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A SIMPLE SOLUTION OF HILBERT'S FOURTEENTH PROBLEM IN DIMENSION FIVE

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Abstract. We give a short proof of a counterexample (due to Daigle and Freudenburg) to Hilbert's fourteenth problem in dimension five.

Introduction. In 1900 at the International Congress of Mathematicians in Paris David Hilbert presented a list of 23 problems, intended to challenge the mathematicians of the new century. The fourteenth problem of this list can be stated as follows: let k be a field, $k[x] := k[x_1, \dots, x_n]$ the polynomial ring, $k(x)$ its quotient field and L a subfield containing k.

Is $L \cap k[x]$ a finitely generated k-algebra?

A positive answer was given by Zariski ([7]) in case $\operatorname{trdeg}_k L \le 2$. However in 1958 Nagata ([5]) constructed a counterexample in dimension 32. Then in 1988 Roberts ([6]) found a new counterexample in dimension 7. Recently, in 1998 Freudenburg ([2]), studying Robert's example, found a 6-dimensional counterexample, from which a 5-dimensional example was obtained in 1999 by Daigle and Freudenburg in [1]: they consider on $B := k[X, S, T, U, V]$ the derivation $D := X^3\partial_S + S\partial_T + T\partial_U + X^2\partial_V$ and show that $B^D := \ker D : B \to B$ is not finitely generated over k (then the quotient field L of B^D is a counterexample to Hilbert fourteen, since $L \cap B = B^D$).

The main aim of this note is to give a short proof of this result, by substantially simplifying the arguments given in [1] and [2].

Finally, I would like to mention that recently S. Kuroda has constructed new counterexamples to Hilbert fourteen in the missing dimensions 4 and 3 ([3], [4]).

1. The main result. Throughout this paper we use the following notations: k is a field of characteristic zero,

$$B := k[X, S, T, U, V], \quad D_0 := X^3\partial_S + S\partial_T + T\partial_U, \quad D := D_0 + X^2\partial_V.$$

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Furthermore,

$$A := k[S,T,U], \qquad D_1 := \partial_S + S\partial_T + T\partial_U.$$

Finally, for any $0 \neq f \in B$, $\deg f$ denotes the usual degree of f. We also use another grading on A given by a vector $w \in \mathbb{N}^3$ and we write w-deg to denote the degree with respect to this grading. The main aim of this note is to give a short proof of

THEOREM 1.1 (Daigle–Freudenburg). *B^D is not a finitely generated k-algebra.*

The proof is based on the following result which will be proved in the next section.

PROPOSITION 1.2. *Let $e : \mathbb{N} \to \mathbb{N}$ be defined by $e(3l) = 2l$, $e(3l+1) = e(3l+2) = 2l+1$ for all $l \geq 0$. There exist $c_0 = 1, c_1, c_2, \ldots$ in A with $D_1 c_i = c_{i-1}$ and $\deg c_i \leq e(i)$ for all $i \geq 1$*

Proof of Theorem 1.1. (i) Define

$$a_i := X^{2i+1} c_i\left(\frac{S}{X^3}, \frac{T}{X^3}, \frac{U}{X^3}\right) \quad \text{for } i \geq 0.$$

Then one easily verifies that $D_0 a_i = X^2 a_{i-1}$ for all $i \geq 1$ and that

$$F_n := \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!}\, a_i V^{n-i} \in B^D \quad \text{for all } n \geq 1.$$

Suppose now that B^D is finitely generated by $g_1, \ldots, g_s$ over k. We may assume that $g_i(0) = 0$ for all i. Write $g_i = \sum g_{ij} V^j$ with $g_{ij} \in k[X,S,T,U]$. By (ii) below we find that $g_{ij} \in (X,S,T,U)$ for all i,j. Let d denote the maximum of the V-degrees of all g_i. Consider $F_{d+1} = XV^{d+1}$+lower degree V-terms as above. So $F_{d+1} \in B^D = k[g_1, \ldots, g_s]$. Looking at the coefficient of V^{d+1}, we deduce that $X \in (X,S,T,U)^2$, a contradiction.

(ii) To prove that $g_{ij} \in (X,S,T,U)$ for all i,j it suffices to show that if $g = \sum g_j V^j \in B^D$ satisfies $g(0) = 0$ then each $g_j \in (X,S,T,U)$. First, clearly $g_0 \in (X,S,T,U)$. So let $j \geq 1$. From $Dg = 0$ we get $jg_j X^2 = D_0(-g_{j-1}) \in D_0(k[X,S,T,U]) \subset (X^3,S,T)$ for all $j \geq 1$. If $g_j(0) \in k^*$, then $X^2 \in (X^3,S,T,UX^2)$, contradiction. So $g_j(0) = 0$, i.e. $g_j \in (X,S,T,U)$.

2. The proof of Proposition 1.2. Put

$$T_1 := T - \frac{1}{2} S^2, \qquad U_1 := U - ST + \frac{1}{3} S^3.$$

Then $A = k[T_1, U_1][S]$. Since $D_1 T_1 = D_1 U_1 = 0$ and $D_1 S = 1$ we get $A_1^D = k[T_1, U_1]$. Consider on A the grading defined by $w(S) = 1, w(T) = 2$ and $w(U) = 3$. Then $D_1(A_n) \subset A_{n-1}$ for all $n \geq 1$, where A_n is the k-span of all monomials of A of w-degree n. By induction on n we construct $c_n \in A$.

So assume that c_n is already constructed. Write $c_n = \sum_{i=0}^{n} H_{n-i}S^i$ with $H_{n-i} \in A_{n-i} \cap A^{D_1}$ (this is possible since $A = A^{D_1}[S]$ and $c_n \in A_n$). Then

$$\widetilde{c}_{n+1} := \sum_{i=0}^{n} \frac{1}{i+1} H_{n-i}S^{i+1} \in A_{n+1}$$

and $D_1(\widetilde{c}_{n+1}) = c_n$. Finally, by Lemma 2.1 below, there exists $h \in A_{n+1} \cap A^{D_1}$ such that $\widetilde{c}_{n+1} := c_{n+1} - h$ satisfies $\deg c_{n+1} \le e(n+1)$.

LEMMA 2.1. *If* $f \in A_{n+1}$ *is such that* $\deg D_1 f \le e(n)$, *then there exists* $h \in A_{n+1} \cap A^{D_1}$ *such that* $\deg(f-h) \le e(n+1)$.

Proof. (i) Let $n = 3l$ (the cases $n = 3l+1$ and $n = 3l+2$ are treated similarly) and let M be the k-span of all $f \in A_{n+1}$ such that $\deg D_1 f \le 2l$ $(= e(3l))$. Write $f = \sum \alpha_{ijk} S^i T^j U^k$ with $i + 2j + 3k = 3l + 1$ and $\alpha_{ijk} \in k$. Then

$$D_1 f = \sum_{i+2j+3k=3l+1} (i\alpha_{ijk} + (j+1)\alpha_{i-2,j+1,k} + (k+1)\alpha_{i-1,j-1,k+1}) S^{i-1}T^jU^k.$$

So

$$(*) \quad \deg D_1 f \le 2l \quad \text{iff} \quad i\alpha_{ijk} + (k+1)\alpha_{i-1,j-1,k+1} + (j+1)\alpha_{i-2,j+1,k} = 0$$

for all i, j, k satisfying $i + 2j + 3k = 3l + 1$ and $(i-1) + j + k \ge 2l + 1$, i.e. $i + j + k \ge 2l + 2$. For such a triple we have $i > 0$. Hence by $(*)$ each α_{ijk} is a linear combination of certain α_{pqr}'s with $p + q + r < i + j + k$. Consequently, each α_{ijk} is a linear combination of the α_{pqr}'s satisfying $p + q + r = 2l + 2$. Since there are $[(l-1)/2] + 1$ of them (just solve the equations $p + 2q + 3r = 0$ and $p + q + r = 2l + 2$) it follows that $\dim \pi(M) \le [(l-1)/2] + 1$, where for any $g \in A$, $\pi(g)$ denotes the sum of all monomials of g of degree $\ge 2l + 2$.

(ii) Put $N := A^{D_1} \cap A_{n+1}$. Then N is the k-span of all "monomials"

$$n_p := {T_1}^{3p+2}{U_1}^{l-(2p+1)}, \qquad \text{where } 0 \le p \le [(l-1)/2].$$

CLAIM. *The* $\pi(n_p)$ *are linearly independent over* k.

It then follows from (i) and the inclusion $\pi(N) \subset \pi(M)$ that $\pi(N) = \pi(M)$, which proves the lemma.

(iii) To see the claim put

$$w_p := (-2)^{3p+2} 3^{l-(2p+1)} \pi(n_p)_{|T=0,\, U=\frac{1}{3}S} = \pi((S^2)^{3p+2}(S+S^3)^{l-(2p+1)}).$$

Observe that

$$(S^2)^{3p+2}(S+S^3)^{l-(2p+1)} = \sum_{j=0}^{l-(2p+1)} \binom{l-(2p+1)}{j} S^{3l+1-2j}.$$

Since $3l + 1 - 2j \geq 2l + 2$ iff $0 \leq j \leq [(l-1)/2]$ we get

$$w_p = \sum_{j=0}^{[(l-1)/2]} \binom{l-(2p+1)}{j} S^{3l+1-2j}.$$

Then the linear independence of the w_p (and hence of the $\pi(n_p)$) follows since

$$\det \left(\binom{l-(2p+1)}{j} \right)_{0 \leq p,j \leq [(l-1)/2]} \neq 0.$$

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