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WEAK CONVERGENCE OF ITERATES OF LASOTA–MACKEY–TYRCHA OPERATORS*

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Dedicated to the memory of Anzelm Iwanik

Abstract. We show that Lasota–Mackey–Tyrcha stochastic operators, which are used in mathematical modeling of cell cycles, have weak* convergent iterates.

1. Introduction. Let (X, d) be a separable metric space such that all finite closed balls $K(x_0, r) = \{x \in X : \mathrm{d}(x, x_0) \le r\}$ are compact. Given a σ-finite measure μ on the Borel σ-algebra $\mathcal{B}$ of subsets of (X, d) we denote by $(L^1(\mu), \|\cdot\|)$ the Banach lattice of μ-integrable functions on X. Functions from $L^1(\mu)$ which are equal μ-almost everywhere are identified. Then instead of $\mathcal{B}$ we will rather think of its μ-completion $\widetilde{\mathcal{B}}$. If not stated otherwise also all inequalities are in the μ-a.e. sense. The convex set $\{f \in L^1(\mu) : f \ge 0, \int_X f\, d\mu = 1\}$ of all *densities* is denoted by $\mathcal{D}_\mu$.

A linear operator $P : L^1(\mu) \to L^1(\mu)$ which preserves $\mathcal{D}_\mu$ (i.e. $P(\mathcal{D}_\mu) \subseteq \mathcal{D}_\mu$) is called *markovian* (or a *Markov operator*). If there exists a Borel measurable function $k : X \times X \to \mathbb{R}_+$ such that $Pf(x) = \int_X k(x, y) f(y)\, d\mu(y)$, then the Markov operator P is called a *kernel operator*. Clearly, in this case, for every $y \in X$ we have $\int k(x, y)\, d\mu(x) = 1$. Here we require $k(\,\cdot\,, y) \in \mathcal{D}_\mu$ for all y, instead of almost all, to extend P to all finite measures on X. Otherwise, this extension would be defined only on a set of full μ-measure. Now P is extended to a positive contraction (also denoted by P), acting on the Banach lattice $(\mathcal{M}(X), \|\cdot\|)$ of all bounded signed measures ν on $(X, \mathcal{B})$, by

$$P\nu(A) = \int_X \int_X k(x, y) \mathbf{1}_A(x)\, d\mu(x)\, d\nu(y).$$

Obviously $P\nu \in L^1(\mu)$.

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The paper is devoted to Markov operators on $L^1([0,\infty))$, the Banach lattice of Lebesgue integrable functions on $[0,\infty)$, with kernels

$$(1)\qquad k(x,y)=\begin{cases} -\frac{\partial}{\partial x}\,H(Q(\lambda(x))-Q(y)) & \text{if } 0\le y\le \lambda(x),\\ 0 & \text{otherwise.}\end{cases}$$

The functions $H,Q,\lambda:[0,\infty)\to[0,\infty)$ are absolutely continuous and satisfy:

(H) $H(0)=1$, $\lim_{x\to\infty}H(x)=0$, H is nonincreasing,

(Qλ) $Q(0)=\lambda(0)=0$, $\lim_{x\to\infty}Q(x)=\lim_{x\to\infty}\lambda(x)=\infty$, and Q, λ are nondecreasing.

The class of Markov operators with kernels (1) such that H, Q, λ satisfy conditions (H), (Qλ) is denoted by LMT (after Lasota, Mackey and Tyrcha whose contribution to mathematical modeling of cell cycles is crucial; see [GL], [LM1], [LM2], [LM3], [LMT], [KT], [R2] and [T] for more details).

A kernel stochastic operator on $L^1(\mu)$ is called *strong Feller in the strict sense* if the mapping

$$\text{(SFS)}\qquad (X,\mathrm{d})\ni y\mapsto k(\,\cdot\,,y)\in(\mathcal{D}_\mu,\|\cdot\|)$$

is continuous. We note that for the adjoint operator P^* the condition (SFS) implies that, whenever $h\in L^\infty(\mu)$, then the image P^*h is continuous. This easily follows from $P^*h(y)=\int_X k(x,y)h(x)\,d\mu(x)$.

In this paper Markov operators P are (SFS) with P^* preserving $C_0(X)$, the Banach lattice of continuous functions h vanishing at infinity (i.e. h is continuous and for every $\varepsilon>0$ there exists a compact set $E_\varepsilon\subseteq X$ satisfying $|h(x)|\le\varepsilon$ for all $x\notin E_\varepsilon$). As usual $C_0(X)$ is endowed with the sup-norm $\|\cdot\|_{\sup}$. If there exists x_0 such that for every $\varepsilon>0$ there is a function $r_\varepsilon:[0,\infty)\to[0,\infty)$ such that $\lim_{t\to\infty}(t-r_\varepsilon(t))=\infty$ and

$$\inf_{y\in X}\int\limits_{K(y,r_\varepsilon(\mathrm{d}(x_0,y)))}k(x,y)\,d\mu(x)>1-\varepsilon,$$

and the kernel $k(x,y)$ has property (SFS), then $P^*C_0(X)\subseteq C_0(X)$. It has been noticed in [B] that LMT operators satisfy (SFS) and P^* preserves $C_0([0,\infty))$.

We recall that the set of all subprobabilistic positive measures on X is compact for the vague topology. We say that a variation norm bounded sequence of measures ν_n is *vaguely convergent* to ν if $\lim_{n\to\infty}\int_X h\,d\nu_n=\int_X h\,d\nu$ for all $h\in C_0(X)$. Since $\mathcal{M}(X)$ may be identified with the adjoint space $C_0(X)^*$, this is simply weak* convergence.

A Markov operator P on $L^1(\mu)$ is said to be *asymptotically stable* if there exists a (unique) $f_*\in\mathcal{D}_\mu$ such that $\lim_{n\to\infty}\|P^nf-(\int f\,d\mu)f_*\|=0$ for all $f\in L^1(\mu)$. Obviously f_* is P-invariant, i.e. $Pf_*=f_*$. The Banach sublattice of all P-invariant functions is denoted by $L^1_*(\mu)$.

The opposite concept to stability is sweeping. Given a family $\mathcal{A}_*$ of subsets of $\mathcal{B}$ we say that P is $\mathcal{A}_*$-*sweeping* if $\lim_{n\to\infty}\int_A P^n f\,d\mu = 0$ for all $A\in\mathcal{A}_*$ and $f\in\mathcal{D}_\mu$. Here $\mathcal{A}_*$ satisfies the natural conditions that $0<\mu(A)<\infty$ for all $A\in\mathcal{A}_*$, $\mathcal{A}_*$ is closed under finite unions, and $\bigcup_{j=1}^{\infty}A_j = X$ for some sequence $A_j\in\mathcal{A}_*$. In our case $\mathcal{A}_*$ is the family of all compact subsets (balls) of (X,d). It has recently been proved in [M] (see also [KM], [KT], [ŁR], [R1] and [LM2]) that an LMT operator either has an invariant density f_* or it is sweeping with respect to compact sets (thus the Foguel alternative holds).

In this paper we obtain a stronger result. Namely, we show that the iterates $P^n f$ of an LMT operator P are weak* convergent to Qf, where Q is a submarkovian projection onto $L^1_*(\mu)$ (i.e. $Q^2=Q\ge 0$ and $\int_X Qf\,d\mu\le 1$ for every $f\in\mathcal{D}_\mu$). Moreover, if $F\in\mathcal{B}$ denotes the minimal (modulo sets of measure zero) set which carries supports of all P-invariant densities, then $Q_F = Q\restriction_{L^1(F,\mu)}$ is a markovian projection onto P-invariant functions, and $\|P^n f - Q_F f\|\to 0$ for all $f\in L^1(F,\mu)$. We note that $L^1(F,\mu)$ is P-invariant and that F is well defined, because $L^1(\mu)$ is separable. If $F_j=\mathrm{supp}(f_{*j})$, where f_{*j} is an ergodic invariant density, then $P_j = P\restriction_{L^1(F_j,\mu)}$ is simply asymptotically stable on its domain. We note that for each $f\in\mathcal{D}_\mu$ we have $\int_{F_j}Qf\,d\mu = \lim_{n\to\infty}\int_{F_j}P^n f\,d\mu = \Lambda_j(f)$, where the last sequence does converge as it is bounded and nondecreasing (F_j is invariant). This implies $Qf=\Lambda_j(f)f_{*j}$ on F_j. If there are no invariant densities at all, then obviously $Q\equiv 0$, and $\lim_{n\to\infty}\int_0^\infty P^n f h\,d\mu = 0$ for every $h\in C_0[0,\infty)$. In particular, P is sweeping with respect to the family of compact sets.

The rest of our notation is consistent with [F1]. We also use some of its results. Let us briefly recall them. Given a stochastic operator P on $L^1(\mu)$, the space X may be divided into two disjoint parts $C\,\dot\cup\, D = X$. The *conservative part* C is characterized by $C=\{x\in X: f\ge 0\Rightarrow \sum_{n=0}^\infty P^n f(x)$ is either 0 or $\infty\}$. Because $P^*\mathbf{1}_C\ge\mathbf{1}_C$, the markovian operator $P_C f = P(\mathbf{1}_C f)$ is well defined. D is called the *dissipative part.* Obviously $P^*\mathbf{1}_D\le\mathbf{1}_D$ and $\sum_{n=0}^\infty P^n f(x)<\infty$ for all $f\in L^1(\mu)$ and $x\in D$. In particular, $\lim_{n\to\infty}P^n f(x)=0$ for $x\in D$. If $C=X$ then we say that the operator P is *conservative.* We denote by $\Sigma_{\mathrm{i}}(P)$ the σ-algebra of all invariant sets A, i.e. such that $P^*\mathbf{1}_A=\mathbf{1}_A$. If $\Sigma_{\mathrm{i}}(P)=\{\emptyset, X\}$ then P is called *ergodic.* The *deterministic σ-algebra* $\Sigma_{\mathrm{d}}(P)$ is defined as $\{B\in\mathcal{B}: P^{*n}\mathbf{1}_B=\mathbf{1}_{B_n}$ for every natural $n\}$. We say that $B_0,B_1,\ldots,B_{d-1}$ (from $\Sigma_{\mathrm{d}}(P)$) form a *cycle* if $P^{*d}\mathbf{1}_{B_0}=\mathbf{1}_{B_0}$ and $P^*\mathbf{1}_{B_j}=\mathbf{1}_{B_{j+1}}$ for all $0\le j\le d-2$. Clearly $\Sigma_{\mathrm{i}}(P)\subseteq\Sigma_{\mathrm{d}}(P)$, but in general these two σ-algebras may differ. If it happens that $\Sigma_{\mathrm{i}}(P)=\Sigma_{\mathrm{d}}(P)$ then we say that the Markov operator P *does not allow cycles.*

2. Main result. In this section we describe the asymptotic properties of the iterates of (SFS) Markov operators acting on abstract $L^1(\mu)$ spaces. We start with

LEMMA 1. *Let P be a (SFS) kernel Markov operator on $L^1(X,\mathcal{B},\mu)$ such that P^* preserves $C_0(X)$. If P_C does not allow cycles then*

$$\lim_{n\to\infty} \int\limits_{K\cap(C\setminus F)} P^n f\,d\mu = 0 \tag{2}$$

for every compact set $K\subseteq X$ and arbitrary $f\in L^1(X,\mathcal{B},\mu)$.

Proof. Obviously $F\subseteq C$. Since $P_C^*\mathbf{1}_F\ge \mathbf{1}_F$, we have $P_C^*\mathbf{1}_{C\setminus F}\le \mathbf{1}_{C\setminus F}$. It is well known that P_C, considered as a Markov operator on $L^1(C,\mathcal{B}\cap C,\mu\restriction_C)$, is conservative. In particular, $P_C^*\mathbf{1}_{C\setminus F}=\mathbf{1}_{C\setminus F}$ and $P_C^*\mathbf{1}_F=\mathbf{1}_F$. Clearly P_C is a kernel operator. Hence P_C is Harris. This implies that $\Sigma_{\mathrm{d}}(P_C)$ is atomic (see [F1] for all details). Let $B\subseteq C$ be an atom of $\Sigma_{\mathrm{d}}(P_C)$. Consider the sequence $P_C^{*n}\mathbf{1}_B=\mathbf{1}_{B_n}$. Clearly $B_n\in\Sigma_{\mathrm{d}}(P_C)$ are atoms as well. Since P_C is conservative, it follows that $\sum_{n=0}^{\infty}P_C^{*n}\mathbf{1}_B=\infty$ on B. In particular, for some $n>0$ we have $\mu(B_n\cap B)>0$. Hence $B_n\cap B=B$. This implies that $P_C^{*n}\mathbf{1}_B\ge\mathbf{1}_B$. By conservativity $P_C^{*n}\mathbf{1}_B=\mathbf{1}_B$. Since P_C does not allow cycles, we have $P_C\mathbf{1}_B=\mathbf{1}_B$. Now let P_B denote the restriction of P to $L^1(B,\mathcal{B}_B,\mu|_B)$. Again P_B is conservative.

We show that P_B is totally ergodic (i.e. for each natural n the operator P_B^n is ergodic). In fact, if $P_B^{*n}\mathbf{1}_A=\mathbf{1}_A$ for some $0<\mu(A)<\mu(B)$ then $P_B^{*j}\mathbf{1}_A=\mathbf{1}_{A_j}$, as P_B is nondisappearing (see Lemma 0 of [KL]). Hence $A\in\Sigma_{\mathrm{d}}(P_C)$, contradicting the assumption that B is an atom. A conservative and totally ergodic Markov operator P_B satisfies the assumptions of the 0-2 law, which says that either $|||P_B^n-P_B^{n+1}|||\to 0$ in the operator norm as $n\to\infty$, or for a fixed $y\in B$ the densities $k_n(\cdot,y)$, corresponding to $P^n\delta_y$, are pairwise orthogonal (see [OS] and [F2] for all details). The latter is excluded as P_B is conservative, and $P^n\delta_y\in\mathcal{D}_\mu$ for all $n\ge 1$. In particular

$$\sum_{n=1}^{\infty}P_B^n\delta_y(\cdot)=\infty \quad \text{on } B.$$

Hence, for all atoms $B\in\Sigma_{\mathrm{d}}(P_C)$,

$$\lim_{n\to\infty}\sup_{f\in\mathcal{D}_\mu}\|P_B^n f-P_B^{n+1}f\|=0. \tag{3}$$

Now assume $B\subseteq C\setminus F$, and let K be an arbitrary compact subset of B. Suppose that

$$\overline{\lim_{n\to\infty}}\int\limits_K P^n f\,d\mu>0 \quad \text{for some } f\in\mathcal{D}_\mu.$$

Since B is invariant we may assume that f is concentrated on B. Let

$P^{n_j} f \to \nu$ as $j \to \infty$ for the vague topology, where ν is a nonzero positive measure. By (3) we have $\|P\nu - \nu\| = 0$. In particular, $d\nu/d\mu = f_\nu \in L^1(\mu)$ and $\int_B f_\nu \, d\mu \ge \int_K d\nu > 0$, contradicting the inclusion $B \subseteq C \setminus F$. We conclude that

$$\lim_{n\to\infty} \int_K P^n f \, d\mu = 0 \tag{4}$$

for every compact set K included in an atom $B \subseteq C \setminus F$. This convergence may be extended to (2) in the following way. Suppose that for some K, $f \in \mathcal{D}_\mu$ and $\varepsilon > 0$ we have

$$\overline{\lim_{n\to\infty}} \int_{K\cap C\setminus F} P^n f \, d\mu \ge \varepsilon.$$

Let $B_1, B_2, \ldots \in \Sigma_{\mathrm{i}}(P_C)$ be atoms included in $C \setminus F$. Because they are P-invariant and disjoint, it follows that there exists a natural m such that

$$\overline{\lim_{n\to\infty}} \sum_{j=m+1}^{\infty} \int_{B_j} P^n f \, d\mu < \frac{\varepsilon}{2}.$$

Without loss of generality we may assume that

$$\overline{\lim_{n\to\infty}} \int_{K\cap B_1} P^n f \, d\mu > \frac{\varepsilon}{2m}.$$

Using again the fact that B_1 is invariant, we may assume that the density f is concentrated on B_1. Now we choose a subsequence $n_j \nearrow \infty$ such that $\int_{K\cap B_1} P^{n_j} f \, d\mu > \varepsilon/(2m)$, and $P^{n_j} f \to \nu$ vaguely. We have already noticed that ν is absolutely continuous and its density f_ν is P-invariant. Hence $\nu(B_1) = 0$. Now let $\mathrm{supp}_{\mathrm{top}}(\nu)$ be the topological support of ν, and let $y_0 \in \mathrm{supp}_{\mathrm{top}}(\nu)$. Using the (SFS) condition we may find $\delta > 0$ small enough that

$$k(\,\cdot\,, y') \wedge k(\,\cdot\,, y'') \ne 0$$

for all $y', y'' \in K(y_0, \delta)$. Choosing j large enough we get

$$\int_{K(y_0,\delta)\cap K\cap B_1} P^{n_j} f \, d\mu > 0.$$

Hence

$$(P^{n_j+1} f) \wedge k(\,\cdot\,, y'') \ge \int_{K(y_0,\delta)\cap K\cap B_1} k(\,\cdot\,, y') P^{n_j} f(y') \, d\mu(y') \wedge k(\,\cdot\,, y'') \ne 0$$

for all $y'' \in K(y_0, \delta)$. Now, if f_1 is an arbitrary density concentrated on $K(y_0, \delta) \cap \mathrm{supp}\, f_\nu \subseteq F$, then

$$P^{n_j+1} f \wedge P f_1 = (P^{n_j+1} f) \wedge \int k(\,\cdot\,, y'') f_1(y'') \, d\mu(y'') \ne 0,$$

contradicting the fact that B_1 and F are both invariant. ∎

The following lemma is an easy consequence of a result from [BB].

LEMMA 2. *Let P be a kernel Markov operator. If $\Sigma_{\rm d}(P)\cap F=\Sigma_{\rm i}(P)\cap F$ then P_F has strong operator convergent iterates.*

Proof. It is enough to show that each P_B is asymptotically stable on $L^1(B,\mathcal{B}\cap B,\mu)$, where $B\in\Sigma_{\rm d}(P)\cap F$ is an atom. For this we notice that P_B satisfies the "0" alternative of the "0-2" law, and clearly it is mean ergodic (as P_B has an invariant density). Now by Corollary on page 22 of [BB] we conclude that for every density f concentrated on B, we have $P_B^n f\to f_{B*}$ in the L^1 norm, where f_{B*} is a unique P-invariant density concentrated on B. Finally we get $P_F^n f\to\sum_j(\int_{B_j} f\,d\mu)f_{B_j*}$ in the L^1 norm. ■

The next lemma describes the dynamics of the process $P^n f$ on the dissipative part.

LEMMA 3. *Let P be a (SFS) Markov operator acting on $L^1(X,\mathcal{B},\mu)$ such that P^* preserves $C_0(X)$. Then for every compact set $K\subseteq X$ and $f\in\mathcal{D}_\mu$ we have*

$$\lim_{n\to\infty}\int\limits_{K\cap D} P^n f\,d\mu=0.$$

Proof. By [F1] there exists an increasing sequence of measurable sets $B_m\subseteq D$ such that $\bigcup_{m=1}^\infty B_m=D$ and $P^{*n}\mathbf{1}_{B_m}\to 0$ pointwise as $n\to\infty$. In particular, $\mathbf{1}_{B_m\cap K}(x)\nearrow\mathbf{1}_{D\cap K}(x)$ for μ-almost all x. Since P is a kernel operator, it follows that

$$\begin{aligned}P^*\mathbf{1}_{B_m\cap K}(y)&=\int k(x,y)\mathbf{1}_{B_m\cap K}(x)\,d\mu(x)\\&\nearrow\int\limits_X k(x,y)\mathbf{1}_{B\cap K}(y)\,d\mu(x)=P^*\mathbf{1}_{D\cap K}(y)\end{aligned}$$

as $m\to\infty$, for all $y\in X$. We notice that $P^*\mathbf{1}_{B_m\cap K}\le P^*\mathbf{1}_{D\cap K}$ and both functions belong to $C_0(X)$, as K is compact and the kernel k satisfies the (SFS) condition. By the Dini theorem and the assumption that all closed balls in $(X,{\rm d})$ are compact we get the uniform convergence $P^*\mathbf{1}_{B_m\cap K}\rightrightarrows P^*\mathbf{1}_{D\cap K}$. Suppose that $\int_{D\cap K}P^{n_j}f\,d\mu\ge\varepsilon$ for some $f\in\mathcal{D}_\mu$, where $n_j\nearrow\infty$ and $\varepsilon>0$. Choosing m large enough we obtain $\|P^*\mathbf{1}_{B_m\cap K}-P^*\mathbf{1}_{D\cap K}\|_{\sup}\le\varepsilon/2$. Hence

$$\begin{aligned}\int\limits_{B_m\cap K}P^{n_j}f\,d\mu&=\Big(\int\limits_{B_m\cap K}-\int\limits_{D\cap K}+\int\limits_{D\cap K}\Big)P^{n_j}f\,d\mu\\&\ge\varepsilon-\Big|\int P^{n_j-1}f\cdot(P^*\mathbf{1}_{B_m\cap K}-P^*\mathbf{1}_{D\cap K})\,d\mu\Big|\\&\ge\varepsilon-\int P^{n_j-1}f\cdot\|P^*\mathbf{1}_{B_m\cap K}-P^*\mathbf{1}_{D\cap K}\|_{\sup}\,d\mu\\&\ge\varepsilon/2\quad\text{ for all } j=1,2,\ldots\end{aligned}$$

On the other hand, $f \cdot P^{*n_j} \mathbf{1}_{B_m \cap K} \to 0$ as $j \to \infty$ for μ almost all x. By the Lebesgue dominated convergence theorem we get

$$\lim_{j\to\infty} \int_{B_m \cap K} P^{n_j} f \, d\mu = 0,$$

a contradiction. ■

THEOREM 1. *Let (X, d) be a metric space such that all closed balls are compact. If a kernel Markov operator P on $L^1(X, \mathcal{B}, \mu)$ satisfies (SFS), P^* preserves $C_0(X)$, and $\Sigma_{\mathrm{i}}(P_C) = \Sigma_{\mathrm{d}}(P_C)$, then for every compact set $K \subseteq X$ and every $f \in L^1(X, \mathcal{B}, \mu)$ we have*

$$\lim_{n\to\infty} \int_K P^n f \, d\mu = \int_K Sf \, d\mu,$$

where S is a substochastic projection onto the sublattice of P-invariant functions. Moreover, on $L^1(F, \mathcal{B}_F, \mu|_F)$ the above convergence holds for the L^1 norm.

Proof. Given a density $f \in \mathcal{D}_\mu$ and an atom B in $\Sigma_{\mathrm{d}}(P_F) = \Sigma_{\mathrm{i}}(P_F)$, we define

$$S_B f = \Big(\lim_{n\to\infty} \int_B P^n f \, d\mu\Big) \cdot f_{B*},$$

where f_{B*} is a (unique) P-invariant density concentrated on B. The limit $\lim_{n\to\infty} \int_B P^n f \, d\mu$ is well defined, as the sequence $\int_B P^n f \, d\mu$ is nondecreasing and bounded by 1. Finally we set

$$Sf = \sum_{B \in \Sigma_{\mathrm{i}}(P_F)} S_B f.$$

On each atom $B \in \Sigma_{\mathrm{i}}(P_F)$ the operator P_B is asymptotically stable. We obtain $\lim_{n\to\infty} \|(P^n f - S_B f) \cdot \mathbf{1}_B\| = 0$. This convergence can be extended to the whole of F in an obvious way. Finally by Lemmas 1 and 3 we get

$$\begin{aligned}\lim_{n\to\infty} \int_K P^n f \, d\mu &= \lim_{n\to\infty} \Big(\int_{K\cap F} + \int_{K \cap (C\setminus F)} + \int_{K\cap D} \Big) P^n f \, d\mu \\ &= \lim_{n\to\infty} \int_{K\cap F} P^n f \, d\mu = \int_{K\cap F} Sf \, d\mu = \int_K Sf \, d\mu.\end{aligned}$$

Since each $f \in L^1(\mu)$ is a linear combination of densities the theorem is proved. ■

The following corollary is an immediate consequence of (SFS) and $P^* C_0(X) \subseteq C_0(X)$. Such operators are actually defined on $\mathcal{M}(X)$. We have

COROLLARY 1. *Let P be a (SFS) kernel Markov operator defined on $L^1(\mu)$ such that $P^* C_0(X) \subseteq C_0(X)$. If P_C does not allow cycles, then there* *exists a substochastic projection $S^{**} = S : \mathcal{M}(X) \to L^1_*(\mu)$ onto the sublattice of P-invariant functions such that $P^n\mu \cong P^{**n}\mu \to S\mu$ as $n \to \infty$ for the vague* (*weak**) *topology.*

3. LMT operators. The last section of the paper is devoted to LMT operators. The result we present has been proved in [B] with some restrictions on P. This also generalizes [KM], [M] as it is obvious that in the absence of invariant densities, the projection S is zero. In particular, for every compact K we have the convergence $\lim_{n\to\infty} \int_K P^n f\,d\mu = 0$ whenever $L^1_*(\mu)$ is trivial. In other words, an LMT operator P satisfies the Foguel alternative, i.e. either P has an invariant density, or it is sweeping with respect to the family of compact sets. We have

THEOREM 2. *Let P be an LMT operator on $L^1([0,\infty))$, where H, Q, λ satisfy conditions* (H) *and* (Qλ). *Then for every compact set $K \subseteq [0,\infty)$ we have*

$$\lim_{n\to\infty} \int_K P^n f\,d\mu = \int_K Sf\,d\mu,$$

where S is a substochastic projection onto the Banach sublattice of P-invariant functions. Moreover $\|(P^n f - Sf)\mathbf{1}_F\| \to 0$ as $n \to \infty$.

Proof. By Theorem 1 it is sufficient to show that P_C does not allow cycles. Since P_C is conservative it follows that given an atom $B \in \Sigma_{\mathrm{d}}(P_C)$, there exists n such that $P_C^{*n}\mathbf{1}_B = \mathbf{1}_B$. Set $\mathbf{1}_{B_j} = P_C^{*j}\mathbf{1}_B$ for $j = 0,1,\ldots,n-1$ ($B_n = B_0 = B$). To show that the cycle B_j is trivial we use essentially the same arguments as in the proof of Theorem 2 of [B]. Namely, it follows directly from the formula (1) of LMT kernels that

$$P^*\mathbf{1}_{[c,d)}(y) = \begin{cases} H(Q(\lambda(c)) - Q(y)) - H(Q(\lambda(d)) - Q(y)) & \text{if } 0 \le y < \lambda(c), \\ 1 - H(Q(\lambda(d)) - Q(y)) & \text{if } \lambda(c) \le y < \lambda(d), \\ 0 & \text{if } \lambda(d) \le y. \end{cases}$$

Substituting $d = \infty$ we get

$$P^*\mathbf{1}_{[c,\infty)}(y) = \begin{cases} H(Q(\lambda(c)) - Q(y)) & \text{if } 0 \le y < \lambda(c), \\ 1 & \text{otherwise.} \end{cases}$$

Let $c_j = \operatorname{ess\,inf} B_j$, $j = 0,1,\ldots,n-1$. Then we have (by continuity of $P^*\mathbf{1}_{B_j}$)

$$P^*\mathbf{1}_{B_j}(y) = \begin{cases} 1 & \text{if } y \in \overline{B}_{j+1}, \\ 0 & \text{if } y \in \overline{B}_s \text{ for all } 0 \le s \le n-1 \text{ and } s \ne j+1. \end{cases}$$

In particular, all c_j must be different. Without loss of generality we assume that $c_0 = \max\{c_0, c_1, \ldots, c_{n-1}\}$. Note that $c_0 < \lambda(c_0)$. Otherwise we would have $P^*\mathbf{1}_{[c_0,\infty)} \ge \mathbf{1}_{[c_0,\infty)}$. On the conservative part we get $P_C^*\mathbf{1}_{[c_0,\infty)\cap C} =$

$\mathbf{1}_{[c_0,\infty)\cap C}$. This implies $0 \le P_C^*\mathbf{1}_{B_0} = P_C^*\mathbf{1}_{[c_0,\infty)\cap B_0} \le \mathbf{1}_{[c_0,\infty)\cap C}$. Hence $P_C^*\mathbf{1}_{B_0}(y) \equiv 0$ for all $0 \le y < c_0$, contradicting $P_C^*\mathbf{1}_{B_0} = \mathbf{1}_{B_1}$ and $c_1 < c_0$. We also have $1 \ge H(Q(\lambda(c_0)) - Q(c_1)) = P^*\mathbf{1}_{[c_0,\infty)}(c_1) \ge P^*\mathbf{1}_{B_0}(c_1) = 1$, as $P^*\mathbf{1}_{B_0}$ is continuous, $P_C^*\mathbf{1}_{B_0} = \mathbf{1}_{B_1}$ and $c_1 \in \overline{B}_1$. Combining these facts we conclude that $H(Q(\lambda(c_0)) - Q(y)) = 1$ if $c_1 \le y \le \lambda(c_0)$, as H is nonincreasing, and Q and λ are nondecreasing. In particular,

$$P^*\mathbf{1}_{[c_0,\infty)} \ge \mathbf{1}_{[c_1,\infty)} \ge \mathbf{1}_{[c_0,\infty)},$$

which means that $P_C^*\mathbf{1}_{[c_0,\infty)\cap C} = \mathbf{1}_{[c_0,\infty)\cap C}$ on the conservative part. Therefore, $P_C^*\mathbf{1}_{B_0} \le \mathbf{1}_{[c_0,\infty)}$, contradicting $P_C^*\mathbf{1}_{B_0} = \mathbf{1}_{B_1}$ if $n > 1$ and $c_1 < c_0$. The resulting contradiction forces $n = 1$ and the triviality of the cycle $B_0, \ldots, B_{n-1}$. ■

We instantly get

Corollary 2. *Given an LMT Markov operator P on $L^1([0,\infty))$, there exists a substochastic projection $S : \mathcal{M}([0,\infty)) \to L^1([0,\infty))$ onto $L^1_*(\mu)$, the sublattice of all P-invariant functions, such that*

$$\lim_{n\to\infty} P^n\mu(K) = S\mu(K)$$

for every compact set $K \subseteq [0,\infty)$. In particular, if P has no invariant density (i.e. $L^1_*(\mu)$ is trivial), then

$$\lim_{n\to\infty} P^n\mu(K) = 0$$

for every compact set $K \subseteq [0,\infty)$; in particular, P is sweeping.

Remark. If there exists a compact set $K(\subseteq F)$ such that for every density $f \in \mathcal{D}_\mu$ we have

$$\overline{\lim_{n\to\infty}} \int_K P^n f\,d\mu > 0 \tag{5}$$

(compare [B] and [BL]) then S is finite-dimensional, being a compact projection. For this we note that $S = S \circ P = P \circ S$. Now the operator $S_K^* : L^\infty(\mu) \to C(K)$ defined by $S_K^* f = (S^* f)|_K$ is compact. Therefore $\Sigma_{\mathrm{i}}(P) \cap K$ must be finite. Clearly $\dim(S^*) = \dim(S_K^*)$ by the assumption (5).

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