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PRODUCT RULE AND CHAIN RULE ESTIMATES FOR THE HAJŁASZ GRADIENT ON DOUBLING METRIC MEASURE SPACES

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Abstract. We use the Calderón Maximal Function to prove the Kato–Ponce Product Rule Estimate and the Christ–Weinstein Chain Rule Estimate for the Hajłasz gradient on doubling measure metric spaces.

1. Introduction and definitions. In [4] it was shown that the Kato–Ponce Product Rule Estimate [9] and the Christ–Weinstein Chain Rule Estimate [3] are valid for fractional derivatives on spaces of homogeneous type [5].

Since the Hajłasz gradient [6] is equivalent in L^p to the Calderón Maximal Function of order one [1], [2] (see Theorem 1 below), and the Calderón Maximal Function can be thought of as a "maximal fractional derivative of integral order", it is reasonable to expect that both estimates mentioned before are also valid for the Calderón Maximal Function and consequently for the Hajłasz gradient on doubling measure metric spaces. The purpose of this paper is to prove this claim.

The equivalence between the Hajłasz gradient and the Calderón Maximal Function is also implicit in [7], where Hajłasz and Kinnunen consider the sharp maximal function. We give here a direct proof.

We shall say that (X,d,μ) is a *metric measure space* if (X,d) is a metric space and μ is a Borel measure on X. Let $B_r(x)$ denote the ball of center x and radius $r>0$. If μ satisfies the condition

$$\mu(B_{2r}(x)) \le C_D \mu(B_r(x))$$

with a constant C_D independent of x and r, then (X,d,μ) will be called a *doubling metric measure space*.

Following Hajłasz [6] we shall say that a measurable function f has a *gradient* g in $L^p(X,d,\mu)$ if

$$|f(x)-f(y)| \le d(x,y)(g(x)+g(y)) \tag{1.1}$$

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for all x, $y \in X$. It was shown in [6] that for $p > 1$ there exists a unique $g = f^\nabla$ in L^p satisfying (1.1) and such that

$$\|f^\nabla\|_p = \inf_g \|g\|_p, \tag{1.2}$$

where the infimum is taken over all g satisfying (1.1). We will call f^∇ the *Hajłasz gradient* of f in L^p.

Let f be a measurable function and $u \geq 1$. We define the *Calderón Maximal Function* $N_u(f)$ of order one [1] by

$$N_u(f,x) = \sup_{r>0} \frac{1}{r}\left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|^u \, d\mu(y)\right)^{1/u}.$$

We denote by M_u, $u \geq 1$, the Hardy–Littlewood maximal function

$$M_u(g)(x) = \sup_{r>0} \left[\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |g(y)|^u \, d\mu(y)\right]^{1/u}.$$

We show in Theorem 1 the relationship between f^∇ and $N_u(f)$. In Theorem 2 we prove the main estimate for $N_u(f)$ that is needed to obtain the rules. We prove in Theorem 3 the Product Rule Estimate and in Theorem 4 the Chain Rule Estimate. The letter C will denote a constant, not necessarily the same at each occurrence.

2. Theorems and proofs

THEOREM 1. *Let (X, d, μ) be a doubling metric measure space, f a measurable function on X, $1 < p < \infty$, and $1 \leq u < p$.*

(a) *If f has the Hajłasz gradient in L^p, then there is a constant $C_1(u,p)$ such that*

$$\|N_u(f)\|_p \leq C_1(u,p)\|f^\nabla\|_p.$$

(b) *If $N_u(f)$ is in L^p, then f has the Hajłasz gradient in L^p and there is a constant $C_2(u,p)$ such that*

$$\|f^\nabla\|_p \leq C_2(u,p)\|N_u(f)\|_p.$$

Proof. We first prove (a). Let g satisfy (1.1). Then

$$\begin{aligned}
&\left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|^u \, d\mu(y)\right)^{1/u} \\
&\qquad \leq r\left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} [g(x) + g(y)]^u \, d\mu(y)\right)^{1/u} \\
&\qquad \leq r\left[g(x) + \left(\frac{1}{\mu(B_r(x))} \int_{B_r(x)} [g(y)]^u \, d\mu(y)\right)^{1/u}\right].
\end{aligned}$$

Dividing both sides by r and taking the supremum over r we get

$$N_u(f,x) \le g(x) + M_u(g)(x).$$

Since $p > u$, we have $\|M_u(g)\|_p \le C(u,p)\|g\|_p$, thus

$$\|N_u(f,x)\|_p \le C_1(u,p)\|f^\nabla\|_p.$$

We now prove (b). Let $x, y \in X$ and $r = d(x,y)$. We have

$$(2.1)\quad |f(x)-f(y)| \le \left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(x)-f(z)|^u\,d\mu(z)\right)^{1/u} + \left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(z)-f(y)|^u\,d\mu(z)\right)^{1/u}$$

$$\le \frac{d(x,y)}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(x)-f(z)|^u\,d\mu(z)\right)^{1/u} + \frac{d(x,y)}{r}\left[\frac{\mu(B_{2r}(y))}{\mu(B_r(x))}\right]^{1/u}\left(\frac{1}{\mu(B_{2r}(y))}\int_{B_{2r}(y)} |f(z)-f(y)|^u\,d\mu(z)\right)^{1/u}$$

$$\le d(x,y)[2C_D^{2/u}N_u(f,x) + 2C_D^{2/u}N_u(f,y)].$$

Therefore, from (1.1) and (1.2) it follows that

$$\|f^\nabla\|_p \le 2C_D^{2/u}\|N_u(f)\|_p. \;\blacksquare$$

THEOREM 2. *Let $1 \le u < s \le \infty$ and $1/s = 1/p + 1/q$, $1 < p \le \infty$, $1 < q \le \infty$. If $N_u(f) \in L^p$ and $h \in L^q$, then*

$$\left\|\sup_{r>0}\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(y)-f(x)|^u|h(y)|^u\,d\mu(y)\right)^{1/u}\right\|_s \le C\|N_u(f)\|_p\|h\|_q,$$

where C is a constant independent of f and h.

Proof. We use inequality (2.1) to get

$$\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(y)-f(x)|^u|h(y)|^u\,d\mu(y)\right)^{1/u}$$

$$\le \frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} 2^u d^u(x,y)C_D^2[N_u(f,x)+N_u(f,y)]^u|h(y)|^u\,d\mu(y)\right)^{1/u}$$

$$\le \left(\frac{2^uC_D^2}{\mu(B_r(x))}\int_{B_r(x)} [N_u(f,x)+N_u(f,y)]^u|h(y)|^u\,d\mu(y)\right)^{1/u}$$

$$\leq 2N_u(f,x)C_D^{2/u}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|h(y)|^u\,d\mu(y)\right)^{1/u}$$

$$+\,2C_D^{2/u}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}N_u^u(f,y)|h(y)|^u\,d\mu(y)\right)^{1/u}$$

$$\leq 2C_D^{2/u}N_u(f,x)M_u(h)(x)+2C_D^{2/u}M_u(N_u(f)\cdot h)(x).$$

Now, taking the supremum on the left hand side, then the L^s-norm of both sides, and using the fact that $1\leq u<s\leq q$ we obtain

$$\left\|\sup_{r>0}\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|^u|h(y)|^u\,d\mu(y)\right)^{1/u}\right\|_s$$

$$\leq C[\|N_u(f)\cdot M_u(h)\|_s+\|M_u(N_u(f)\cdot h)\|_s]$$

$$\leq C[\|N_u(f)\|_p\|M_u(h)\|_q+\|N_u(f)\cdot h\|_s]\leq C\|N_u(f)\|_p\|h\|_q. \ \blacksquare$$

THEOREM 3 (Product Rule Estimate). *Let* $1<p_1,p_2<\infty, 1<q_1,q_2\leq\infty$, $1<s\leq\infty$, $1<u<s$ *and* $1/s=1/p_1+1/q_1=1/p_2+1/q_2$. *If* $N_u(f)\in L^{p_1}$, $N_u(g)\in L^{p_2}$, $f\in L^{q_2}$ *and* $g\in L^{q_1}$, *then* $N_u(fg)\in L^s$, *and there is a constant* C *independent of* f *and* g *such that*

$$\|N_u(fg)\|_s\leq C[\|N_u(f)\|_{p_1}\|g\|_{q_1}+\|N_u(g)\|_{p_2}\|f\|_{q_2}].$$

Proof. We write

$$N_u(fg)(x)=\sup_r\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)g(y)-f(x)g(x)|^u\,d\mu(y)\right)^{1/u}$$

$$\leq\sup_r\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|^u|g(y)|^u\,d\mu(y)\right)^{1/u}$$

$$+\sup_r\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|g(y)-g(x)|^u|f(x)|^u\,d\mu(y)\right)^{1/u}$$

$$\leq\sup_r\frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)}|f(y)-f(x)|^u|g(y)|^u\,d\mu(y)\right)^{1/u}$$

$$+\,N_u(g,x)f(x).$$

We now compute the L^s-norm of both sides, and using Theorem 2 and Hölder's inequality to estimate the terms on the right hand side, we obtain

$$\|N_u(fg)\|_s\leq C[\|N_u(f)\|_{p_1}\|g\|_{q_1}+\|N_u(g)\|_{p_2}\|f\|_{q_2}]. \ \blacksquare$$

THEOREM 4 (Chain Rule Estimate). *Let* $F\in C^1(\mathbb{C})$ *and* $H(z)=\sup_{|w|<|z|}|F'(w)|$, *where* $|F'(w)|$ *denotes* $|\nabla\operatorname{Re}(F)(w)|+|\nabla\operatorname{Im}(F)(w)|$. *Let*

$1/s = 1/p + 1/q$, $1 < s \le \infty$, $1 < p \le \infty$, $1 < q \le \infty$, $1 \le u < s$. If $N_u(g) \in L^q$ and $H \circ g \in L^p$, then $N_u(F \circ g) \in L^s$ and

$$\|N_u(F \circ g)\|_s \le C\|H \circ g\|_p \|N_u(g)\|_q$$

with C independent of F and g.

Proof. Observe that

$$\sup_{0\le\lambda\le1} |F'(\lambda z_1 + (1-\lambda)z_2)| \le H(z_1) + H(z_2)$$

for any z_1, z_2. Then applying the Mean Value Theorem we have

$$\begin{aligned}|F(g(y)) - F(g(x))| &\le \sup_{0\le\lambda\le1} |F'(\lambda g(y) + (1-\lambda)g(x))|\,|g(y) - g(x)| \\ &\le [H(g(y)) + H(g(x))]|g(y) - g(x)|.\end{aligned}$$

Therefore

$$N_u(F \circ g, x) = \sup_{r>0} \frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int |F(g(y)) - F(g(x))|^u \, d\mu(y)\right)^{1/u}$$

$$\le \sup_r \frac{1}{r}\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} |g(y)-g(x)|^u H^u(g(y))\, d\mu(y)\right)^{1/u} + H(g(x))N_u(g, x).$$

We now compute the L^s-norm of both sides, and using Theorem 2 and Hölder's inequality to estimate the terms on the right hand side, we obtain

$$\|N_u(F \circ g)\|_s \le C\|H \circ g\|_p \|N_u(g)\|_q. \ \blacksquare$$

Corollary (of Theorem 3). *Let L_1^p be the space of functions f such that $f \in L^p$ and $N_u(f) \in L^p$, $1 \le u < p$. Then $L_1^p \cap L^\infty$ is closed under pointwise multiplication.*

Proof. Let $f, g \in L_1^p \cap L^\infty$. Clearly $f \cdot g$ is in $L^p \cap L^\infty$. On the other hand, from Theorem 3 with $q_1 = q_2 = \infty$ and $p_1 = p_2 = p$ we get $N_u(fg) \in L^p$. $\blacksquare$

Corollary (of Theorem 4, Power Rule Estimate). *Let $F(z) = z^n$, n integer > 1, $1/s = 1/p + 1/q$, $1 < s \le \infty$, $1 < p \le \infty$, $1 < q \le \infty$, $1 \le u < s$. If $g^{n-1} \in L^p$ and $N_u(g) \in L^q$, then $N_u(g^n) \in L^s$ and*

$$\|N_u(g^n)\|_s \le C_n \|g^{n-1}\|_p ||N_u(g)||_q.$$

Proof. This follows from Theorem 4 by observing that $H(z) \le c_n |z^{n-1}|$. $\blacksquare$

Note 1. The explicit formula for H in Theorem 4 was suggested by Michael Christ (personal communication).

Note 2. The Power Rule Estimate can also be proven directly using Theorem 2 and Hölder's inequality, by observing that

$$g^n(x) - g^n(y) = (g(x) - g(y)) \cdot \left(\sum_{k=0}^{n-1} g^{n-1-k}(x) g^k(y)\right)$$

and that

$$\frac{1}{s} = \frac{1}{p} + \frac{n-k-1}{(n-1)q} + \frac{k}{(n-1)q}.$$

NOTE 3. Observe that the definition of the Calderón Maximal Function and inequality (2.1) imply that "f is a Lipschitz function of order 1 if and only if $N_u(f)$ is in L^∞, and $\|f\|_{\mathrm{Lip}(1)}$ is equivalent to $\|N_u(f)\|_\infty$".

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