16), $Ae_i = \lambda_{ii}e_i$, $A\phi e_i = \lambda_{ii}\phi e_i$, we have finally proved

$$\begin{gathered} Ae_i = -\left(\frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right)e_i, \qquad A\phi e_i = -\left(\frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right)\phi e_i, \\ \kappa = -\left(\frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right)^2 < 0. \end{gathered} \tag{5.17}$$

Differentiating the last equality of (5.17) along ξ we obtain $(\xi\kappa)[2(c/4 + \kappa) + \alpha^2] = 0$. If we had $(\xi\kappa) \neq 0$ we would also have $2(c/4 + \kappa) + \alpha^2 = 0$, which would mean $\kappa = \text{const}$ and $(\xi\kappa) = 0$, thus a contradiction.

Therefore $(\xi\kappa) = 0$ and by a similar reasoning $(e_i\kappa) = (\phi e_i\kappa) = 0$. This means that the real hypersurface M has two constant principal curvatures, α and $-(c/4\alpha + \kappa/\alpha)$.

In case $M_n(c) = \mathbb{C}P^n$, according to [T1], M can only be a geodesic hypersphere, with $\alpha = 2\cot 2r$, $-(c/4\alpha + \kappa/\alpha) = \cot r$. The last two equations lead to $\cot^2 r = -\kappa$.

In case $M_n(c) = \mathbb{C}H^n$, based on [M], M can be a horosphere (type A_0), a geodesic sphere of radius r, $0 < r < \infty$ (type $A_{1,0}$) or a tube of radius r around a totally geodesic $\mathbb{C}H^k$ ($1 \le k \le n-2$), where $0 < r < \infty$ (type $A_{1,1}$). In type A_0 we have

$$\alpha = \sqrt{c}, \qquad -\left(\frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right) = \frac{\sqrt{|c|}}{2}.$$

The last two equations lead to $\kappa = c/4$. In type $A_{1,0}$ we have

$$\alpha = \sqrt{c}\coth(\sqrt{|c|}\,r), \qquad -\left(\frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right) = \frac{\sqrt{|c|}}{2}\coth\left(\frac{\sqrt{|c|}\,r}{2}\right).$$

The last two equations lead to

$$r = \frac{1}{\sqrt{|c|}}\ln\left(\frac{2\sqrt{\kappa/c}+1}{2\sqrt{\kappa/c}-1}\right)$$

where $4\kappa > c$. In type $A_{1,1}$ we have

$$\alpha = \sqrt{c}\coth(\sqrt{|c|}\,r), \qquad -\left(\frac{c}{4\alpha} + \frac{\kappa}{\alpha}\right) = \frac{\sqrt{|c|}}{2}\tanh\left(\frac{\sqrt{|c|}\,r}{2}\right).$$

The last two equations lead to

$$r = \frac{1}{\sqrt{|c|}}\ln\left(\frac{1+2\sqrt{\kappa/c}}{1-2\sqrt{\kappa/c}}\right)$$

where $4\kappa < c$.

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