INDUCED ALMOST CONTINUOUS FUNCTIONS ON HYPERSPACES

BY

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Abstract. For a metric continuum $X$, let $C(X)$ (resp., $2^X$) be the hyperspace of sub-continua (resp., nonempty closed subsets) of $X$. Let $f : X \to Y$ be an almost continuous function. Let $C(f) : C(X) \to C(Y)$ and $2^f : 2^X \to 2^Y$ be the induced functions given by $C(f)(A) = \text{cl}_Y(f(A))$ and $2^f(A) = \text{cl}_Y(f(A))$. In this paper, we prove that:

- If $2^f$ is almost continuous, then $f$ is continuous.
- If $C(f)$ is almost continuous and $X$ is locally connected, then $f$ is continuous.
- If $X$ is not locally connected, then there exists an almost continuous function $f : X \to [0,1]$ such that $C(f)$ is almost continuous and $f$ is not continuous.

Introduction. A continuum is a nonempty, nondegenerate, compact connected metric space. All the spaces considered in this paper are continua. Given a continuum $X$ we consider the following hyperspaces of $X$:

$$2^X = \{ A \subset X : A \text{ is closed and nonempty} \},$$

$$C(X) = \{ A \in 2^X : A \text{ is connected} \}.$$  

Both are considered with the Hausdorff metric $D$.

Given a (not necessarily continuous) function between continua $f : X \to Y$, we can consider its graph $\Lambda(f) = \{(p, f(p)) \in X \times Y : p \in X \}$ and the induced function $2^f : 2^X \to 2^Y$ given by $2^f(A) = \text{cl}_Y(f(A)) (f(A) \text{ is the image of } A \text{ under } f)$. We are interested in functions $f : X \to Y$ for which the natural induced map $C(f) : C(X) \to C(Y)$ is defined. Thus we need to require that, for each $A \in C(X)$, $f(A)$ is connected; we call a function satisfying this condition weakly Darboux (in Real Analysis a Darboux function is a function such that the image of a connected set is a connected set).

Of course, every continuous function is weakly Darboux. It is known that almost continuous functions are weakly Darboux (see Lemma 1). Recall that $f$ is almost continuous provided that, for each open subset $U$ of $X \times Y$ such that $\Lambda(f) \subset U$, there exists a continuous function $g : X \to Y$ such that $\Lambda(g) \subset U$. We say that $f$ is proper almost continuous if $f$ is almost continu-

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uous but not continuous. A simple example of a proper almost continuous function is the function \( h : [0, 1] \to [-1, 1] \) given by \( h(t) = \sin(1/t) \) if \( t > 0 \), and \( h(0) = 0 \).

Almost continuous functions were introduced by J. Stallings in [3] where he used them to generalize some fixed point theorems.

Given a continuous function between continua \( f : X \to Y \) and a class of mappings \( \mathcal{M} \), the problem of determining if one of the following properties implies another has been widely studied:

(a) \( f \) belongs to \( \mathcal{M} \),
(b) \( C(f) \) belongs to \( \mathcal{M} \),
(c) \( 2^f \) belongs to \( \mathcal{M} \).

A discussion on this topic can be found in [2, Section 77].

In this paper, we study this problem for the class \( \mathcal{M} \) of (not necessarily continuous) almost continuous functions. Observe that, to define \( 2^f \) it is not necessary to require that \( f \) is almost continuous. Since the restriction of an almost continuous function to a closed subset of the domain is also almost continuous (see [3, Proposition 2]), if \( 2^f \) is almost continuous, then so is \( 2^f|\{\{p\} \in 2^X : p \in X\} \). This implies that \( f \) is almost continuous. Thus (c) implies (a) and, similarly, (b) implies (a).

The first result we obtain is that if a function \( f : X \to Y \) and its induced function \( 2^f \) are weakly Darboux, then \( f \) is continuous. Thus, for the class of weakly Darboux functions, (a) and (c) together imply (b). The second result says that if \( X \) is locally connected and the functions \( f \) and \( C(f) \) are weakly Darboux, then \( f \) is continuous, and the third result says that if \( X \) is not locally connected, then it is possible to construct a proper almost continuous function \( f : X \to [0, 1] \) such that \( C(f) \) is almost continuous. Thus (a) and (b) together do not imply (c).

**Almost continuity of \( 2^f \).** Throughout this paper \( X \) denotes a continuum with metric \( d \). The symbol \( \mathbb{N} \) denotes the set of positive integers. Given \( \varepsilon > 0 \), \( p \in X \) and \( A \subset X \), let \( B(\varepsilon, p) = \{q \in X : d(p, q) < \varepsilon\} \) and \( N(\varepsilon, A) = \bigcup \{B(\varepsilon, a) \subset X : a \in A\} \). An order arc in \( 2^X \) is a continuous function \( \alpha : [0, 1] \to 2^X \) such that \( \alpha(s) \subset \alpha(t) \) if \( 0 \leq s \leq t \leq 1 \). Conditions for the existence of order arcs are given in Theorem 15.3 of [2]. A Whitney map is a continuous function \( \mu : 2^X \to [0, 1] \) such that \( \mu(X) = 1, \mu(\{p\}) = 0 \) for each \( p \in X \) and, if \( A, B \in 2^X \) and \( A \subsetneq B \), then \( \mu(A) < \mu(B) \). It is known that every continuum \( X \) admits Whitney maps (see [2, Thm. 13.4]).

The following lemma is well known (see [3, Corollary to Proposition 3]). We include it here for completeness.

**Lemma 1.** If \( f : X \to Y \) is almost continuous, then \( f \) is weakly Darboux.
**Proof.** Let $A \subset C(X)$. We need to show that $f(A)$ is connected. Suppose to the contrary that $f(A)$ is not connected. Let $K, L$ be nonempty separated subsets of $f(A)$ such that $f(A) = K \cup L$. Since $Y$ is metric, there exist disjoint open subsets $V$ and $W$ such that $K \subset V$ and $L \subset W$. Fix points $a, b \in A$ such that $f(a) \in K$ and $f(b) \in L$. Consider the set

$$
\mathcal{U} = [(X - A) \times Y] \cup [(X - \{a\}) \times W] \cup [(X - \{b\}) \times V] \\
\cup [(X - \{a, b\}) \times (W \cup V)].
$$

Clearly, $\mathcal{U}$ is an open subset of $X \times Y$ which contains $A(f)$. Since $f$ is almost continuous, there exists a continuous function $g : X \to Y$ such that $A(g) \subset \mathcal{U}$.

Given a point $p \in A$, by the definition of $\mathcal{U}$, $g(p) \in W \cup V$. Moreover, $(a, g(a)) \in \mathcal{U}$ implies that $g(a) \in V$. Similarly, $(b, g(b)) \in \mathcal{U}$ implies that $g(b) \in W$. Therefore, $g(A)$ is a connected subset of $W \cup V$ and $g(A) \cap W \neq \emptyset \neq g(A) \cap V$. This is a contradiction. Hence $f(A)$ is connected.

**Lemma 2.** Let $\alpha : [0, 1] \to 2^X$ be an order arc. Suppose that $F : 2^X \to 2^Y$ is weakly Darboux and such that $A \subset B$ implies $F(A) \subset F(B)$. Then the function $F \circ \alpha : [0, 1] \to 2^Y$ is continuous.

**Proof.** Let $\{t_n\}_{n=1}^\infty$ be a sequence in $[0, 1]$ converging $t \in [0, 1]$. We need to check that $\lim F(\alpha(t_n)) = F(\alpha(t))$. It is enough to consider the case in which the sequence $\{t_n\}_{n=1}^\infty$ is strictly monotone.

**Case 1:** $0 < t_1 < t_2 < \cdots$. In this case, $F(\alpha(t_1)) \subset F(\alpha(t_2)) \subset \cdots$. This implies that $\lim F(\alpha(t_n)) = \cl_Y(\bigcup\{F(\alpha(t_n)) : n \in \mathbb{N}\})$. Hence $\lim F(\alpha(t_n)) \subset F(\alpha(t))$ because each $F(\alpha(t_n))$ is contained in $F(\alpha(t))$. If $\lim F(\alpha(t_n)) \neq F(\alpha(t))$, fix a point $p \in F(\alpha(t)) - \lim F(\alpha(t_n))$. Consider the following sets in $2^Y$: $K = \{A \in 2^Y : A \subset \lim F(\alpha(t_n))\}$ and $L = \{A \in 2^Y : p \in A\}$. It is easy to see that $K$ and $L$ are disjoint closed subsets of $2^Y$ such that $F(\alpha([0, t])) \subset K$ and $F(\alpha(t)) \subset L$. This contradicts the connectedness of $F(\alpha([0, t]))$ and proves that $\lim F(\alpha(t_n)) = F(\alpha(t))$.

**Case 2:** $1 > t_1 > t_2 > \cdots$. In this case, $\lim F(\alpha(t_n)) = \bigcap\{F(\alpha(t_n)) : n \in \mathbb{N}\}$ and $\lim F(\alpha(t_n)) \supset F(\alpha(t))$. If $\lim F(\alpha(t_n)) \neq F(\alpha(t))$, a contradiction can be obtained by considering the sets $K = \{A \in 2^Y : \lim F(\alpha(t_n)) \subset A\}$ and $L = \{F(\alpha(t))\}$. Thus $\lim F(\alpha(t_n)) = F(\alpha(t))$.

This completes the proof of the lemma.

Proceeding as in Lemma 2, one can prove the following lemma.

**Lemma 3.** Let $\alpha : [0, 1] \to C(X)$ be an order arc. Suppose that $F : C(X) \to C(Y)$ is weakly Darboux and such that $A \subset B$ implies $F(A) \subset F(B)$. Then the function $F \circ \alpha : [0, 1] \to C(Y)$ is continuous.
Theorem 1. Suppose that \( f : X \rightarrow Y \) and \( 2^f : 2^X \rightarrow 2^Y \) are weakly Darboux. Then \( f \) is continuous.

Proof. Let \( F = 2^f \). Suppose that \( f \) is not continuous. Then there exist points \( p \in X \), \( q \in Y \) and a sequence \( \{p_n\}_{n=1}^{\infty} \) in \( X \) such that \( \lim p_n = p \), \( \lim f(p_n) = q \) and \( q \neq f(p) \). Let \( \varepsilon = d_Y(f(p), q) > 0 \). Notice that \( f(X) \) is nondegenerate.

Fix a Whitney map \( \mu : 2^Y \rightarrow [0, 1] \). Since the set \( \{B \in 2^Y : \text{diam}(B) \geq \varepsilon/3\} \) is closed in \( 2^Y \) and it does not intersect \( \mu^{-1}(0) \), there exists \( r > 0 \) such that \( r < \mu(F(X)) \) and, if \( B \in \mu^{-1}(r) \), then \( \text{diam}(B) < \varepsilon/3 \). Fix \( \eta > 0 \) such that \( \eta < \varepsilon \) and \( \text{diam}(B) > \eta \) for each \( B \in \mu^{-1}(r) \). Since \( \lim f(p_n) = q \), we may assume that \( f(p_n) \in B(\eta/3, q) \) for each \( n \in \mathbb{N} \).

For each \( n \in \mathbb{N} \), fix an order arc \( \alpha_n : [0, 1] \rightarrow C(X) \) with \( \alpha_n(0) = \{p_n\} \) and \( \alpha_n(1) = X \). By Lemma 2, the function \( \mu \circ F \circ \alpha_n : [0, 1] \rightarrow [0, 1] \) is continuous. Since \( \mu(F(\alpha_n(0))) = \mu(F(\{p_n\})) = 0 \) and \( \mu(F(\alpha_n(1))) = \mu(F(X)) > r \), there exists \( t_n \in [0, 1] \) such that \( \mu(F(\alpha_n(t_n))) = r \). Since \( r > 0 \), \( F(\alpha_n(t_n)) \) is a nondegenerate subcontinuum of \( Y \). Notice that, by the choice of \( r \), \( \text{diam}(F(\alpha_n(t_n))) < \varepsilon/3 \). Since \( f(p_n) = f(\alpha_n(t_n)) \subset F(\alpha_n(t_n)) \) and \( f(p_n) \in B(\varepsilon/3, q) \), it follows that

\[
F(\alpha_n(t_n)) \subset B(2\varepsilon/3, q).
\]

Since \( C(X) \) is compact, we may assume that the sequence \( \{\alpha_n(t_n)\}_{n=1}^{\infty} \) converges to an element \( A \in C(X) \). Let

\[
P = \{p, p_1, p_2, \ldots\} \quad \text{and} \quad E = A \cup \alpha_1(t_1) \cup \alpha_2(t_2) \cup \cdots.
\]

Then \( P, E \in 2^X \) and \( P \subset E \). Let \( \beta : [0, 1] \rightarrow 2^X \) be given by

\[
\beta(t) = \begin{cases} 
P \cup \left( \bigcup_{i=1}^{n-1} \alpha_i(t_i) \right) \cup \alpha_n \left( 2^n \left( t - \left( 1 - \frac{1}{2^{n-1}} \right) \right) t_n \right), & \text{if } t \in [1 - 1/2^{n-1}, 1 - 1/2^n], n \in \mathbb{N}, \\
E & \text{if } t = 1.
\end{cases}
\]

It is easy to check that \( \beta \) is well defined, continuous in \([0, 1]\) and has the property that if \( s \leq t \), then \( \beta(s) \subset \beta(t) \).

To see that \( \beta \) is continuous at 1, take a sequence \( s_1 < s_2 < \cdots \) in \([0, 1]\) such that \( \lim s_m = 1 \). Since \( \beta(s_1) \subset \beta(s_2) \subset \cdots \), we have \( \lim \beta(s_m) = \text{cl}(\bigcup \{ \beta(s_m) : m \in \mathbb{N} \}) \). Hence \( \lim \beta(s_m) \subset E \) as each \( \beta(s_m) \) is a subset of \( E \). Since \( \lim \alpha_n(t_n) = A \), we have \( A \subset \text{cl}(\bigcup \{ \alpha_n(t_n) : n \in \mathbb{N} \}) \).

Given \( n \in \mathbb{N} \), there exists \( m_0 \in \mathbb{N} \) such that \( 1 - 1/2^n < s_{m_0} \). Thus \( \alpha_n(t_n) \subset \beta(1 - 1/2^n) \subset \beta(s_{m_0}) \subset \lim \beta(s_m) \). Therefore, \( E \subset \lim \beta(s_m) \).

We have shown that \( \beta(1) = E = \lim \beta(s_m) \). This completes the proof that \( \beta \) is continuous. Hence \( \beta \) is an order arc in \( 2^X \) such that \( \beta(0) = P \) and \( \beta(1) = E \).
By Lemma 2, the function $F \circ \beta : [0, 1] \to 2^Y$ is continuous. Hence $F(\beta(1)) = \lim_{n \to \infty} F(\beta(1 - 1/2^n))$. Given $n \in \mathbb{N}$, 
\[
F(\beta(1 - 1/2^n)) = F(P \cup \alpha_1(t_1) \cup \cdots \cup \alpha_n(t_n)) 
\subseteq \cl(f(\{p, p_1, p_2, \ldots\}) \cup F(\alpha_1(t_1)) \cup \cdots \cup F(\alpha_n(t_n)) 
\subseteq f(\{p\}) \cup \cl(f(\{p_1, p_2, \ldots\}) \cup F(\alpha_1(t_1)) \cup \cdots \cup F(\alpha_n(t_n)) 
\subseteq \{f(p)\} \cup \cl(B(2\varepsilon/3, q)).
\]
Hence, $F(\beta(1)) \subseteq \{f(p)\} \cup \cl(B(2\varepsilon/3, q))$. That is, 

$$F(E) \subseteq \{f(p)\} \cup \cl(B(2\varepsilon/3, q)).$$

Since $A \subseteq E$, we have $F(A) \subseteq F(E) \subseteq \{f(p)\} \cup \cl(B(2\varepsilon/3, q))$. By the choice of $\varepsilon$, $\{f(p)\}$ and $\cl(B(2\varepsilon/3, q))$ are closed in $Y$ and disjoint. By hypothesis, $F(A)$ is connected. Since $p \in A$, $f(p) \in F(A)$. Thus, $F(A) = \{f(p)\}$.

Define $\gamma : [0, 1] \to 2^X$ by

$$\gamma(t) = \begin{cases} 
A \cup P & \text{if } t = 0, \\
A \cup P \cup \alpha_n(2^n(t - 1/2^n)t_n) \cup \bigcup_{i=n+1}^{\infty} \alpha_i(t_i) & \text{if } t \in [1/2^n, 1/2^{n+1}), n \in \mathbb{N}.
\end{cases}$$

Since $\lim \alpha_n(t_n) = A$, we have $\gamma(t) \in 2^X$ for each $t \in [0, 1]$. It is easy to check that $\gamma$ is well defined, continuous in $(0, 1]$ and has the property that if $s \leq t$, then $\gamma(s) \subset \gamma(t)$.

To prove that $\gamma$ is continuous at 0, take a sequence $s_1 > s_2 > \cdots$ in $[0, 1]$ such that $\lim s_m = 0$. Given $\delta > 0$, let $N \in \mathbb{N}$ be such that $\alpha_n(t_n) \subseteq N(\delta, A)$ for each $n \geq N$. Fix $M \in \mathbb{N}$ such that $s_M < 1/2^N$. Given $m \geq M$, we have $s_m < 1/2^N$, thus $\gamma(s_m) \subseteq \gamma(1/2^N) \subseteq N(\delta, A \cup P)$. Since $A \cup P \subseteq \gamma(s_m) \subseteq N(\delta, \gamma(s_m))$, we conclude that $D(\gamma(s_m), A \cup P) = \delta$. We have shown that $\lim \gamma(s_m) = A \cup P = \gamma(0)$. Hence, $\gamma$ is continuous at 0.

We have proved that $\gamma$ is an order arc from $\gamma(0) = A \cup P$ to $\gamma(1) = E$. By Lemma 2, $F \circ \gamma : [0, 1] \to 2^Y$ is a continuous function. Thus

$$\lim F(\gamma(1/2^n)) = F(\gamma(0)) = F(A \cup P) = \cl(f(A)) \cup \cl(f(P)) 
\subseteq \{f(p)\} \cup \cl(B(\eta/3, q)).$$

Therefore,

$$\lim F(\gamma(1/2^n)) \subseteq \{f(p)\} \cup B(\eta/2, q).$$

On the other hand, $F(\alpha_{n+1}(t_{n+1})) \subset F(\gamma(1/2^n))$, $F(\alpha_{n+1}(t_{n+1}))$ is connected (see [2, Corollary 15.4]) and $\mu(F(\alpha_{n+1}(t_{n+1}))) = r$. Notice that, by the choice of $\eta$, $\diam(F(\alpha_{n+1}(t_{n+1}))) > \eta$. This implies that $F(\alpha_{n+1}(t_{n+1})) \not\subseteq B(\eta/2, q)$, and moreover, $f(p_{n+1}) \in F(\alpha_{n+1}(t_{n+1})) \cap B(\eta/2, q)$. Hence, $\bd(B(\eta/2, q)) \cap F(\alpha_{n+1}(t_{n+1})) \neq \emptyset$. This implies that

$$\bd(B(\eta/2, q)) \cap \lim F(\alpha_{n+1}(t_{n+1})) \neq \emptyset.$$
Hence, \( \text{bd}(B(\eta/2, q)) \cap \lim F(\gamma(1/2^n)) \neq \emptyset \). This contradicts the inclusion \( \lim F(\gamma(1/2^n)) \subset \{ f(p) \} \cup B(\eta/2, q) \) proved above and completes the proof of the theorem. ■

**Corollary 1.** Suppose that \( f : X \to Y \) and \( 2f : 2^X \to 2^Y \) are almost continuous functions. Then \( f \) is continuous.

**Almost continuity of \( C(f) \)**

**Theorem 2.** Suppose that \( X \) is locally connected and the functions \( f : X \to Y \) and \( C(f) : C(X) \to C(Y) \) are weakly Darboux. Then \( f \) is continuous.

**Proof.** Let \( F = C(f) \). In order to prove that \( f \) is continuous take a sequence \( \{ p_n \}_{n=1}^{\infty} \) in \( X \) converging to a point \( p \in X \).

For each \( n \in \mathbb{N} \), let \( A_n = \{ A \in C(X) : p, p_n \in A \} \). Since \( A_n \) is a nonempty and compact subset of \( C(X) \), there exists \( A_n \in A_n \) such that \( D(\{ p \}, A_n) = \min \{ D(\{ p \}, A) : A \in A_n \} \). Then \( p, p_n \in A_n \).

We claim that \( \lim A_n = \{ p \} \). Let \( \varepsilon > 0 \). Since \( X \) is locally connected, there exists an open and connected subset \( U \) of \( X \) with \( \text{diam}(\text{cl}_X(U)) < \varepsilon \) and \( p \in U \). Thus \( D(\{ p \}, \text{cl}_X(U)) < \varepsilon \). Let \( N \in \mathbb{N} \) be such that \( p_n \in U \) for each \( n \geq N \). Then \( D(\{ p \}, A_n) \leq D(\{ p \}, \text{cl}_X(U)) < \varepsilon \) for each \( n \geq N \). Therefore, \( \lim A_n = \{ p \} \).

For each \( n \in \mathbb{N} \), let \( B_n = A_n \cup A_{n+1} \cup \ldots \). Clearly, \( B_n \in C(X) \), \( p, p_n \in B_n \), \( \lim B_n = \{ p \} \) and \( B_1 \supset B_2 \supset \ldots \). Let \( \alpha : [0, 1] \to C(X) \) be an order arc such that \( \alpha(1/n) = B_n \) for each \( n \in \mathbb{N} \) and \( \alpha(0) = \{ p \} \) (such an order arc can be constructed using Theorem 15.3 of [2]).

By Lemma 3, the map \( C(f) \circ \alpha : [0, 1] \to C(Y) \) is continuous. Thus \( \lim C(f)(\alpha(1/n)) = \{ f(p) \} \). Since \( f(p_n) \in C(f)(\alpha(1/n)) \) for each \( n \in \mathbb{N} \), \( \lim f(p_n) = f(p) \). Therefore, \( f \) is continuous. ■

**Corollary 2.** Suppose that \( X \) is locally connected and the functions \( f : X \to Y \) and \( C(f) : C(X) \to C(Y) \) are almost continuous. Then \( f \) is continuous.

**Theorem 3.** Let \( X \) be a non-locally connected continuum. Then there exists a proper almost continuous function \( f : X \to [0, 1] \) such that \( C(f) \) is almost continuous.

**Proof.** Let \( H \) be the harmonic fan defined as a subset of the complex plane by

\[
H = \{ z \in \mathbb{C} : |z| \leq 2 \text{ and } \text{Arg}(z) \in \{ 0, 1/1, 1/2, 1/3, \ldots \} \}
\]

and set

\[
H_n = \{ z \in H : \text{Arg}(z) \in \{ 0, 1/1, 1/2, \ldots, 1/n \} \}.
\]
By [1], there exists a continuous surjection \( g : X \to H \). For each \( n \in \mathbb{N} \), define \( r_n : H \to H_n \) by

\[
r_n(z) = \begin{cases} 
  z & \text{if } z \in H_n, \\
  |z| & \text{if } z \in H - H_n.
\end{cases}
\]

Let \( h : H \to [0, 1] \) be given by

\[
h(z) = \begin{cases} 
  0 & \text{if } |z| \geq 1 \text{ and } \text{Arg}(z) = 0, \\
  |z| - 1 & \text{otherwise}.
\end{cases}
\]

Finally, put \( f = h \circ g \).

The following observations are easy to prove:

(a) \( r_n \) is a continuous retraction.
(b) \( h \) is not continuous.
(c) \( h \circ r_n \) is continuous.
(d) If \( M \subset H \) is connected, then \( h(M) \) is connected. Hence, \( f \) is weakly Darboux.
(e) The function \( f \) is not continuous. Indeed, for each \( k \in \mathbb{N} \), fix a point \( a_k \in g^{-1}(2(\cos(1/k) + i \sin(1/k))) \). By compactness of \( X \) there exists a subsequence \( \{a_{k_n}\}_{n=1}^{\infty} \) tending to a point \( a \in X \). Then, for each \( n \in \mathbb{N} \), \( f(a_{k_n}) = 1 \), and

\[
f(a) = h(\lim_{n \to \infty} (2(\cos(1/k_n) + i \sin(1/k_n)))) = h(2) = 0.
\]

The following observations are also easy to check.

For a connected subset \( A \) of \( X \) we have:

(i) If \( A \cap g^{-1}(0) \neq \emptyset \neq A \cap g^{-1}(\{z : |z| \geq 1\}) \), then \( f(A) = [0, 1] \) and \( (h \circ r_n \circ g)(A) = [0, 1] \) for all \( n \in \mathbb{N} \).
(ii) If \( A \subset g^{-1}(\{z : |z| \leq 1\}) \), then \( (h \circ r_n \circ g)(A) = f(A) \) for all \( n \in \mathbb{N} \).
(iii) If \( A \cap g^{-1}(0) = \emptyset \), then \( g(A) \) is contained in some convex segment contained in \( H \). Hence, there exists \( k \in \mathbb{N} \) such that \( (h \circ r_n \circ g)(A) = f(A) \) for all \( n \geq k \).

We are ready to show that \( C(f) \) is almost continuous. Let \( U \) be an open subset of \( C(X) \times C([0, 1]) \) containing the graph of \( C(f) \). Note that each function \( h \circ r_n \circ g \) is continuous. We claim that the graph of some map \( C(h \circ r_n \circ g) \) is contained in \( U \). Suppose to the contrary that, for each \( n \in \mathbb{N} \), there exists \( A_n \in C(X) \) such that \( (A_n, (h \circ r_n \circ g)(A_n)) \notin U \). In particular, \( (h \circ r_n \circ g)(A_n) \neq f(A_n) \). By (i)–(iii) we know that \( A_n \cap g^{-1}(0) = \emptyset \) and \( g(A_n) \subset H - H_n \) since otherwise \( f(A_n) = (h \circ r_n \circ g)(A_n) \). By compactness of \( C(X) \) we may assume that the sequence \( \{A_n\}_{n=1}^{\infty} \) converges to some \( A \in C(X) \). By continuity of \( g \), we see that \( g(A) \) is contained in the limit convex segment of \( H \). Therefore,

\[
(A, (h \circ r_1 \circ g)(A)) = (A, (h \circ g)(A)) = (A, f(A)) \in U.
\]
By continuity of $C(h \circ r_1 \circ g)$, it follows that there exists $k \in \mathbb{N}$ such that $(A_n, (h \circ r_1 \circ g)(A_n)) \in U$ for all $n \geq k$. Since $g(A_n) \subset H - H_n$, we have $r_1(g(A_n)) = r_n(g(A_n))$. Hence $(A_n, (h \circ r_n \circ g)(A_n)) \in U$ for all $n \geq k$. This contradicts the choice of the sets $A_n$ and completes the proof that $C(f)$ is almost continuous.

Finally, since $\{\{p\} \in C(X) : p \in X\}$ is closed in $C(X)$, by [3, Proposition 2], it follows that $f$ is almost continuous.

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**REFERENCES**

