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ON CYCLIC VERTICES IN VALUED TRANSLATION QUIVERS

BY

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Abstract. Let x and y be two vertices lying on an oriented cycle in a connected valued translation quiver (Γ,τ,δ). We prove that, under certain conditions, x and y belong to the same cyclic component of (Γ,τ,δ) if and only if there is an oriented cycle in (Γ,τ,δ) passing through x and y.

Before we state our combinatorial result, we fix some terminology.

Let $\Gamma=(\Gamma_0,\Gamma_1)$ be a *quiver*, that is, a locally finite oriented graph with the set of *vertices* Γ_0 and the set of *arrows* Γ_1. Assume that Γ contains neither loops nor multiple arrows. Given a vertex x of Γ, denote by x^+ the set of all vertices y of Γ such that there is an arrow $x\to y$, and by x^- the set of all vertices y such that there is an arrow $y\to x$.

A pair (Γ,τ) is called a *translation quiver* if $\tau:\Gamma_0'\to\Gamma_0$ is an injective map, for some subset $\Gamma_0'\subseteq\Gamma_0$, satisfying $(\tau x)^+=x^-$ for all $x\in\Gamma_0'$. The vertices in $\Gamma_0\setminus\Gamma_0'$ are said to be *projective*, and those in $\Gamma_0\setminus\tau\Gamma_0'$ *injective*. A vertex x is said to be τ-*periodic* if $\tau^t x=x$ for some $t\geq 1$. A τ-orbit without a projective or an injective vertex is called *stable*. In case all τ-orbits of (Γ,τ) are stable, (Γ,τ) itself is said to be stable. Note that (Γ,τ) is stable if and only if $\Gamma_0'=\Gamma_0$ and $\tau:\Gamma_0'\to\Gamma_0$ is bijective. Given a translation quiver (Γ,τ) a *translation subquiver* is a translation quiver of the form (Λ,τ') with $\Lambda_0\subseteq\Gamma_0,\Lambda_1\subseteq\Gamma_1$, $\Lambda_0'\subseteq\Gamma_0'$, and with τ' being the restriction of τ to Λ_0'.

Recall that a path $x_0\to x_1\to\cdots\to x_r$ in Γ is called *sectional* if $x_{i-2}\neq\tau x_i$ for each i, $2\leq i\leq r$. Let $\delta:\Gamma_1\to\mathbb{N}\times\mathbb{N}$ be a map and write $\delta(\alpha)=(\delta_{x,y},\delta'_{x,y})$ for all arrows $\alpha:x\to y$ of Γ. The triple (Γ,τ,δ) is called a *valued translation quiver* if the following conditions are satisfied for all non-projective vertices x of Γ:

(i) $\delta'_{\tau x,y}=\delta_{y,x}$ for all $y\in x^-$.
(ii) $\delta_{\tau x,y}=\delta'_{y,x}$ for all $y\in x^-$.

Recall that a map $\ell:\Gamma_0\to\mathbb{N}$ is called an *additive length function* for (Γ,τ,δ) if (cf. [3]):

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(i) for any vertex x which is not projective,

$$\ell(x) + \ell(\tau x) = \sum_{y \in x^-} \delta_{y,x} \ell(y);$$

(ii) for any vertex x which is projective,

$$\ell(x) > \sum_{y \in x^-} \delta_{y,x} \ell(y);$$

(iii) for any vertex x which is injective,

$$\ell(x) > \sum_{y \in x^+} \delta'_{y,x} \ell(y).$$

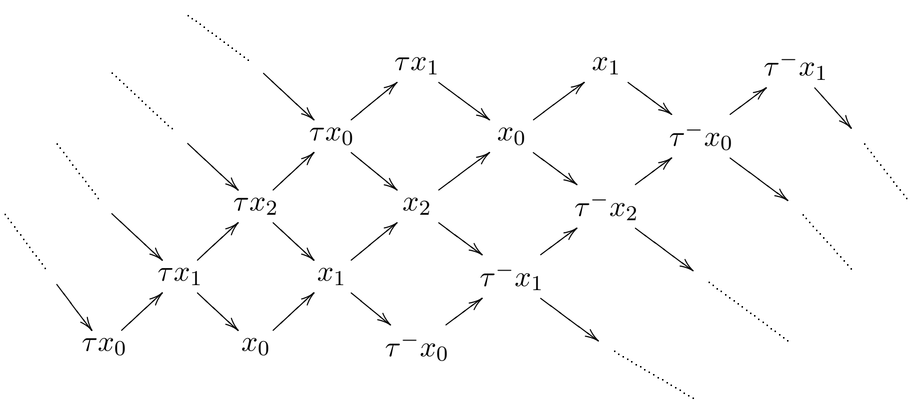
We denote by $(\Gamma, \tau, \delta)_c$ the valued translation subquiver of (Γ, τ, δ) obtained by removing from (Γ, τ, δ) all vertices which do not lie on oriented cycles, and the arrows attached to them. The connected components of $(\Gamma, \tau, \delta)_c$ are said to be *cyclic components* of (Γ, τ, δ).

The following theorem is the main result of this note.

Theorem. *Let (Γ, τ, δ) be a connected valued translation quiver and ℓ be an additive length function with $\ell(u) \neq \ell(v)$ for any arrow $u \to v$ in (Γ, τ, δ). Moreover, let x and y be two vertices lying on an oriented cycle of (Γ, τ, δ), and assume that (Γ, τ, δ) is not stable. Then x and y belong to the same cyclic component of (Γ, τ, δ) if and only if there is an oriented cycle in (Γ, τ, δ) passing through x and y.*

The following example shows that our assumption on the non-stability of the translation quiver is essential for the validity of the theorem.

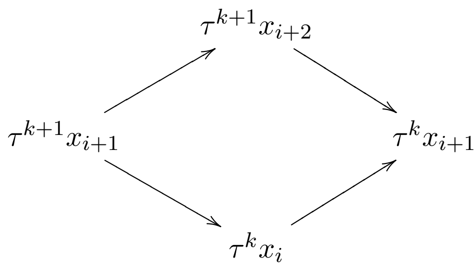
Example. Let (Γ, τ, δ) be the valued translation quiver of the form



Observe that every vertex lies on a unique sectional cycle. On the other hand, no two vertices of (Γ, τ, δ) from different sectional cycles lie on a common oriented cycle.

In the proof of our main result, an essential role will be played by the following preliminary fact (cf. [5, Lemmas 1 and 2]).

PROPOSITION. *Let (Γ, τ, δ) be a connected valued translation quiver and ℓ be an additive length function such that $\ell(u) \neq \ell(v)$ for any arrow $u \to v$ in (Γ, τ, δ). Assume that there exists a sectional cycle in (Γ, τ, δ). Then the meshes in (Γ, τ, δ) are of the form*



for $k \in \mathbb{Z}$. Thus, (Γ, τ, δ) contains neither projective nor injective vertices, all vertices in (Γ, τ, δ) are of the form $\tau^k x_i$, $i = 0, 1, \ldots, n-1$, $k \in \mathbb{Z}$, and (Γ, τ, δ) is trivially valued.

Proof. Let

$$(*) \qquad x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n = x_0$$

be a sectional cycle in (Γ, τ, δ). By assumption, $\ell(x_i) \neq \ell(x_{i+1})$ for any arrow $x_i \to x_{i+1}$, $i = 0, 1, \ldots, n-1$, so there exists a minimal element among the numbers $\ell(x_0), \ell(x_1), \ldots, \ell(x_{n-1})$. Without loss of generality we may assume that $\ell(x_0)$ is minimal. Then x_0 is not projective, and hence τx_0 exists. Moreover, $\tau x_0 \neq x_{n-2}$, because the cycle $(*)$ is sectional. Then $\ell(x_0) + \ell(\tau x_0) \geq \delta_{x_{n-1}, x_0} \ell(x_{n-1}) \geq \ell(x_{n-1})$, and, combining this with $\ell(x_{n-2}) > \ell(x_0)$, we get $\ell(x_{n-2}) + \ell(\tau x_0) > \ell(x_{n-1})$. Hence x_{n-1} is not projective and τx_{n-1} exists. Let $k = n-1$. Again, since the cycle $(*)$ is sectional, we have $\ell(\tau x_k) \neq \ell(x_{k-2})$. Then

$$\ell(x_k) + \ell(\tau x_k) \geq \delta_{\tau x_{k+1}, x_k} \ell(\tau x_{k+1}) + \delta_{x_{k-1}, x_k} \ell(x_{k-1}) \geq \ell(\tau x_{k+1}) + \ell(x_{k-1}),$$

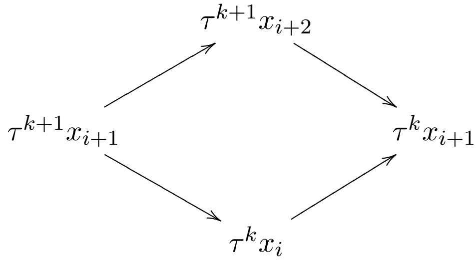
and combining this with the inequalities $\ell(\tau x_{k+1}) + \ell(x_0) \geq \ell(x_k)$ and $\ell(x_{k-2}) > \ell(x_0)$, we get $\ell(x_{k-2}) + \ell(\tau x_k) > \ell(x_{k-1})$. Hence x_{k-1} is not projective and τx_{k-1} exists. Repeating the above arguments for each $k = n-2, n-3, \ldots, 3$, we conclude that x_k is projective and τx_k exists for each $k = n-2, n-3, \ldots, 2$. Now, $\ell(\tau x_2) \neq \ell(x_0)$, because the cycle $(*)$ is sectional. Then we have

$$\ell(x_2) + \ell(\tau x_2) \geq \delta_{\tau x_3, x_2} \ell(\tau x_3) + \delta_{x_1, x_2} \ell(x_1) \geq \ell(\tau x_3) + \ell(x_1),$$

and combining this with the inequality $\ell(\tau x_3) + \ell(x_0) \geq \ell(x_2)$, we get $\ell(x_0) + \ell(\tau x_2) \geq \ell(x_1)$. Hence x_1 is not projective and τx_1 exists. Moreover, $\tau x_1 \neq$

x_{n-1}, because the cycle $(*)$ is sectional. Dually, one shows that no vertex x_i, $i = 0, 1, \ldots, n-1$, is injective.

Since x_i is neither projective nor injective, we conclude that the translations τ and τ^- of the sectional cycle $(*)$ are also sectional cycles. So, by induction on k, we infer that the translations τ^k and τ^{-k} of $(*)$ are sectional cycles. Therefore, for any $k \in \mathbb{Z}$ and $i = 0, 1, \ldots, n-1$, the vertex $\tau^k x_i$ is neither projective nor injective. Moreover, for any $k \in \mathbb{Z}$ and $i = 0, 1, \ldots, n-1$, we have the following subquiver of Γ:



Thus, we have

$$\ell(\tau^k x_{i+1}) + \ell(\tau^{k+1} x_{i+1}) \geq \delta_{\tau^k x_i, \tau^k x_{i+1}} \ell(\tau^k x_i) + \delta_{\tau^{k+1} x_{i+2}, \tau^k x_{i+1}} \ell(\tau^{k+1} x_{i+2})$$

for all $k \in \mathbb{Z}$, $i = 0, 1, \ldots, n-1$, where $\delta_{\tau^k x_i, \tau^k x_{i+1}} \geq 1$ and $\delta_{\tau^{k+1} x_{i+2}, \tau^k x_{i+1}} \geq 1$. Now, by keeping k fixed and summing over all indices $i \in \{0, 1, \ldots, n-1\}$, we see that this has to be an equality, showing that the meshes are complete and $\delta_{\tau^k x_i, \tau^k x_{i+1}} = \delta_{\tau^{k+1} x_{i+2}, \tau^k x_{i+1}} = 1$. Since the set $S = \{\tau^k x_i \mid i = 0, 1, \ldots, n-1,\ k \in \mathbb{Z}\}$ contains neither injective nor projective vertices, is closed with respect to the meshes, and (Γ, τ, δ) is trivially valued, we conclude that S is the whole set of vertices of (Γ, τ, δ) ■

Proof of Theorem. It is sufficient to show that if x and y are connected by an arrow in (Γ, τ, δ), then there is an oriented cycle in (Γ, τ, δ) passing through x and y.

Assume first that (Γ, τ, δ) does not contain a sectional cycle. Suppose that there is an arrow $y \to x$ in (Γ, τ, δ) but x and y do not lie on a common oriented cycle in (Γ, τ, δ). It follows from our assumption that $\ell(y) \neq \ell(x)$. Assume $\ell(y) > \ell(x)$. Since x lies on an oriented cycle of (Γ, τ, δ), we have

$$(*) \qquad x = x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n = x_0.$$

Observe that $x = x_0$ is not projective, because $\ell(y) > \ell(x)$ and y is a direct predecessor of x in (Γ, τ, δ), and hence τx_0 exists. Moreover, $x_{n-1} \neq y$, because x and y do not lie on a common oriented cycle in (Γ, τ, δ). Then

$$\ell(\tau x_0) + \ell(x_0) \geq \delta_{x_{n-1}, x_0} \ell(x_{n-1}) + \delta_{y, x_0} \ell(y) \geq \ell(x_{n-1}) + \ell(y),$$

and since $\ell(y) > \ell(x_0)$, we get $\ell(\tau x_0) > \ell(x_{n-1})$. Hence x_{n-1} is not projective and τx_{n-1} exists. Again, since x and y do not lie on a common oriented cycle in (Γ, τ, δ), we have $x_{n-2} \neq \tau x_0$, and hence

$$\begin{aligned} \ell(\tau x_{n-1}) + \ell(x_{n-1}) &\geq \delta_{x_{n-2},x_{n-1}} \ell(x_{n-2}) + \delta_{\tau x_n, x_{n-1}} \ell(\tau x_n) \\ &\geq \ell(x_{n-2}) + \ell(\tau x_n) > \ell(x_{n-2}) + \ell(x_{n-1}) \end{aligned}$$

implies $\ell(\tau x_{n-1}) > \ell(x_{n-2})$, because $x_n = x_0$. Repeating the above arguments we conclude that, for each $k = 1, \ldots, n-2$, x_k is not projective (hence τx_k exists), $x_{k-1} \neq \tau x_{k+1}$, and $\ell(\tau x_k) > \ell(x_{k-1})$. Finally, observe that $\tau x_1 \neq y$. Indeed, if $\tau x_1 = y$, we get a sectional cycle

$$\tau x_0 \to y \to \tau x_2 \to \cdots \to \tau x_{n-1} \to \tau x_n = \tau x_0,$$

a contradiction. Therefore, since the cycle $(*)$ is not sectional, we have $x_{n-1} = \tau x_1$. But then (Γ, τ, δ) contains the oriented cycle

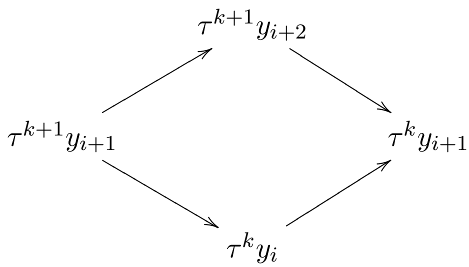
$$\tau x_0 \to y \to x \to x_1 \to \cdots \to x_{n-1} = \tau x_1 \to \cdots \to \tau x_n = \tau x_0,$$

contrary to assumption. In the case when $\ell(x) > \ell(y)$, invoking an oriented cycle of (Γ, τ, δ) passing through y, we get a similar contradiction.

Assume now that (Γ, τ, δ) contains a sectional cycle

$$y_0 \to y_1 \to \cdots \to y_{m-1} \to y_m = y_0.$$

By the proposition above, the meshes in (Γ, τ, δ) are of the form



with $k \in \mathbb{Z}$, (Γ, τ, δ) contains neither projective nor injective vertices, and all vertices in (Γ, τ, δ) are of the form $\tau^k y_i$, $i = 0, 1, \ldots m-1$, $k \in \mathbb{Z}$. Moreover, (Γ, τ, δ) is trivially valued. Therefore, the translation quiver is stable, a contradiction. ∎

As a direct consequence of the above proof we obtain the following fact.

COROLLARY 1. *Let (Γ, τ, δ) be a connected valued translation quiver and ℓ be an additive length function such that $\ell(u) \neq \ell(v)$ for any arrow $u \to v$ in (Γ, τ, δ). Moreover, let x and y be two vertices lying on an oriented cycle of (Γ, τ, δ), and assume that (Γ, τ, δ) has no sectional cycles. Then x and y belong to the same cyclic component $(\Gamma, \tau, \delta)_c$ of (Γ, τ, δ) if and only if there is an oriented cycle in (Γ, τ, δ) passing through x and y.*

Note that if A is an artin algebra over a commutative artin ring R, then the Auslander–Reiten quiver Γ_A has no sectional cycles [1]. Moreover, Γ_A is a valued translation quiver with an additive length function defined as follows: $\ell(X) = l_R(X)$, where $l_R(X)$ is the length of the composition sequence of an R-module $X \in \operatorname{mod} A$, and $\operatorname{mod} A$ is the category of all finitely generated right A-modules. We also know that for any irreducible morphism $f : X \to Y$ in Γ_A, f is either an epimorphism or a monomorphism, and thus $\ell(X) \neq \ell(Y)$. So, the above corollary is a generalization of the analogous fact proved in [4] for the Auslander–Reiten quiver Γ_A.

COROLLARY 2. *Let (Γ, τ, δ) be a connected valued translation quiver and ℓ be an additive length function such that $\ell(u) \neq \ell(v)$ for any arrow $u \to v$ in (Γ, τ, δ). Moreover, let x and y be two vertices lying on an oriented cycle of (Γ, τ, δ), and assume that there exists a τ-periodic vertex in (Γ, τ, δ). Then x and y belong to the same cyclic component $(\Gamma, \tau, \delta)_c$ of (Γ, τ, δ) if and only if there is an oriented cycle in (Γ, τ, δ) passing through x and y.*

Proof. If (Γ, τ, δ) does not contain a sectional cycle, then the proof is identical as for the theorem above. Again, if (Γ, τ, δ) contains at least one sectional cycle, then (Γ, τ, δ) is stable. Since there exists a τ-periodic vertex, the Happel–Preiser–Ringel theorem [2, Section 2] and the proposition above show that $(\Gamma, \tau, \delta) \cong \mathbb{Z}\Delta/(\tau^n)$, where Δ is a quiver of Euclidean type $\widetilde{\mathbb{A}}_t$ with the cyclic orientation. Hence, there is an oriented cycle in (Γ, τ, δ) passing through x and y. ∎

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