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SUBCATEGORIES OF THE DERIVED CATEGORY AND COTILTING COMPLEXES

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Abstract. We show that there is a one-to-one correspondence between basic cotilting complexes and certain contravariantly finite subcategories of the bounded derived category of an artin algebra. This is a triangulated version of a result by Auslander and Reiten. We use this to find an existence criterion for complements to exceptional complexes.

Introduction. Homologically finite subcategories were introduced by Auslander and Smalø [3], and they have proved to be important in the study of artin algebras. Homologically finite subcategories of the category of finitely generated modules have been studied by several authors. In [1], Auslander and Reiten showed that there is a correspondence between certain contravariantly finite subcategories and basic cotilting modules. In this paper we consider some subcategories of the bounded derived category of an artin algebra that are associated with cotilting complexes. In the first section we give definitions and basic results that we use in the second section, where we show that there is a correspondence between cotilting complexes and certain contravariantly finite subcategories of the derived category. The third section is devoted to examples. In the fourth section we use the correspondence to prove an existence criterion for complements of exceptional complexes.

1. Subcategories of the derived category. Let Λ be an artin algebra. Let $\operatorname{mod} \Lambda$ be the category of finitely generated left Λ-modules, and let $\mathcal{D} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ be the bounded derived category. This is a triangulated category. We denote the shift functor by $[1]$, and its inverse by $[-1]$. We let $\mathcal{I}(\Lambda)$ be the full subcategory of $\operatorname{mod} \Lambda$ formed by the injective objects, and, similarly, $\mathcal{P}(\Lambda)$ stands for the projectives. Then $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ is equivalent to $K^{+,\mathrm{b}}(\mathcal{I}(\Lambda))$. We consider this an identification, and let $K^{\mathrm{b}}(\mathcal{I}(\Lambda))$ denote the coperfect complexes. By a subcategory, we will always mean a full additive

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subcategory of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. Triangles will be denoted by $A \to B \to C \to A[1]$, or sometimes by $A \to B \to C \to$. By $\operatorname{Hom}(\ ,\ )$ we always mean homomorphisms in the derived category.

The existence of base change diagrams in triangulated categories is an important tool. For any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ D & \end{tikzcd}$$

there exists a commutative diagram

$$\begin{tikzcd} & X[-1] \arrow[r, equal] \arrow[d] & X[-1] \arrow[d] & \\ C[-1] \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d, equal] \\ C[-1] \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & C \\ & X \arrow[r, equal] & X & \end{tikzcd}$$

where the rows and columns are triangles. This is called a *cobase change*, and there is a dual notion of *base change*.

We also need the following basic facts about maps in the derived category.

LEMMA 1.1. *Let Λ be an artin algebra over a commutative artin ring R and let A and B be in $\mathcal{D}$.*

(a) $\operatorname{Hom}_{\mathcal{D}}(A, B)$ *is a finitely generated R-module.*
(b) $\operatorname{End}_{\mathcal{D}}(A)$ *and* $\operatorname{End}_{\mathcal{D}}(A)^{\mathrm{op}}$ *are artin algebras.*
(c) $\operatorname{Hom}_{\mathcal{D}}(A, B)$ *is a finitely generated* $\operatorname{End}_{\mathcal{D}}(A)^{\mathrm{op}}$*-module.*
(d) $\operatorname{Hom}_{\mathcal{D}}(A,\ )$ *induces an equivalence* $\operatorname{add} A \to \mathcal{P}(\operatorname{End}_{\mathcal{D}}(A)^{\mathrm{op}})$.

Proof. We only prove (a). To see that $\operatorname{Hom}_{\mathcal{D}}(A, B)$ is an R-module is easy. We show that it is finitely generated. Consider $A = (a^i)$ and $B = (b^i)$ as objects in $K^{+,\mathrm{b}}(\mathcal{I}(\Lambda))$. Then maps are represented by chain maps modulo homotopy. There are numbers M and N such that $a^m = b^m = 0$ for $m \leq M$ and $H^n(A) = H^n(B) = 0$ for $n \geq N$. Let $A_{>N}$ be the complex

$$\ldots \to 0 \to 0 \to a_{N+1} \to a_{N+2} \to a_{N+3} \to \ldots$$

and let $A_{\leq N}$ be the complex

$$\ldots \to a_{N-2} \to a_{N-1} \to a_N \to 0 \to 0 \to \ldots$$

Let $X_A = \ker(a_{N+1} \to a_{N+2})$. Similarly define $B_{>N}$, $B_{\le N}$ and X_B. Then $\mathrm{Hom}_{\mathcal{D}}(A,B)$ is an R-submodule of $\mathrm{Hom}_{\mathcal{D}}(A_{>N},B_{>N}) \oplus \mathrm{Hom}_{\mathcal{D}}(A_{\le N},B_{\le N})$. We see that $\mathrm{Hom}_{\mathcal{D}}(A_{>N},B_{>N})$ is finitely generated, because of the isomorphism $\mathrm{Hom}_{\mathcal{D}}(A_{>N},B_{>N}) \simeq \mathrm{Hom}_{\Lambda}(X_A,X_B)$. Moreover $\mathrm{Hom}_{\mathcal{D}}(A_{\le N},B_{\le N})$ is obviously finitely generated. The claim now follows. ■

We also need minimal versions of maps. A map $f: A \to B$ is called *right minimal* if for any $g: A \to A$ such that $fg = f$, the map g must be an automorphism. There is a dual notion of *left minimal.*

LEMMA 1.2. *Let $f: A \to B$ be a map in $\mathcal{D}$. Then there is a decomposition $A = A' \oplus A''$ such that $f|_{A'}$ is right minimal and $f|_{A''} = 0$.*

Proof. Apply the functor $\mathrm{Hom}(A,\ )$ to f. Choose A' such that $\mathrm{Hom}(A,A') \to \mathrm{Im}\,\mathrm{Hom}(A,f)$ is a projective cover, and let A'' be such that $A' \oplus A'' = A$. The claim now follows from Lemma 1.1(d). ■

A subcategory $\mathcal{X}$ is called *extension closed* if for any triangle $A \to B \to C \to$ in $D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$, B is in $\mathcal{X}$ whenever A and C are in $\mathcal{X}$. It is called *resolving* if it is extension closed and closed under $[-1]$ (that means $X[-1]$ is in $\mathcal{X}$ whenever X is). For any subcategory $\mathcal{X}$ let ${}^{\perp}\mathcal{X} = \{Y \in D^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \mid \mathrm{Hom}(Y,\mathcal{X}[i]) = 0 \text{ for all } i > 0\}$. When T is a complex we denote ${}^{\perp}(\mathrm{add}\,T)$ by ${}^{\perp}T$. For any subcategory $\mathcal{X}$, the category ${}^{\perp}\mathcal{X}$ is obviously resolving. We also define $\mathcal{X}^{\perp} = \{Y \in D^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \mid \mathrm{Hom}(\mathcal{X},Y[i]) = 0 \text{ for all } i > 0\}$, and this is always a coresolving category (that is, closed under extensions and $[1]$).

Let $\mathcal{X}$ denote a subcategory, and let C be any object in $D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$. A *right $\mathcal{X}$-approximation* to C is a map $f: X \to C$ with X in $\mathcal{X}$ such that all maps from an object in $\mathcal{X}$ to C factor through f. If this map is right minimal, it is called a *minimal right $\mathcal{X}$-approximation.* A subcategory is *contravariantly finite* if there exists a right $\mathcal{X}$-approximation for any object C in $D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$. We have dual notions of *left approximations*, and *covariantly finite* subcategories. It follows easily from Lemma 1.1 that finite subcategories of $\mathcal{D}$ are contravariantly finite and covariantly finite. The next lemma is a triangulated version of Wakamatsu's Lemma [6].

LEMMA 1.3. *Let $\mathcal{X}$ be an extension closed subcategory of $D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$. Let $Y_C \to X_C \xrightarrow{f} C \to$ be a triangle, where f is a minimal right $\mathcal{X}$-approximation. Then $\mathrm{Hom}(X,Y_C[1]) = 0$ for all X in $\mathcal{X}$. If $\mathcal{X}$ is resolving, then Y_C is in $\mathcal{X}^{\perp}$.*

Proof. Assume there is a map $X[-1] \to Y_C$, where X is in $\mathcal{X}$. Then there is a cobase change diagram

$$\begin{tikzcd}
 & X[-1] \arrow[r, equal] \arrow[d] & X[-1] \arrow[d] & \\
C[-1] \arrow[r] \arrow[d, equal] & Y_C \arrow[r] \arrow[d] & X_C \arrow[r] \arrow[d] & C \arrow[d, equal] \\
C[-1] \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & C \\
 & X \arrow[r, equal] & X &
\end{tikzcd}$$

Since $\mathcal{X}$ is extension closed, E is in $\mathcal{X}$ and the map $E \to C$ factors through f. By this we get the commutative diagram of triangles

$$\begin{tikzcd}
Y_C \arrow[r] \arrow[d] & X_C \arrow[r] \arrow[d] & C \arrow[d, equal] \\
B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & C \arrow[d, equal] \\
Y_C \arrow[r] & X_C \arrow[r] & C
\end{tikzcd}$$

Since $X_C \to C$ is minimal, the composition $X_C \to E \to X_C$ is an isomorphism, and therefore also the composition $Y_C \to B \to Y_C$ is an isomorphism. Thus $Y_C \to B$ is a split monomorphism, and thus the map $X[-1] \to Y_C$ is 0. The second statement follows now by definition. ∎

Let $\mathcal{X}$ be any subcategory of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. Let $\widehat{\mathcal{X}}$ be the full subcategory of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ with objects C such that there is an integer n and a sequence of triangles

$$\begin{gathered}
X_n \to X_{n-1} \to K_{n-1} \to \\
K_{n-1} \to X_{n-2} \to K_{n-2} \to \\
\vdots \\
K_1 \to X_0 \to C \to
\end{gathered}$$

with all the X_i in $\mathcal{X}$.

When $\mathcal{X}$ is resolving, the category $\widehat{\mathcal{X}}$ has a particularly nice description.

Lemma 1.4. *Let $\mathcal{X}$ be a resolving subcategory. Then C is in $\widehat{\mathcal{X}}$ if and only if there is an $n > 0$ such that $C[-n]$ is in $\mathcal{X}$.*

Proof. Observe that when we have a triangle $A \to B \to C \to$ with B and C in $\mathcal{X}$, then also A is in $\mathcal{X}$. Assume C is in $\widehat{\mathcal{X}}$, and that there is a sequence of triangles as in the definition of $\widehat{\mathcal{X}}$. Then $K_{n-i}[-i]$ is in $\mathcal{X}$ for $i = 1, \ldots, n-1$, and $C[-n]$ is in $\mathcal{X}$.

Assume $C[-n]$ is in $\mathcal{X}$. Consider the triangle

$$C[-1] \to 0 \to C \to .$$

Since 0 is in $\mathcal{X}$, and $C[-n]$ is in $\mathcal{X}$, the claim follows by considering shifts of this triangle. ∎

The following two lemmas will be useful in the next section, where we consider the subcategory $\omega = \mathcal{X} \cap \mathcal{X}^{\perp}$.

Lemma 1.5. *Let $\mathcal{X}$ be resolving. If $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$, then $\mathcal{X}^{\perp}$ consists of coperfect complexes.*

Proof. Assume Λ has r simple modules up to isomorphism, and let $C_1, \dots, C_r$ be stalk complexes with the simples in degree 0. Then there is an N, such that $C_i[-n]$ is in $\mathcal{X}$ for $n > N$ and all $i = 1, \dots, r$. Let Y be an object in $\mathcal{X}^{\perp}$. Then Y can be represented by a complex of injective modules y^i, with differential δ^i, bounded to the left, and there is a t such that $s > t$ implies $H^s(Y) = 0$ and $H^t(Y) \neq 0$. We can also assume that

$$0 \to \ker \delta^t \to y^t \to y^{t+1} \to \dots$$

is a minimal injective resolution of $\ker \delta^t$. To prove our claim, we need to show that this resolution is always finite. So assume $\ker \delta^t$ has a summand of infinite injective dimension. Choose $k > N + 1$ such that $H^s(Y) = 0$ for $s \geq k-1$. If we can show that $\operatorname{Hom}(C[-(k-1)], Y[1]) = \operatorname{Hom}(C[-k], Y) \neq 0$, then $C[-(k-1)]$ is not in $\mathcal{X}$, and we have a contradiction, since $k - 1 > N$. Let m be a non-injective indecomposable summand of $\operatorname{Im} \delta^{k-1}$, and let s be a simple submodule of m. The inclusion map $s \hookrightarrow m$ induces a chain map β

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & j^0 \arrow[r, "\alpha^0"] \arrow[d, "\beta^0"'] & j^1 \arrow[r, "\alpha^1"] \arrow[d, "\beta^1"'] & j^2 \arrow[r, "\alpha^2"] \arrow[d, "\beta^2"'] & \cdots \\
\cdots \arrow[r] & y^{k-1} \arrow[r, "\delta^{k-1}"] & y^k \arrow[r, "\delta^k"] & y^{k+1} \arrow[r, "\delta^{k+1}"] & y^{k+2} \arrow[r, "\delta^{k+2}"] & \cdots
\end{tikzcd}$$

where the upper row is a minimal injective resolution of s. We need to show that this map is not nullhomotopic. Assume there is a homotopy γ. Since $\gamma^1\alpha^0|_s = 0$ and $\beta^0|_s = \delta^{k-1}\gamma^0|_s + \gamma^1\alpha^0|_s \neq 0$, the composition $(s \hookrightarrow j^0 \xrightarrow{\gamma^0} y^{k-1} \xrightarrow{\delta^{k-1}} m) \neq 0$, and thus j^0 must be isomorphic to m, and we have a contradiction. ∎

Lemma 1.6. *Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$, and assume $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. Then $\widehat{\omega} = \mathcal{X}^{\perp}$.*

Proof. Observe that for a triangle $A \to B \to C \to$, if A and B are in $\mathcal{X}^{\perp}$, then so is C. Therefore $\widehat{\omega}$ is in $\mathcal{X}^{\perp}$, since ω obviously is. Let C be in $\mathcal{X}^{\perp}$, and assume $C[-n]$ is in $\mathcal{X}$, where $n > 0$. Choose minimal right $\mathcal{X}$-approximations to find triangles

$$K_1 \to X_0 \to C \to$$
$$K_2 \to X_1 \to K_1 \to$$
$$\vdots$$
$$K_n \to X_{n-1} \to K_{n-1} \to$$

with X_i in $\mathcal{X}$. Then the K_i are in $\mathcal{X}^\perp$ by Lemma 1.3. Then also the X_i are in $\mathcal{X}^\perp$ since $\mathcal{X}^\perp$ is closed under extensions. Thus, we need only show that K_n is in $\mathcal{X}$. But since $\mathcal{X}$ is resolving and $C[-n]$ is in $\mathcal{X}$, we see that $K_i[-n+i]$ is in $\mathcal{X}$. ∎

Let $\mathcal{X}$ be any subcategory of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ and let C be any object in $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. If C is in $\widehat{\mathcal{X}}$, then we define the *resolution dimension* $\mathcal{X}\text{-resdim}\, C$ to be the smallest integer n such that there are triangles

$$X_n \to X_{n-1} \to K_{n-1} \to$$
$$K_{n-1} \to X_{n-2} \to K_{n-2} \to$$
$$\vdots$$
$$K_1 \to X_0 \to C \to$$

with all the X_i in $\mathcal{X}$. If C is not in $\widehat{\mathcal{X}}$, we let $\mathcal{X}\text{-resdim}\, C = \infty$. We then have the following.

LEMMA 1.7. *Let $\mathcal{X}$ be closed under shift. Then for any triangle $A \to B \to C \to$, we have $\mathcal{X}\text{-resdim}\, C \leq \mathcal{X}\text{-resdim}\, A + \mathcal{X}\text{-resdim}\, B + 1$.*

Proof. Let $a = \mathcal{X}\text{-resdim}\, A$, $b = \mathcal{X}\text{-resdim}\, B$ and $c = \mathcal{X}\text{-resdim}\, C$. The proof is by induction on b. For $b = 0$, we have $c \leq 1 + a = 1 + a + b$, by the definition of the resolution dimension. Assume $1 \leq b < \infty$. Then there is a triangle $Q \to X \to B \to$, with X in $\mathcal{X}$, and $\mathcal{X}\text{-resdim}\, Q \leq b - 1$. Consider the diagram

$$\begin{tikzcd}
 & & A \arrow[r, equal] \arrow[d] & A \arrow[d] \\
Q \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & Q[1] \arrow[d] \\
K \arrow[r] & X \arrow[r] & C \arrow[r] \arrow[d] & K[1] \arrow[d] \\
 & & A[1] \arrow[r, equal] & A[1]
\end{tikzcd}$$

Then (by the octahedral axiom) $A \to Q[1] \to K[1] \to$ is a triangle. By induction we have $\mathcal{X}\text{-resdim}\, K \leq a + b - 1 + 1$, and therefore $\mathcal{X}\text{-resdim}\, C \leq a + b + 1$. ∎

For any subcategory ω let $\widetilde{\omega}$ denote the closure under shift. A full subcategory $\mathcal{T}'$ of a triangulated category $\mathcal{T}$ is said to *generate* $\mathcal{T}$ if the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{T}'$ is equal to $\mathcal{T}$. We have the following obvious consequence of the preceding lemma.

PROPOSITION 1.8. *Let $\mathcal{X}$ be a triangulated subcategory of $D^{\mathrm{b}}(\operatorname{mod}\Lambda)$. Assume $\omega \subseteq \mathcal{X} \subseteq D^{\mathrm{b}}(\operatorname{mod}\Lambda)$ and that ω generates $\mathcal{X}$. Then for any object X in $\mathcal{X}$, we have $\widetilde{\omega}\text{-resdim}\, X < \infty$.*

2. Subcategories and cotilting complexes. In this section we show that there is a 1-1 correspondence between certain contravariantly finite subcategories of $D^{\mathrm{b}}(\operatorname{mod}\Lambda)$ and basic cotilting complexes. A *cotilting complex* is a complex T in $D^{\mathrm{b}}(\operatorname{mod}\Lambda)$ with the following properties:

- T is in $K^{\mathrm{b}}(\mathcal{I}(\Lambda))$.
- $\operatorname{Hom}(T, T[i]) = 0$ for all $i \neq 0$.
- T generates $K^{\mathrm{b}}(\mathcal{I}(\Lambda))$.

The dual notion of tilting complexes was introduced by Rickard [5]. If we write T as a sum of indecomposable objects $T_1 \oplus \ldots \oplus T_r$, then T is called *basic* if T_i is not isomorphic to T_j, when $i \neq j$.

To show our promised correspondence we will need the following lemmas.

LEMMA 2.1. *Let $\mathcal{X}$ be a contravariantly finite, resolving subcategory with $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod}\Lambda)$. Assume $\mathcal{X}^{\perp}$ generates $K^{\mathrm{b}}(\mathcal{I}(\Lambda))$ and that $\omega = \mathcal{X} \cap \mathcal{X}^{\perp}$ is selforthogonal. Then $\omega = \operatorname{add} T$ for a cotilting complex T.*

Proof. Since $\mathcal{X}^{\perp} = \widehat{\omega}$ by Lemma 1.6, we find that ω generates $K^{\mathrm{b}}(\mathcal{I}(\Lambda))$. By Lemma 1.5, ω is coperfect. Thus, we need only show that ω is of finite type. By Proposition 1.8 the $\widetilde{\omega}$-resdim of I is finite, where I is the stalk complex with an injective cogenerator of $\operatorname{mod}\Lambda$ in degree zero. Thus, there exist triangles

$$W_n[i_n] \to W_{n-1}[i_{n-1}] \to K_{n-1} \to$$

$$K_{n-1} \to W_{n-2}[i_{n-2}] \to K_{n-2} \to$$

$$\vdots$$

$$K_1 \to W_0[i_0] \to I \to$$

with the W_i in ω. The direct sum $T = \bigoplus_{i=0}^{n} W_i$ is a cotilting complex. For any Y in ω, also $T \oplus Y$ is a cotilting complex. This means that Y is in $\operatorname{add} T$, since the number of nonisomorphic indecomposable summands for all cotilting complexes over a fixed algebra Λ is constant. ■

LEMMA 2.2. *Let T be a selforthogonal coperfect complex. Then ${}^{\perp}T$ is contravariantly finite.*

Proof. Choose an object X_0 in $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. Since T is coperfect, there is an integer r and an integer $s \geq 0$ such that $\operatorname{Hom}(X_0, T[i]) = 0$ when $i \notin [r-s, r]$. If $r \leq 0$, then X_0 is in ${}^{\perp}T$. Assume $r \geq 1$. Choose a minimal left $\operatorname{add} T[r]$-approximation such that we have a triangle $X_0 \to T_0[r] \to X_1 \to$. Then, apply the functor $\operatorname{Hom}(\ ,T)$ to this triangle, and consider the corresponding long exact sequence. By applying Wakamatsu's Lemma we then have $\operatorname{Hom}(X_1, T[i]) = 0$ when $i \notin [r-s+1, r]$ if $s \neq 0$. If $s = 0$, then also $\operatorname{Hom}(X_1, T[i]) = 0$ for $i \neq r$. Repeat this r times. If $2r - s \leq r$, then $\operatorname{Hom}(X_r, T[i]) = 0$ for $i \notin [2r-s, r]$. Otherwise, $\operatorname{Hom}(X_r, T[i]) = 0$ for $i \neq r$. In both cases $X_r[-r]$ is in $\mathcal{X} = {}^{\perp}T$. We have a composition of maps

$$X_r[-r] \to X_{r-1}[-r+1] \to \ldots \to X_1[-1] \to X_0.$$

This map is a right $\mathcal{X}$-approximation. To prove this, assume there is a map $f : X \to X_0$ with X in $\mathcal{X}$. Since the composition $X \to X_0 \to T_0[r]$ is 0, the map f factors through $X_1[-1] \to X_0$. Repeat this argument for $X_i[-i]$ for $i = 1, \ldots, r-1$. ∎

Lemma 2.3. *Let T be a selforthogonal coperfect complex. Then ${}^{\perp}T$ is resolving, and ${}^{\perp}T \cap ({}^{\perp}T)^{\perp} = \operatorname{add} T$. Also $\widehat{{}^{\perp}T} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ and $({}^{\perp}T)^{\perp} = \widehat{\operatorname{add} T}$.*

Proof. The subcategory ${}^{\perp}T = \mathcal{X}$ is obviously resolving, and $\operatorname{add} T \subseteq \mathcal{X} \cap \mathcal{X}^{\perp}$ by assumption. Let X be in $\mathcal{X} \cap \mathcal{X}^{\perp}$, and choose a minimal left $\operatorname{add} T$-approximation $X \to T_0 \to X_1 \to$. Then X_1 is in ${}^{\perp}T$ by the dual of Wakamatsu's Lemma. Therefore $X_1 \to X[1]$ is the 0 map, and X is in $\operatorname{add} T$. Thus $\operatorname{add} T = {}^{\perp}T \cap ({}^{\perp}T)^{\perp}$. It follows from Lemma 1.4 that $\widehat{{}^{\perp}T} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. It remains to show that $({}^{\perp}T)^{\perp} = \widehat{\operatorname{add} T}$, but this follows from Lemma 1.6. ∎

We are now ready to prove the main theorem.

Theorem 2.4. *There is a* 1-1 *correspondence between the basic cotilting complexes, and the subcategories $\mathcal{X}$ of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ with the following properties*:

(a) *$\mathcal{X}$ is contravariantly finite.*
(b) *$\mathcal{X}$ is resolving.*
(c) $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$.
(d) *$\mathcal{X} \cap \mathcal{X}^{\perp}$ is selforthogonal.*
(e) *$\mathcal{X}^{\perp}$ generates $K^{\mathrm{b}}(\mathcal{I}(\Lambda))$.*

The correspondence is given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathcal{X}^{\perp}$ and $T \mapsto {}^{\perp}T$.

Proof. By Lemmas 2.1, 2.2 and 2.3 we only need to show that the given maps are bijections. Also, by Lemma 2.3 we know that given a cotilting complex T, we have ${}^{\perp}T \cap ({}^{\perp}T)^{\perp} = \operatorname{add} T$, so it is enough to show that given an $\mathcal{X}$ such that (a)–(e) are satisfied, we have ${}^{\perp}(\mathcal{X} \cap \mathcal{X}^{\perp}) = \mathcal{X}$. The inclusion $\mathcal{X} \subseteq {}^{\perp}(\mathcal{X} \cap \mathcal{X}^{\perp})$ is obvious. Let $\omega = \mathcal{X} \cap \mathcal{X}^{\perp}$. Then it is easy to see that ${}^{\perp}\omega = {}^{\perp}(\widehat{\omega})$. Since $\widehat{\omega} = \mathcal{X}^{\perp}$, we only need to show that ${}^{\perp}(\mathcal{X}^{\perp}) \subseteq \mathcal{X}$. For this, let Z be in ${}^{\perp}(\mathcal{X}^{\perp})$ and choose a minimal right $\mathcal{X}$-approximation $X \to Z$. Consider the triangle $Y \to X \to Z \to$. By Wakamatsu's Lemma, Y is in $\mathcal{X}^{\perp}$, so that the map $Z \to Y[1]$ is the 0 map. But then Z is a summand in X. ■

3. Examples. In this section we give examples showing that none of the properties (a)–(e) in Theorem 2.4 can be left out.

Example 1. Let T be any *exceptional complex*. This means that T is coperfect and $\operatorname{Hom}(T, T[i]) = 0$ when $i \neq 0$. Assume that T is not a cotilting complex. Then ${}^{\perp}T$ satisfies the first four conditions in Theorem 2.4, but not (e).

Example 2. Let Λ be the lower triangular 3×3 matrix algebra over a field k. The algebra has 6 indecomposable modules; the three simples S_1, S_2 and S_3, the projectives P_1 and P_2 of length 3 and 2 and an injective I_2 of length 2. Since Λ is hereditary, the indecomposable complexes are just stalk complexes of indecomposable modules. It is easy to see that any (full) subcategory of $D^{\mathrm{b}}(\operatorname{mod} \Lambda)$ is contravariantly finite.

Let

$$\mathcal{X} = \Big(\bigcup_{i \leq 0,\, X \in \operatorname{ind} \Lambda} X[i] \Big) \setminus P_1[-1].$$

Then $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod} \Lambda)$. We also see that

$$\mathcal{X}^{\perp} = \Big(\bigcup_{i \geq 1,\, Y \in \operatorname{ind} \Lambda} Y[i] \Big) \cup \{P_1, I_2, S_1\},$$

so $\mathcal{X}^{\perp}$ obviously generates the derived category, and we also have $\mathcal{X} \cap \mathcal{X}^{\perp} = \{P_1, I_2, S_1\}$ which is selforthogonal. But $\mathcal{X}$ is not resolving, since $P_1[-1]$ is not in $\mathcal{X}$. We underline the objects in $\mathcal{X}$, and overline the objects in $\mathcal{X}^{\perp}$ in the Auslander–Reiten quiver of the derived category.

$$\begin{tikzcd}
\underline{P_3[-1]} \arrow[dr] & & \underline{S_2[-1]} \arrow[dr] & & \underline{S_1[-1]} \arrow[dr] & & \overline{\underline{P_1}} \arrow[dr] & & \overline{P_3[1]} \arrow[dr] & & \overline{S_2[1]} \arrow[dr] & & \overline{S_1[1]} \\
\cdots & \underline{P_2[-1]} \arrow[ur] \arrow[dr] & & \underline{I_2[-1]} \arrow[ur] \arrow[dr] & & \underline{P_2} \arrow[ur] \arrow[dr] & & \overline{\underline{I_2}} \arrow[ur] \arrow[dr] & & \overline{P_2[1]} \arrow[ur] \arrow[dr] & & \overline{I_2[1]} \arrow[ur] \arrow[dr] & \cdots \\
\underline{S_1[-2]} \arrow[ur] & & P_1[-1] \arrow[ur] & & \underline{P_3} \arrow[ur] & & \underline{S_2} \arrow[ur] & & \overline{\underline{S_1}} \arrow[ur] & & \overline{P_1[1]} \arrow[ur] & & \overline{P_3[2]}
\end{tikzcd}$$

Example 3. Let Λ be as in the above example, and use the same notation for the modules. Let

$$\mathcal{X} = \Big(\Big(\bigcup_{i\leq -1,\, X\in \operatorname{ind}\Lambda} X[i]\Big) \setminus S_1[-1]\Big) \cup \{P_3[1], P_3\}.$$

Then $\mathcal{X}$ is resolving and by Lemma 1.4 we also see that $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod}\Lambda)$. We have

$$\mathcal{X}^{\perp} = \Big(\bigcup_{i\geq 1,\, X\in \operatorname{ind}\Lambda} X[i]\Big) \cup \{S_2, I_2, S_1, S_2[-1], I_2[-1], S_1[-1]\}.$$

Therefore, $\mathcal{X}^{\perp}$ generates the derived category. But

$$\mathcal{X} \cap \mathcal{X}^{\perp} = \{S_2[-1], I_2[-1], P_3[1]\}$$

is not selforthogonal.

$$\begin{tikzcd}[column sep=tiny]
 & \underline{P_3[-1]} \arrow[dr] & & \overline{\underline{S_2[-1]}} \arrow[dr] & & \overline{S_1[-1]} \arrow[dr] & & P_1 \arrow[dr] & & \overline{\underline{P_3[1]}} \arrow[dr] & & \overline{S_2[1]} \arrow[dr] & & \overline{S_1[1]} & \\
\cdots & & \underline{P_2[-1]} \arrow[ur] \arrow[dr] & & \overline{\underline{I_2[-1]}} \arrow[ur] \arrow[dr] & & P_2 \arrow[ur] \arrow[dr] & & \overline{I_2} \arrow[ur] \arrow[dr] & & \overline{P_2[1]} \arrow[ur] \arrow[dr] & & \overline{I_2[1]} \arrow[ur] \arrow[dr] & & \cdots \\
 & \underline{S_1[-2]} \arrow[ur] & & \underline{P_1[-1]} \arrow[ur] & & \underline{P_3} \arrow[ur] & & \overline{S_2} \arrow[ur] & & \overline{S_1} \arrow[ur] & & \overline{P_1[1]} \arrow[ur] & & \overline{P_3[2]} &
\end{tikzcd}$$

Example 4. Let Λ be the Kronecker algebra. This is again a hereditary algebra, such that the indecomposable complexes are just stalk complexes with indecomposable modules. There are three types of indecomposable modules (for details, see [2]): the preprojectives $\mathcal{P}$, the regular modules $\mathfrak{R}$ and the preinjectives $\mathcal{I}$. Let

$$\mathcal{X} = \Big(\bigcup_{i<0,\, X\in \operatorname{ind}\Lambda} X[i]\Big) \cup \Big(\bigcup_{Y\in \mathcal{P}\cup\mathfrak{R}} Y[0]\Big).$$

Then $\mathcal{X}$ is resolving and $\widehat{\mathcal{X}} = D^{\mathrm{b}}(\operatorname{mod}\Lambda)$. In this case, we have

$$\mathcal{X}^{\perp} = \Big(\bigcup_{i>0,\, X\in \operatorname{ind}\Lambda} X[i]\Big) \cup \Big(\bigcup_{Y\in \mathcal{I}} Y[0]\Big).$$

Thus, $\mathcal{X}^{\perp}$ generates $D^{\mathrm{b}}(\operatorname{mod}\Lambda)$, and $\mathcal{X}\cap\mathcal{X}^{\perp} = 0$. But $\mathcal{X}$ is not contravariantly finite, since for example the stalk complex with the simple injective in degree zero, does not have any $\mathcal{X}$-approximation.

Example 5. Let Λ be the Kronecker algebra. Let $\mathcal{X} = \bigcup_{i\leq 0} P[i]$, where P is the simple projective module. Then $\mathcal{X}$ is resolving and contravariantly finite. We have

$$\mathcal{X}^{\perp} = \Big(\bigcup_{j\in\mathbb{Z}} I[j]\Big) \cup \Big(\bigcup_{i\geq 0,\, X\in \operatorname{ind}\Lambda} X[i]\Big),$$

where I is the simple injective module. Then obviously $\mathcal{X}^{\perp}$ generates $D^{\mathrm{b}}(\operatorname{mod}\Lambda)$, and $\mathcal{X}\cap\mathcal{X}^{\perp}=\{P[0]\}$ is selforthogonal. Since $\mathcal{X}$ is resolving, we can use Lemma 1.4 to see that $\widehat{\mathcal{X}}\neq D^{\mathrm{b}}(\operatorname{mod}\Lambda)$.

4. Partial cotilting complexes. In this section we show that the correspondence given in section 2 can be used to give an existence criterion for complements of exceptional complexes. If C is an exceptional complex, then a complex X is called a *complement* if $C\oplus X$ is a cotilting complex. There is a similar result for module categories in [4].

PROPOSITION 4.1. *Let C be an exceptional complex, and let $\mathcal{Y}={}^{\perp}C$. Then C has a complement if and only if there is a category $\mathcal{X}\subseteq\mathcal{Y}$ such that $\mathcal{X}$ satisfies* (a)–(e) *in Theorem* 2.4 *and C is in $\mathcal{X}$.*

Proof. If C has a complement X, then choose $\mathcal{X}={}^{\perp}(C\oplus X)$. Conversely, let $\mathcal{X}$ be such that $\mathcal{X}\subseteq\mathcal{Y}$ and C is in $\mathcal{X}$ and such that $\mathcal{X}$ satisfies (a)–(e) in Theorem 2.4. Then $\mathcal{X}={}^{\perp}T$ for a cotilting complex T. Since ${}^{\perp}T\subseteq{}^{\perp}C$, we have $({}^{\perp}C)^{\perp}\subseteq({}^{\perp}T)^{\perp}$. But then C is in $({}^{\perp}T)^{\perp}\cap{}^{\perp}T=\operatorname{add}T$. This means that C has a complement. ■

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