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APPROXIMATIONS OF STONE–ČECH COMPACTIFICATIONS BY HIGSON COMPACTIFICATIONS

BY

KAZUHIRO KAWAMURA (Tsukuba) and
KAZUO TOMOYASU (Miyakonojo)

Abstract. The Higson compactification $\overline{X}^d$ of a non-compact proper metric space (X,d) is rarely equivalent to the Stone–Čech compactification βX. We give a characterization of such spaces. Also, we show that for each non-compact locally compact separable metric space, βX is equivalent to $\varprojlim\{\overline{X}^d : d$ is a proper metric on X which is compatible with the topology of $X\}$. The approximation method of the above type is illustrated by some examples and applications.

1. Introduction and preliminaries. In this paper all spaces are assumed to be locally compact Hausdorff. For compactifications αX and γX of a non-compact space X we write $\alpha X \geq \gamma X$ if there exists a continuous map $f : \alpha X \to \gamma X$ such that $f{\restriction}X$ is the identity on X. If such an f can be chosen to be a homeomorphism, we write $\alpha X \approx \gamma X$ and two compactifications αX and γX are said to be *equivalent* or *αX is equivalent to γX*.

The Higson compactification is defined for all proper metric spaces [15] as follows. Here a *proper* metric space is a space X with a specific metric (called a proper metric) d such that each d-bounded set has compact closure. Note that every non-compact proper metric space (X,d) has infinite diameter. For a metric space (X,d) and for $r > 0$, $B_r(x,d)$ denotes the open r-ball $\{y \in X : d(x,y) < r\}$. A continuous map $f : X \to Y$ of a non-compact proper metric space (X,d) to a metric space (Y,ϱ) is said to satisfy the *$(*)_d$-condition* (with respect to ϱ) if $\lim_{x\to\infty} \operatorname{diam}_\varrho(f(B_r(x,d))) = 0$ for each $r > 0$, that is, for each $r > 0$ and for each $\varepsilon > 0$, there is a compact set $K = K_{r,\varepsilon}$ in X such that for each point $x \notin K$, $\operatorname{diam}_\varrho(f(B_r(x,d))) < \varepsilon$. For a space X, $C(X)$ (resp. $C^*(X)$) denotes the set of all real-valued (resp. bounded real-valued) continuous functions on X. The set $C(X)$ is a ring under pointwise addition and multiplication with $C^*(X)$ being a subring of $C(X)$. We define $C_d(X) = \{f \in C(X) : f$ satisfies the $(*)_d$-condition$\}$ and

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$C_d^*(X) = C^*(X) \cap C_d(X)$. With the supremum norm on $C^*(X)$, $C_d^*(X)$ is a closed subring of $C^*(X)$ containing all constant functions. Since the metric d on X is proper, $C_d^*(X)$ generates the topology of X. The *Higson compactification* of a proper metric space (X, d) is the unique compactification associated with the closed subring $C_d^*(X)$ (cf. [15]), denoted by $\overline{X}^d$. Note that this is a metric-dependent compactification. The remainder $\overline{X}^d - X$ of the compactification is called the *corona* and is denoted by $\nu_d X$.

It is well known that the Higson compactification behaves like the Stone–Čech compactification (cf. [13]–[16], [20]). However, the Higson compactification of a proper metric space is rarely equivalent to the Stone–Čech compactification of the space. For example, the Higson compactification of the positive integers $\mathbb{N}$ endowed with the standard metric is not equivalent to the Stone–Čech compactification of $\mathbb{N}$ (cf. [15]). In Section 2, we give a necessary and sufficient condition on a proper metric space X that the Stone–Čech compactification βX is equivalent to $\overline{X}^d$. In Section 3, we will prove that the Stone–Čech compactification of a locally compact separable metric space X is characterized as the limit space of Higson compactifications of proper metrics on X. This enables us to derive certain information on the Stone–Čech compactification from that on the Higson compactification and vice versa.

In the remaining part of this section we state some preliminary facts on $\overline{X}^d$. First of all, the compactification $\overline{X}^d$ is characterized as follows:

PROPOSITION 1.1 ([15], Proposition 1). *Let (X, d) be a non-compact proper metric space. The Higson compactification $\overline{X}^d$ is the unique compactification of X which satisfies the following condition: for each continuous map $f : X \to Y$ of X to a compact metric space Y, f has a continuous extension to $\overline{X}^d$ if and only if f satisfies the $(*)_d$-condition.*

Note here that the compactness of Y guarantees that $(*)_d$-condition does not depend on the choice of compatible metrics on Y.

NOTATION. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the standard metric $d_n(x, y) = (\sum_{i=1}^n (x_i - y_i)^2)^{1/2}$, ω the first infinite ordinal, $\mathbb{Q}$ the set of all rationals, $[0, 1]$ the closed unit interval, and $J = [0, \infty)$.

A finite system $\{E_1, \ldots, E_n\}$ of subsets of a proper metric space (X, d) is said to *diverge* if, for each $R > 0$, the intersection of the R-neighborhoods of the sets E_i, $i = 1, \ldots, n$, is a bounded subset of X. Equivalently, a system $\{E_1, \ldots, E_n\}$ diverges if and only if $\lim_{x \to \infty} \sum_{i=1}^n d(x, E_i) = \infty$. In what follows we will frequently make use of the Taĭmanov theorem stated in the following form.

THEOREM 1.2 (cf. [9], Theorem 3.5.5). *Let X be a Tikhonov space and let αX and γX be compactifications of X. The following conditions are equivalent:*

(1) $\alpha X \geq \gamma X$,

(2) *if* A *and* B *are disjoint closed subsets of* X *such that* $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \emptyset$, *then* $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$.

The above yields the following characterization which was proved by A. N. Dranishnikov, J. Keesling and V. V. Uspenskij.

PROPOSITION 1.3 (cf. [8], Proposition 2.3). *Let* (X, d) *be a proper metric space. Then the following conditions are equivalent*:

(1) $\alpha X \approx \overline{X}^d$,

(2) *Let* A *and* B *be disjoint closed subsets of* (X, d). *Then* $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$ *if and only if the system* $\{A, B\}$ *diverges.*

For undefined notation and terminology, see [4], [9], [18] and [19] .

2. The Stone–Čech compactification versus Higson compactification. In this section we will characterize the non-compact locally compact separable metric spaces (X, d) such that $\beta X \approx \overline{X}^d$.

DEFINITION 2.1. Let (X, d) be a metric space and $r > 0$. A subset D of X is said to be r-*discrete* if $B_r(x, d) \cap D = \{x\}$ for each $x \in D$.

PROPOSITION 2.2. *Let* (X, d) *be a non-compact proper metric space. Then the following conditions are equivalent*:

(1) $\overline{X}^d \approx \beta X$,

(2) $C^*_d(X) = C^*(X)$, *and*

(3) *for each* $r > 0$ *there exists a compact subset* $K_r \subset X$ *such that* $X - K_r$ *is* r*-discrete.*

Proof. Clearly, (1) is equivalent to (2), and (3) implies (2). So it remains to show that (2) implies (3). Let X be a proper metric space with $C^*_d(X) = C^*(X)$. Since X is σ-compact, it is represented as $X = \bigcup_{n<\omega} C_n$ such that C_n is a compact subset of X for each $n < \omega$. Assume that there exists an $r > 0$ such that $X - K$ is not r-discrete for each compact subset K of X. Then we can choose a point $x_0 \in X - C_0$ such that $B_r(x_0, d) \cap (X - C_0) \neq \{x_0\}$. Put $K_0 = C_0$ and choose a point $y_0 \in (B_r(x_0, d) - \{x_0\}) \cap (X - C_0)$. There exists a continuous function $f_0 : X \to [0, 1]$ such that $f_0(\{x_0\} \cup (X - B_r(x_0, d))) = \{0\}$ and $f_0(y_0) = 1$. Next we take a compact subset K_1 such that $K_1 \supset B_r(B_r(x_0, d) \cup K_0 \cup C_1, d)$. From the assumption we can take a point $x_1 \in X - K_1$ such that $B_r(x_1, d) \cap (X - K_1) \neq \{x_1\}$. Then there exist a point $y_1 \in (B_r(x_1, d) - \{x_1\}) \cap (X - K_1)$ and a continuous function $f_1 : X \to [0, 1]$ such that $f_1(\{x_1\} \cup (X - B_r(x_1, d))) = \{0\}$ and $f_1(y_1) = 1$. Continuing this process, we obtain a sequence $\{K_n\}_{n<\omega}$ of compact sets, sequences $\{x_n\}_{n<\omega}$, $\{y_n\}_{n<\omega}$ of points and a sequence $\{f_n\}_{n<\omega}$ of continuous functions satisfying the following conditions for each $n < \omega$:

(1) $x_n \in X - K_n$,
(2) $y_n \in (B_r(x_n, d) - \{x_n\}) \cap (X - K_n)$,
(3) $K_{n+1} \supset B_r(B_r(x_n, d) \cup K_n \cup C_{n+1}, d)$, and
(4) $f_n(\{x_n\} \cup (X - B_r(x_n, d))) = \{0\}$ and $f_n(y_n) = 1$.

Put $f = \sum_{n<\omega} f_n$. Since $\{B_r(x_n, d) : n < \omega\}$ is discrete, we see that $f \in C^*(X)$. We show that $f \notin C^*_d(X)$. Let K be a compact subset of X. From the condition (3) above, there exists an $n < \omega$ such that $K \subset K_n$. The conditions (1), (2), (3) and (4) above imply that $\operatorname{diam}(f(B_r(x_n, d))) \geq 1$ and hence $f \notin C^*_d(X)$. This means that $C^*(X) \neq C^*_d(X)$, contradicting the hypothesis (2). ∎

Now let (X, d) be a fixed metric space. Define a binary relation δ_d on the power set $\mathcal{P}(X)$ of X as follows:

$$A\delta_d B \quad \text{if and only if} \quad d(A, B) = 0.$$

It is well known that δ_d is a separated Efremovich proximity on X. The theory of proximities then tells us that there exists a compactification $u_d X$ of X, called the *Smirnov* or *Samuel compactification* of X.

THEOREM 2.3 ([21], Theorem 2.5). *Let (X, d) be a metric space and αX a compactification of X. Then the following conditions are equivalent*:

(1) $\alpha X \approx u_d X$,
(2) *if $A, B \subset X$, then $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B \neq \emptyset$ if and only if $d(A, B) = 0$.*

EXAMPLE 2.4. $u_d\mathbb{R}$ is not equivalent to $\overline{\mathbb{R}}^d$, where d is a standard metric on $\mathbb{R}$ defined by $d(x, y) = |x - y|$. In fact, put $A = \{2n : n < \omega\}$ and $B = \{2n + 1 : n < \omega\}$. Then we note that $d(A, B) = 1$ and $d(x, A) + d(x, B) = 1$ for each $x \in \mathbb{R}$ and thus the system $\{A, B\}$ does not diverge. From Proposition 1.3, we have $\mathrm{cl}_{\overline{\mathbb{R}}^d} A \cap \mathrm{cl}_{\overline{\mathbb{R}}^d} B \neq \emptyset$, while $\mathrm{cl}_{u_d\mathbb{R}} A \cap \mathrm{cl}_{u_d\mathbb{R}} B = \emptyset$ by Theorem 2.3. Hence $u_d\mathbb{R}$ is not equivalent to $\overline{\mathbb{R}}^d$.

If a system $\{A, B\}$ of closed subssets of a proper metric space (X, d) diverges, then it is clear that $d(A, B) > 0$. Hence we have in general:

LEMMA 2.5. *Let (X, d) be a non-compact proper metric space. Then $\overline{X}^d \leq u_d X$.*

PROPOSITION 2.6. *Let X be a non-compact locally compact separable metric space. Then the following conditions are equivalent*:

(1) *there exists a compatible proper metric ϱ on X such that $\overline{X}^\varrho \approx \beta X$*,
(2) *there exists a compatible metric ϱ' on X such that $u_{\varrho'} X \approx \beta X$*,
(3) *βX is an* Oz *space*, *that is*, *each regular closed subset of βX is a zero set of βX* (cf. [2], Theorem 5.1),
(4) *the set of non-isolated points of X is compact*,

(5) *there exists a proper metric d on X such that, for each $r > 0$, there exists a compact subset $K_r \subset X$ with $X - K_r$ being r-discrete, and*

(6) *there exists a compact subset K of X such that $X = K \oplus (X - K)$ and $X - K$ is discrete.*

Proof. The implication (1)⇒(2) follows from Lemma 2.5. Also, by [21], Corollary 3.5, (2)⇔(3)⇔(4). Next we show that (4) implies (5). Let K be the set of non-isolated points of X and d a compatible metric on X. Put $K_{n+1} = B_{n+1}(K, d)$ for $n < \omega$. If $\{x, y\} \not\subset K$, then put

$$n(x, y) = \min\{n < \omega : \{x, y\} \subset K_n\}.$$

We define a metric on X as follows:

$$\varrho(x,y) = \begin{cases} d(x,y)/(\mathrm{diam}(K,d)) & \text{if } \{x,y\} \subset K, \\ n(x,y) & \text{if } \{x,y\} \not\subset K \text{ with } x \neq y, \\ 0 & \text{if } \{x,y\} \not\subset K \text{ with } x = y. \end{cases}$$

One can verify that ϱ is a compatible proper metric on X and satisfies the condition (5). From Proposition 2.2 we note that (5) implies (1). The proof of the implication (6)⇒(5) is similar to that of (4)⇒(5). Thus it remains to prove that (5) implies (6). By the hypothesis (5), there exist a proper metric d on X and a compact subset K_1 of X such that $X - K_1$ is 1-discrete with respect to d. It is easy to show that $X - K_1$ is closed in (X, d) and hence $X = K_1 \oplus (X - K_1)$. ∎

3. The least upper bound of Higson compactifications. We show that the Stone–Čech compactification is equivalent to the limit space of Higson compactifications. The following is well known and the proof is provided for completeness.

LEMMA 3.1. *For each locally compact separable metric space X, there exists a proper metric on X compatible with the topology.*

Proof. We may clearly assume that X is not compact. Since X is σ-compact, there exists an increasing sequence $\{K_n : n < \omega\}$ of non-empty compact subsets of X such that K_n is a compact subset of int K_{n+1} for each $n < \omega$. For the sake of convenience, let $K_{-2} = K_{-1} = \emptyset$. We can construct a collection $\{U_n : n < \omega\}$ of open subsets of X as follows:

(1) $U_n \supset K_n - \mathrm{int}_X K_{n-1}$ for each $n < \omega$,
(2) $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Let d be a compatible metric on X. For each $n < \omega$, there exists a continuous function $f_n : X \to [0, n]$ such that

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin U_n, \\ n & \text{if } x \in K_n - \mathrm{int}_X K_{n-1}, \end{cases}$$

Since $\{U_n : n < \omega\}$ is locally finite in X (actually, it is of order 2), we note that $f = \sum_{n<\omega} f_n$ is a continuous function on X. Define $\varrho : X \times X \to \mathbb{R}$ by

$$\varrho(x, y) = \max\{|f(x) - f(y)|, d(x, y)\}.$$

It is easy to see that ϱ is compatible with the topology on X. It remains to prove that ϱ is a proper metric. Fix a point $x \in X$ and let $r > 0$. There exists an $n < \omega$ such that $x \in K_n - \mathrm{int}_X K_{n-1}$. Choose a natural number $m < \omega$ such that $m - 3n > r$. Then we show that $B_r(x, \varrho) \subset K_{m-1}$. Assume that $y \notin K_{m-1}$. There exists an $l \geq m$ such that $y \in K_l - \mathrm{int}_X K_{l-1}$. Then $f(y) \geq l$ and $f(x) \leq 3n$. Thus $f(y) - f(x) \geq l - 3n \geq r > 0$, and we have

$$\begin{aligned} \varrho(x, y) &\geq |f(x) - f(y)| \ (= f(y) - f(x)) \\ &\geq l - 3n \geq m - 3n > r \end{aligned}$$

Thus, $y \notin B_r(x, \varrho)$. This shows that $B_r(x, \varrho) \subset K_{m-1}$ and $\mathrm{cl}_X B_r(x, \varrho)$ is compact. ∎

Put

$$\mathrm{PM}(X) = \{d : d \text{ is a proper metric compatible with the topology on } X\}.$$

THEOREM 3.2. *Let X be a non-compact locally compact separable metric space. Then $\beta X \approx \sup_{d \in \mathrm{PM}(X)} \overline{X}^d$.*

Proof. Let $\gamma X = \sup_{d \in \mathrm{PM}(X)} \overline{X}^d$. Since X is σ-compact, there exists a sequence $\{K_n : n < \omega\}$ of non-empty compact subsets of X such that $K_n \subsetneq \mathrm{int}\, K_{n+1}$ for every $n < \omega$. For convenience, let $K_{-2} = K_{-1} = \emptyset$. Take a collection $\{U_n : n < \omega\}$ of open subsets of X as in Lemma 3.1. Let A and B be disjoint non-empty closed sets of X. Below, we find a proper metric $d_{A,B}$ such that $\mathrm{cl}_{\overline{X}^{d_{A,B}}} A \cap \mathrm{cl}_{\overline{X}^{d_{A,B}}} B = \emptyset$. We may assume that A and B are non-compact. Put $A_n = (K_n - \mathrm{int}\, K_{n-1}) \cap A$, $B_n = (K_n - \mathrm{int}\, K_{n-1}) \cap B$ and $\varepsilon_n = \min\{d(A_n, B_n), 1\} > 0$ for each $n < \omega$. Taking a subsequence of $\{K_n\}_{n<\omega}$ if necessary, we may further assume that $A_n \neq \emptyset$, $B_n \neq \emptyset$. For each $n < \omega$, there exists a continuous function $f_n : X \to [0, n/\varepsilon_n]$ such that

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin U_n, \\ n/\varepsilon_n & \text{if } x \in B_n, \\ 0 & \text{if } x \in A. \end{cases}$$

Since $\{U_n : n < \omega\}$ is locally finite in X, we note that $f = \sum_{n<\omega} f_n$ is a continuous function on X. Fix a compatible proper metric d. Define $d_{A,B} : X \times X \to \mathbb{R}$ by

$$d_{A,B}(x, y) = \max\{|f(x) - f(y)|, d(x, y)\}.$$

It is easy to see that the metric $d_{A,B}$ is compatible with the topology on X and $d_{A,B}$ is proper since $d_{A,B} \geq d$. We claim that

$$\lim_{x\to\infty}(d_{A,B}(x,A)+d_{A,B}(x,B))=\infty,$$

which shows that $\mathrm{cl}_{\overline{X}^{d_{A,B}}} A\cap \mathrm{cl}_{\overline{X}^{d_{A,B}}} B=\emptyset$ by Proposition 1.3.

Fix a natural number n. We need to find a compact set K such that $d_{A,B}(x,A)+d_{A,B}(x,B)\geq n$ for $x\in X-K$. Let $K=\mathrm{cl}B_n(\bigcup_{k\leq n}(A_k\cup B_k), d_{A,B})$ and note that K is compact in X. Assume that $x\notin K$. Since $d_{A,B}$ is a proper metric, there exist $a(x)\in A$ and $b(x)\in B$ such that

$$d_{A,B}(x,a(x))+d_{A,B}(x,b(x))=d_{A,B}(x,A)+d_{A,B}(x,B).$$

We divide our considerations into two cases. If either $a(x)\in\bigcup_{k\leq n}A_k$ or $b(x)\in\bigcup_{k\leq n}B_k$, then either $d_{A,B}(x,a(x))\geq n$ or $d_{A,B}(x,b(x))\geq n$ and we have the desired conclusion. If $a(x)\in A_p$ and $b(x)\in B_q$ for some $p,q\geq n+1$, then

$$\begin{aligned}d_{A,B}(x,a(x))+d_{A,B}(x,b(x))&\geq d_{A,B}(a(x),b(x))\geq|f(a(x))-f(b(x))|\\&=|f(b(x))|=q/\varepsilon_q\geq q>n,\end{aligned}$$

as desired. Noticing that $\overline{X}^{d_{A,B}}\leq\gamma X$, we obtain $\mathrm{cl}_{\gamma X}A\cap\mathrm{cl}_{\gamma X}B=\emptyset$ for each pair of disjoint closed sets A, B of X. This is the characterizing property of the Stone–Čech compactification of a normal space, and therefore $\beta X\approx\gamma X=\sup_{d\in\mathrm{PM}(X)}\overline{X}^d$. ∎

Let $\{\alpha_iX:i\in I\}$ be a family of compactifications of X. Suppose that I is a directed set, where the order $\succeq$ is defined as $i\succeq j$ iff $\alpha_iX\geq\alpha_jX$. For each pair $i\succeq j$, let $\pi_{ij}:\alpha_iX\to\alpha_jX$ be the natural projection. Then $\{\alpha_iX,\pi_{ij}:i\in I,\ i\succeq j\}$ forms an inverse system. Let $\varphi_i:\sup_{i\in I}\alpha_iX\to\alpha_iX$ be the natural projection. Hence for each $i\succeq j$ we have $\pi_{ij}\circ\varphi_i=\varphi_j$. The limit map

$$\varprojlim\varphi_i:\sup_{i\in I}\alpha_iX\to\varprojlim\{\alpha_iX,\pi_{ij}:i,j\in I\text{ with }i\succeq j\}$$

is known to be a homeomorphism (cf. [17], Proposition 1.7). Now, let $d,\varrho\in\mathrm{PM}(X)$ and declare that $d\leq\varrho$ (resp. $d\preceq\varrho$) iff $d(x,y)\leq\varrho(x,y)$ for each pair of points $x,y\in X$ (resp. $C_d^*(X)\subset C_\varrho^*(X)$, i.e., $\overline{X}^d\leq\overline{X}^\varrho$). Note that $(\mathrm{PM}(X),\leq)$ (resp. $(\mathrm{PM}(X),\preceq)$) is a directed set. Furthermore, if $d\leq\varrho$, then $d\preceq\varrho$. From Theorem 3.2 and the above remark we obtain the following corollary:

Corollary 3.3. *Let X be a non-compact locally compact separable metrizable space. Then $\beta X\approx\varprojlim\{\overline{X}^d,\pi_{d\varrho}:d,\varrho\in\mathrm{PM}(X)$ and $d\geq\varrho\}$, where $\pi_{d\varrho}:\overline{X}^d\to\overline{X}^\varrho$ is the natural projection for each d,ϱ with $\varrho\leq d$. That is, βX is approximated arbitrarily closely by the Higson compactification of suitable proper metrics.*

REMARK 3.4. Higson compactifications are mainly studied in the context of the "coarse geometry", in which the asymptotic behavior of spaces is the main object of study. In view of this geometry, the definition of the partial order $\leq$ on $\mathrm{PM}(X)$ above is too restrictive. To make the definition fit into the scheme of the coarse geometry, one should define $d \leq \varrho$ to hold if the inequality $d(x,y) \leq \varrho(x,y)$ holds "up to a bounded error" and for "x, y sufficiently close to infinity". Although it is possible to make this statement rigorous, we will keep the original definition in order to avoid technical complexity.

Now, we will give an application of the above corollary. R. F. Dickman Jr. proved that for $n \geq 2$, the Stone–Čech remainder $(\mathbb{R}^n)^*$ of the n-dimensional Euclidean space is a non-metric decomposable continuum (cf. [7]). From Corollary 3.3 we obtain the following result.

COROLLARY 3.5. *Let n be a natural number with $n \geq 2$. There exists a metric $d \in \mathrm{PM}(\mathbb{R}^n)$ such that, for each proper metric $\varrho \in \mathrm{PM}(\mathbb{R}^n)$ with $\varrho \geq d$, the corona $\nu_\varrho \mathbb{R}^n$ is a decomposable continuum.*

Proof. Assume the contrary that for each $d \in \mathrm{PM}(\mathbb{R}^n)$, there exists a proper metric $\varrho \in \mathrm{PM}(\mathbb{R}^n)$ such that $\varrho \geq d$ and $\nu_\varrho \mathbb{R}^n$ is an indecomposable continuum. Put

$$I = \{\varrho \in \mathrm{PM}(\mathbb{R}^n) : \nu_\varrho \mathbb{R}^n \text{ is an indecomposable continuum}\}.$$

Since I is cofinal in $\mathrm{PM}(\mathbb{R}^n)$, and by Corollary 3.3, we note that

$$(\mathbb{R}^n)^* = \varprojlim\{\nu_\varrho \mathbb{R}^n, \pi_{\varrho d} \restriction \nu_\varrho \mathbb{R}^n : d, \varrho \in I \text{ and } \varrho \geq d\},$$

where $\pi_{\varrho d} : \overline{\mathbb{R}^n}^{\varrho} \to \overline{\mathbb{R}^n}^{d}$ is the natural projection for each ϱ, d with $\varrho \geq d$. Since $\nu_\varrho \mathbb{R}^n$ is an indecomposable continuum for each $\varrho \in I$, it follows that $(\mathbb{R}^n)^*$ is an indecomposable continuum, which contradicts the Dickman theorem above. ∎

The following result is a Higson compactification analogue of Glicksberg's theorem on the Stone–Čech compactification of product spaces [11]. A weaker version can be obtained as an application of Corollary 3.3. See Appendix (c).

PROPOSITION 3.6. *Let (X, d_X) and (Y, d_Y) be non-compact proper metric spaces and suppose Y is non-discrete. Then for each proper metric ϱ on $X \times Y$, $\overline{X \times Y}^{\varrho}$ is not equivalent to $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$.*

Proof. Suppose that there exists a proper metric ϱ on $X \times Y$ such that $\overline{X \times Y}^{\varrho} \approx \overline{X}^{d_X} \times \overline{Y}^{d_Y}$. Then the corona $\nu_\varrho(X \times Y)$ contains a copy of Y which contains a convergent sequence S with limit point y. Applying Theorem 1 of [16] to the σ-compact subset $S \setminus \{y\}$, we see that S [= the closure of $S \setminus \{y\}$ in $\nu_\varrho(X \times Y)$] is homeomorphic to $\beta(S \setminus \{y\}) \approx \beta\omega$, a contradiction. ∎

The first Čech cohomology $\check{\mathrm{H}}^1(X)$ of X with integer coefficients is identified with the homotopy classes $[X, S^1]$. Let $\check{\mathrm{H}}^1_d(X)$ be the subgroup of $[X, S^1]$ defined by

$$\check{\mathrm{H}}^1_d(X) := [X, S^1]_d := \{[f]_d : f : X \to S^1 \text{ satisfies the } (*)_d\text{-condition}\}.$$

The following result was proved by J. Keesling.

THEOREM 3.7 ([15], Theorem 1). *Suppose that (X, d) is a non-compact connected proper metric space. Then we have an exact sequence*

$$(*) \qquad 0 \to C^*_d(X) \to C_d(X) \to \check{\mathrm{H}}^1(\overline{X}^d).$$

If, in addition, for every $r > 0$, there is a compact set $K_r \subset X$ such that the ball $B_r(x, d)$ is connected for each $x \in X - K_r$, then the above sequence extends to an exact sequence

$$(**) \qquad 0 \to C^*_d(X) \to C_d(X) \to \check{\mathrm{H}}^1(\overline{X}^d) \to \check{\mathrm{H}}^1_d(X) \to 0.$$

Let X be a non-compact connected completely regular Hausdorff space. The following exact sequence is classical:

$$(***) \qquad 0 \to C^*(X) \to C(X) \to [\beta X, S^1] \xrightarrow{b} [X, S^1] \to 0.$$

The connection of the sequence $(**)$ with $(***)$ fits very well the view point represented in Corollary 3.3. That is, taking the limit of the sequence $(**)$ with d varying, we almost recover the sequence $(***)$ as follows.

Let $\mathcal{F} \subset C^*(X)$ be a subcollection which separates points and closed sets (in X). The evaluation map of $\mathcal{F}$,

$$e_\mathcal{F} : X \to \prod_\mathcal{F} \mathrm{cl}_\mathbb{R} f(X),$$

is defined by $(e_\mathcal{F}(x))_f = f(x)$ for each $x \in X$, $f \in \mathcal{F}$. It is well known that $e_\mathcal{F}$ is an embedding and then the closure of $e_\mathcal{F}(X)$ in $\prod_\mathcal{F} \mathrm{cl}_\mathbb{R} f(X)$ is a compactification of X, called the compactification associated with the collection $\mathcal{F}$ and denoted by $e_\mathcal{F} X$. Let

$$\mathrm{PM}_C(X) = \{d \in \mathrm{PM}(X) : \text{for each } r > 0, \text{there is a compact set } K_r \subset X \text{ such that } B_r(x, d) \text{ is connected for each } x \in X - K_r\}$$

PROPOSITION 3.8. *Suppose that X is a non-compact connected locally compact separable metric space. Then $\bigcup_{d \in \mathrm{PM}(X)} C^*_d(X)$ is dense in $C^*(X)$ with respect to the uniform convergence topology and there is the following exact sequence*:

$$0 \to \bigcup_{d \in \mathrm{PM}(X)} C^*_d(X) \to \bigcup_{d \in \mathrm{PM}(X)} C_d(X) \to [\beta X, S^1].$$

If $\mathrm{PM}_C(X)$ *is cofinal in* $\mathrm{PM}(X)$, *then the above extends to an exact sequence as follows*:

$$0 \to \bigcup_{d\in \mathrm{PM}(X)} C_d^*(X) \to \bigcup_{d\in \mathrm{PM}(X)} C_d(X) \to [\beta X, S^1] \to [X, S^1] \to 0.$$

Proof. First, we show that $\bigcup_{d\in \mathrm{PM}(X)} C_d^*(X)$ is dense in $C^*(X)$ with respect to the uniform topology. Let $\mathcal{G} = \bigcup_{d\in \mathrm{PM}(X)} C_d^*(X)$ and note that $\mathcal{G}$ separates points and closed sets in X. It follows from Theorem 3.2 that $e_{\mathcal{G}}X \approx \beta X$. Let $\mathcal{H}$ be the closure of $\mathcal{G}$ in $C^*(X)$ with the uniform convergence topology. Then $\mathcal{H}$ is a closed subring that contains all constant functions and $e_{\mathcal{H}}X \geq e_{\mathcal{G}}X$. Since $e_{\mathcal{G}}X \approx \beta X$, we have $e_{\mathcal{H}}X \approx \beta X$. From [1], Theorem 3.7 (cf. [9], Problem 3.12.22(e)), we conclude that $C^*(X) = \mathcal{H}$. Thus, $\bigcup_{d\in \mathrm{PM}(X)} C_d^*(X)$ is dense in $C^*(X)$ with the uniform convergence topology.

Let $d, \varrho \in \mathrm{PM}(X)$ with $d \leq \varrho$. Let $\iota_d : C_d^*(X) \to C_d(X)$, $f_{d\varrho} : C_d^*(X) \to C_\varrho^*(X)$, and $g_{d\varrho} : C_d(X) \to C_\varrho(X)$ be the inclusions. Let $\pi_{\varrho d} : \overline{X}^\varrho \to \overline{X}^d$ be the natural projection and $e : \mathbb{R} \to S^1$ the covering map. We now define homomorphisms $a_d : C_d(X) \to [\overline{X}^d, S^1]$ and $h_{d\varrho} : [\overline{X}^d, S^1] \to [\overline{X}^\varrho, S^1]$ as follows: $a_d(f) = [\overline{e \circ f}]$ and $h_{d\varrho}([g]) = [g \circ \pi_{\varrho d}]$ for $f \in C_d(X)$ and $[g] \in [\overline{X}^d, S^1]$, respectively. Clearly, the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & C_d^*(X) \arrow[r, "\iota_d"] \arrow[d, "f_{d\varrho}"'] & C_d(X) \arrow[r, "a_d"] \arrow[d, "g_{d\varrho}"'] & {[\overline{X}^d, S^1]} \arrow[d, "h_{d\varrho}"'] \\
0 \arrow[r] & C_\varrho^*(X) \arrow[r, "\iota_\varrho"] & C_\varrho(X) \arrow[r, "a_\varrho"] & {[\overline{X}^\varrho, S^1]}
\end{tikzcd}$$

We consider direct systems $\{C_d^*(X), f_{d\varrho} : d, \varrho \in \mathrm{PM}(X),\ d \leq \varrho\}$, $\{C_d(X), g_{d\varrho} : d, \varrho \in \mathrm{PM}(X),\ d \leq \varrho\}$, and $\{[\overline{X}^d, S^1], h_{d\varrho} : d, \varrho \in \mathrm{PM}(X), d \leq \varrho\}$. Then from Theorem 3.7 and the fact that the direct limit of exact sequences is exact (cf. [19], Chap. 4, Sec. 5, Theorem 7) we have an exact sequence

$$0 \to \varinjlim_{d\in \mathrm{PM}(X)} C_d^*(X) \to \varinjlim_{d\in \mathrm{PM}(X)} C_d(X) \to \varinjlim_{d\in \mathrm{PM}(X)} [\overline{X}^d, S^1].$$

Clearly,

$$\varinjlim_{d\in \mathrm{PM}(X)} C_d^*(X) \cong \bigcup_{d\in \mathrm{PM}(X)} C_d^*(X)$$

and

$$\varinjlim_{d\in \mathrm{PM}(X)} C_d(X) \cong \bigcup_{d\in \mathrm{PM}(X)} C_d(X).$$

Furthermore, from Corollary 3.3,

$$\varinjlim_{d\in \mathrm{PM}(X)} [\overline{X}^d, S^1] \cong \varinjlim_{d\in \mathrm{PM}(X)} \check{\mathrm{H}}^1(\overline{X}^d) \cong \check{\mathrm{H}}^1(\beta X) \cong [\beta X, S^1].$$

This completes the first part of the proof.

Next, we assume that $\mathrm{PM}_C(X)$ is cofinal in $\mathrm{PM}(X)$. Let $d, \varrho \in \mathrm{PM}(X)$ with $d \le \varrho$. Let $b_d : [\overline{X}^d, S^1] \to [X, S^1]_d$ be defined by $b_d([f]) = [f{\restriction}X]_d$ for each $[f] \in [\overline{X}^d, S^1]$ and let $i_{d\varrho} : [X, S^1]_d \to [X, S^1]_\varrho$ be defined by $i_{d\varrho}([f]_d) = [f]_\varrho$ for each $[f]_d \in [X, S^1]_d$. These maps form a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C^*_d(X) \arrow[r, "\iota_d"] \arrow[d, "f_{d\varrho}"'] & C_d(X) \arrow[r, "a_d"] \arrow[d, "g_{d\varrho}"'] & {[\overline{X}^d, S^1]} \arrow[r, "b_d"] \arrow[d, "h_{d\varrho}"'] & {[X, S^1]_d} \arrow[r] \arrow[d, "i_{d\varrho}"'] & 0 \\
0 \arrow[r] & C^*_\varrho(X) \arrow[r, "\iota_\varrho"] & C_\varrho(X) \arrow[r, "a_\varrho"] & {[\overline{X}^\varrho, S^1]} \arrow[r, "b_\varrho"] & {[X, S^1]_\varrho} \arrow[r] & 0
\end{tikzcd}$$

The limit of the above sequence reduces to the following exact sequence:

$$0 \to \bigcup_{d\in \mathrm{PM}(X)} C^*_d(X) \to \bigcup_{d\in \mathrm{PM}(X)} C_d(X) \to [\beta X, S^1] \to \varinjlim_{d\in \mathrm{PM}(X)} [X, S^1]_d \to 0.$$

Clearly, $\varinjlim_{d\in \mathrm{PM}(X)}[X, S^1]_d \cong \bigcup_{d\in \mathrm{PM}(X)}[X, S^1]_d$. Now there exists the natural inclusion $i : \bigcup_{d\in \mathrm{PM}(X)}[X, S^1]_d \to [X, S^1]$ and it is easy to see that $i \circ \varinjlim b_d = b$. It follows that i is a bijection and hence $\bigcup_{d\in \mathrm{PM}(X)}[X, S^1]_d = [X, S^1]$. Thus the second part of the proof is complete. ∎

Let (X, d) be a proper metric space. We improve Theorem 3.2 by proving that βX is the supremum of Higson compactifications, whose coronas have the covering dimension $\le 2(\mathrm{asdim}(X, d)) + 1$, where $\mathrm{asdim}(X, d)$ denotes the asymptotic dimension of a metric space (X, d) introduced by M. Gromov [12] (it is denoted by $\mathrm{asdim}_+ X$ in [12]).

Definition 3.9. A family μ of subsets of X is *uniformly bounded* if there is a constant $C > 0$ such that the diameters of all members of μ are less than C. The *asymptotic dimension* $\mathrm{asdim}(X, d)$ of a proper metric space (X, d) is at most n if, for each $R > 0$, there exist uniformly bounded families $\mu_1, \dots, \mu_{n+1}$ of subsets of X such that, for every $i = 1, \dots, n+1$, all the pairwise distances between members of μ_i are $> R$ and $\bigcup_{i=1}^{n+1} \mu_i$ covers X.

Remark 3.10. It is known that $\mathrm{asdim}(\mathbb{R}^n, d_n) = n$, where d_n is the standard metric on $\mathbb{R}^n$ (cf. [8]). Furthermore, if Y is a subspace of a non-compact proper metric space (X, d), then $\mathrm{asdim}(Y, d{\restriction}Y) \le \mathrm{asdim}(X, d)$.

The asymptotic dimensions of product spaces are (roughly) estimated as follows.

LEMMA 3.11. *Let* (X, d_X) *and* (Y, d_Y) *be proper metric spaces. If* $\mathrm{asdim}(X, d_X) \le m$ *and* $\mathrm{asdim}(Y, d_Y) \le n$, *then* $\mathrm{asdim}(X \times Y, \max\{d_X, d_Y\}) \le mn + m + n$.

Proof. Fix an $R > 0$. Since $\mathrm{asdim}(X, d_X) \le m$ there exist uniformly bounded families $\mu_1, \ldots, \mu_{m+1}$ of subsets of X such that $\bigcup_{i=1}^{m+1} \mu_i = X$ and, for every $i = 1, \ldots, m+1$, all the pairwise distances between members of μ_i are $> R$. Similarly, there exist uniformly bounded families $\nu_1, \ldots, \nu_{n+1}$ of subsets of Y such that $\bigcup_{i=1}^{n+1} \nu_i = Y$ and, for every $i = 1, \ldots, n+1$, all the pairwise distances between members of ν_i are $> R$. For simplicity let $\varrho = \max\{d_X, d_Y\}$ on $X \times Y$. Put $\upsilon_{ij} = \{U \times V : U \in \mu_i \text{ and } V \in \nu_j\}$ for $i = 1, \ldots, m+1$ and $j = 1, \ldots, n+1$. Clearly,

$$\bigcup_{\substack{1 \le i \le m+1 \\ 1 \le j \le n+1}} \upsilon_{ij} = X \times Y.$$

Since for every $U \times V \in \upsilon_{ij}$, $\mathrm{diam}(U \times V, \varrho) = \max\{\mathrm{diam}(U, d_X), \mathrm{diam}(V, d_Y)\}$, υ_{ij} is uniformly bounded in $X \times Y$. Furthermore, for all distinct members $U \times V, U' \times V' \in \upsilon_{ij}$,

$$\varrho(U \times V, U' \times V') \ge \max\{d_X(U, U'), d_Y(V, V')\} > R.$$

Therefore we have $\mathrm{asdim}(X \times Y, \varrho) \le mn + m + n$. ■

It is known that the asymptotic dimension is a quasi-isometry invariant of X (cf. [12], p. 29, Remark). In other words,

LEMMA 3.12. *Let* (X, d_X) *and* (Y, d_Y) *be proper metric spaces. If there exist a* (*not necessarily continuous*) *map* $f : X \to Y$ *and positive numbers* $\lambda \ge 1$, $C \ge 0$ *such that, for each* $x, y \in X$, $(1/\lambda) d_X(x, y) - C \le d_Y(f(x), f(y)) \le \lambda d_X(x, y) + C$ *and* $Y = B_C(f(X), d_Y)$, *then* $\mathrm{asdim}(X, d_X) = \mathrm{asdim}(Y, d_Y)$.

The following theorem was proved in [8].

THEOREM 3.13 ([8], Theorem 1.1). *Let* (X, d) *be a proper metric space. Then* $\dim \nu_d X \le \mathrm{asdim}(X, d)$.

We improve Theorem 3.2 in the following:

PROPOSITION 3.14. *If* (X, d) *is a proper metric space, then* $\beta X \approx \sup\{\overline{X}^{\varrho} : \varrho \in \mathrm{PM}(X)$ *with* $\varrho \ge d$ *and* $\dim \nu_\varrho X \le 2\,\mathrm{asdim}(X, d) + 1\}$.

Proof. Let $\mathrm{asdim}(X, d) = n$ and we will show that $\beta X \approx \sup\{\overline{X}^{\varrho} : \varrho \in \mathrm{PM}(X)$ and $\dim \nu_\varrho X \le 2n + 1\}$. Since X is σ-compact, there exists a sequence $\{K_n : n < \omega\}$ of non-empty compact subsets of X such that K_n is a proper subset of $\mathrm{int}\, K_{n+1}$ for every $n < \omega$. For convenience, let $K_{-2} = K_{-1} = \emptyset$. Take a collection $\{U_n : n < \omega\}$ of open subsets of X as in Lemma 3.1. Let A and B be disjoint non-empty closed subsets of X.

Without loss of generality, we may assume that A and B are non-compact. Let $d_{A,B}$ be a proper metric on X as constructed in Theorem 3.2. It is of the form $d_{A,B}(x,y) = \max\{d(x,y), |f(x) - f(y)|\}$ for a continuous function $f : X \to [0,\infty) = J$ and has the property that $\mathrm{cl}_{\overline{X}^{d_{A,B}}} A \cap \mathrm{cl}_{\overline{X}^{d_{A,B}}} B = \emptyset$. We claim that $\mathrm{asdim}(X, d_{A,B}) \le 2n+1$, which implies that $\dim \nu_{d_{A,B}} X \le 2n+1$, and thus completes the proof. Now, put $Y = \{(x, f(x)) : x \in X\} \subset X \times J$ and let d_Y be the metric on Y induced by the metric $\max\{d, d_J\}$ on $X \times J$, where d_J is the standard metric $d_J(x,y) = |x-y|$ on $J = [0,\infty)$. Clearly, (Y, d_Y) is isometric to $(X, d_{A,B})$. From Lemma 3.11 and 3.12, $\mathrm{asdim}(X, d_{A,B}) = \mathrm{asdim}(Y, d_Y) \le 2n+1$. From Theorem 3.13 we see that $\dim \nu_{d_{A,B}} X \le 2n+1$. ■

When $X = \mathbb{R}^n$ with the standard metric, the above Y is a subset of $\mathbb{R}^n \times \mathbb{R}$ with the metric $\sigma((x,s),(y,t)) = \max\{d_n(x,y), |s-t|\}$ (for $(x,s),(y,t) \in \mathbb{R}^n \times \mathbb{R}$), which is clearly quasi-isometric to the standard metric d_{n+1} on $\mathbb{R}^{n+1}$. Thus

PROPOSITION 3.15. *The following statement holds*:

$$\beta\mathbb{R}^n \approx \sup\{\overline{\mathbb{R}^n}^d : d \ge \textit{the standard metric and } \dim \nu_d \mathbb{R}^n \le n+1\}.$$

It is easy to construct a proper metric d on a locally compact non-compact separable metric space X such that (X,d) is quasi-isometric to (J, d_J). For such a metric, $\mathrm{asdim}(X,d) = 1$. Therefore we have:

COROLLARY 3.16. *For each locally compact separable metric space X, $\beta X \approx \sup\{\overline{X}^d : \dim \nu_d X \le 3\}$.*

4. Appendix: Approximation of compactifications by metrizable compactifications. The purpose of this appendix is to illustrate some examples and applications of approximation methods of compactifications by metrizable compactifications.

(a) *Higson compactifications and singular compactifications.* It is easy to see that each compactification αX of a locally compact separable metrizable space X is the inverse limit of some metrizable compactifications. A result of Chandler and Faulkner (cf. [5]) implies that βX is the supremum of (all) compactifications with the remainder being homeomorphic to $[0,1]$. We prove an analogue of Chandler–Faulkner's theorem for Higson compactifications. The notion of singular compactification is necessary for this purpose. Let X be a non-compact space, Y a compact space and $f : X \to Y$ a continuous map. The *singular set* $S(f)$ of f is the subset of Y defined by

$$S(f) = \{y \in Y : \text{for every open neighborhood } U \text{ of } y \text{ in } Y,\ \mathrm{cl}_X f^{-1}(U) \text{ is not compact}\}$$

(cf. [3]). We say that f is *singular* if $S(f) = Y$ (cf. [10]). For each singular map f, we construct a *singular compactification* of X as follows: The underlying set is the disjoint union $X \cup Y$. Basic neighborhoods of points of X are the same as those with respect to the original topology. Points in Y have neighborhoods of the form $U \cup (f^{-1}(U) - F)$, where U is open in Y and F is compact in X. Then $X \cup Y$ with this topology is a compactification of X, denoted by $X \cup_f S(f)$. A compactification αX of X is called a *singular compactification* if $\alpha X \approx X \cup_f S(f)$ for some singular map f (cf. [5] and [10]). This compactification originates with [6].

For each compactification αX, let C_α be the set of all functions f in $C^*(X)$ which admit an extension f^α to αX. Also, S^α denotes the set of all singular functions of C_α.

The proof of Theorem 3 of [15] shows the following.

Lemma 4.1. *Let (X, d) be a non-compact proper metric space. Then there exists a C-embedded copy N of ω in X such that $C^*_{d_N}(N) = C^*(N)$ and N is C^*-embedded in $\overline{X}^d$, where $d_N = d \restriction N$.*

The proof of [5], Theorem 2, works to prove the following straightforward generalization.

Lemma 4.2 (cf. [5], Theorem 2). *Let X be a non-pseudocompact locally compact Hausdorff space and suppose a closed subring $\mathcal{F}$ of $C^*(X)$ containing all constant functions separates points and closed sets in X. If there exists a C-embedded copy N of ω such that $\mathcal{F}(N) = C^*(N)$, where $\mathcal{F}(N) = \{f \restriction N : f \in \mathcal{F}\}$, then $e_{\mathcal{F}} X \approx \sup\{X \cup_f S(f) : f \in \mathcal{F}$ and f is singular$\}$.*

Proof. Let $\alpha X = e_{\mathcal{F}} X$ and $\mathcal{G} = \{f \in \mathcal{F} : f \text{ is singular}\}$, and let $p, q \in \alpha X - X$ be distinct points. Since $\mathcal{F}(N) = C^*(N)$, we note that N is C^*-embedded in αX and it follows that $\mathrm{cl}_{\alpha X} N \approx \beta N$. Take a point $r \in \mathrm{cl}_{\alpha X} N \cap (\alpha X - X)$ with $r \neq p, q$. Let U be a neighborhood of r with $p, q \notin \mathrm{cl}_{\alpha X} U$, and put $N' = U \cap N$. Note that N' is a C^*-embedded copy of ω such that $\mathcal{F}(N') = C^*(N')$. In particular, N' is C^*-embedded in αX. By Lemma 3 of [5], $N' \cup \{p, q\}$ is C^*-embedded in αX. Now, enumerate N' as $\{x_n : n < \omega\}$ and $\mathbb{Q} \cap [0, 1]$ as $\{q_n : n < \omega\}$ and define a function $f^* : N' \cup \{p, q\} \to \mathbb{R}$ as follows:

$$f^*(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & \text{if } x = q, \\ q_n & \text{if } x = x_n. \end{cases}$$

Since $N' \cup \{p, q\}$ is C^*-embedded in αX, there exists a continuous extension $F : \alpha X \to \mathbb{R}$ of f^*. Let $f = F \restriction X$. From [1], Theorem 3.7 we note that $f \in \mathcal{G}$ and $f^\alpha(p) \neq f^\alpha(q)$. Thus $\mathcal{G}$ separates points in $\alpha X - X$, which, by Theorem 1 of [5], completes the proof. ∎

Applying Lemma 4.2 via Lemma 4.1, we obtain the following:

THEOREM 4.3. *Let* (X,d) *be a non-compact proper metric space. Then* $\overline{X}^d \approx \sup\{X \cup_f S(f) : f \in S^d\}$, *where* $S^d = \{f \in C^*_d(X) : f$ *is singular*$\}$.

REMARK 4.4. *In the above theorem,* $X \cup_f S(f)$ *is always metrizable.* To see this, let $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a net of $X\} + \omega$, where a *net* for a topological space X is a collection $\mathcal{N}$ of subsets of X such that every open set in X is the union of elements of $\mathcal{N}$. It is well known that $w(X) = nw(X)$ for each compact Hausdorff space X (cf. [9], Theorem 3.1.19). Thus $w(X \cup_f S(f)) = nw(X \cup_f S(f)) = \omega$ and the claim follows.

(b) *Products of metrizable compactifications.* We are grateful to the referee for pointing out the following result. It provides another application of the approximation method.

THEOREM 4.5. *Let* X *and* Y *be locally compact non-compact separable metrizable. Then there exists a metric compactification* $\alpha(X\times Y)$ *of* $X \times Y$ *such that no metric compactifications* μX *of* X *and* νY *of* Y *satisfy* $\mu X \times \nu Y \geq \alpha(X \times Y)$.

Proof. Suppose that, for each metric compactification $\alpha(X\times Y)$ of $X\times Y$, there exist metric compactifications $\mu_\alpha X$ and $\nu_\alpha Y$ of X and Y such that $\mu_\alpha X \times \nu_\alpha Y \geq \alpha(X \times Y)$. It follows from Lemma 4.2 and Remark 4.4 that $\sup\{\alpha(X\times Y) : \alpha(X\times Y)$ is a metric compactification of $X\times Y\} \approx \beta(X \times Y)$. Then

$$\beta X \times \beta Y \geq \sup_{\alpha(X\times Y)} (\mu_\alpha X \times \nu_\alpha Y) \geq \sup_{\alpha(X\times Y)} \alpha(X\times Y) \approx \beta(X\times Y)$$

where the supremums are taken over all metric compactifications of $X \times Y$. By the maximality of the Stone–Čech compactification, we see that $\beta X \times \beta Y \approx \beta(X \times Y)$, which contradicts the Glicksberg theorem. ∎

(c) *A weaker version of Proposition 3.6.* The following is a weaker version of Proposition 3.6. We decided to include it here since the proof seems to give a good illustration of the approximation method developed in this paper.

PROPOSITION 4.6. *Let* X *and* Y *be locally compact non-compact separable metrizable spaces. There exist proper metrics* $\varrho \in \mathrm{PM}(X\times Y)$, $\varrho_X \in \mathrm{PM}(X)$ *and* $\varrho_Y \in \mathrm{PM}(Y)$ *such that, if* $d \succeq \varrho$, $d_X \succeq \varrho_X$ *and* $d_Y \succeq \varrho_Y$ *with* $d \in \mathrm{PM}(X\times Y)$, $d_X \in \mathrm{PM}(X)$ *and* $d_Y \in \mathrm{PM}(Y)$, *then* $\overline{X\times Y}^d$ *is not equivalent to* $\overline{X}^{d_X} \times \overline{Y}^{d_Y}$.

Proof. Assume on the contrary that, for each $\varrho \in \mathrm{PM}(X\times Y)$, for each $\varrho_X \in \mathrm{PM}(X)$ and for each $\varrho_Y \in \mathrm{PM}(Y)$, there exist $d \in \mathrm{PM}(X\times Y)$, $d_X \in \mathrm{PM}(X)$ and $d_Y \in \mathrm{PM}(Y)$ with $d \succeq \varrho$, $d_X \succeq \varrho_X$ and $d_Y \succeq \varrho_Y$ such that $\overline{X\times Y}^d \approx \overline{X}^{d_X} \times \overline{Y}^{d_Y}$. Then the subset

$$I(X \times Y) = \{d \in \mathrm{PM}(X \times Y) : \overline{X \times Y}^{d} \approx \overline{X}^{d_X} \times \overline{Y}^{d_Y} \text{ for some } d_X \in \mathrm{PM}(X) \text{ and } d_Y \in \mathrm{PM}(Y)\}$$

is cofinal in $\mathrm{PM}(X \times Y)$ with respect to the order $\preceq$. For each $d \in I(X \times Y)$, we fix metrics $d_X \in \mathrm{PM}(X)$ and $d_Y \in \mathrm{PM}(Y)$ and a homeomorphism $\varphi_d : \overline{X \times Y}^{d} \to \overline{X}^{d_X} \times \overline{Y}^{d_Y}$. Let $I(X) = \{d_X : d \in I(X \times Y)\}$ and $I(Y) = \{d_Y : d \in I(X \times Y)\}$, which are cofinal in $\mathrm{PM}(X)$ and $\mathrm{PM}(Y)$ respectively (with respect to $\preceq$). An important observation here is: for compactifications $\gamma_i X,\ \delta_i Y\ (i = 1, 2)$,

$$\text{(1)} \quad \gamma_1 X \times \delta_1 Y \geq \gamma_2 X \times \delta_2 Y \quad \text{if and only if} \quad \gamma_1 X \geq \gamma_2 X \text{ and } \delta_1 Y \geq \delta_2 Y.$$

Indeed, given a continuous surjection $g : \gamma_1 X \times \delta_1 Y \to \gamma_2 X \times \delta_2 Y$, we choose a point $y_0 \in Y$ and consider the following composition of maps:

$$f_X : \gamma_1 X \xrightarrow{\iota_{y_0}} \gamma_1 X \times \delta_1 Y \xrightarrow{g} \gamma_2 X \times \delta_2 Y \xrightarrow{p_X} \gamma_2 X,$$

where ι_{y_0} is the natural embedding defined by $\iota_{y_0}(x) = (x, y_0)$ whenever $x \in \gamma_1 X$, and p_X denotes the standard projection. Since $g \restriction X \times Y = \mathrm{id}_{X \times Y}$, it is easily seen that $f_X \restriction X = \mathrm{id}_X$ and hence $f_X(\gamma_1 X) = \gamma_2 X$. The same argument works for $\delta_i Y\ (i = 1, 2)$.

Now, for metrics $d, \varrho \in I(X \times Y)$ with $d \preceq \varrho$, we have

$$\overline{X}^{d_X} \times \overline{Y}^{d_Y} \approx \overline{X \times Y}^{d} \preceq \overline{X \times Y}^{\varrho} \approx \overline{X}^{\varrho_X} \times \overline{Y}^{\varrho_Y}.$$

It follows from (1) that $\overline{X}^{d_X} \leq \overline{X}^{\varrho_X}$ and $\overline{Y}^{d_Y} \leq \overline{Y}^{\varrho_Y}$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\overline{X \times Y}^{\varrho} \arrow[r, "\varphi_\varrho"] \arrow[d, "\pi_{\varrho d}"'] & \overline{X}^{\varrho_X} \times \overline{Y}^{\varrho_Y} \arrow[r, "p_{\varrho_X}"] \arrow[d, "\pi_{\varrho_X d_X} \times \pi_{\varrho_Y d_Y}"'] & \overline{X}^{\varrho_X} \arrow[d, "\pi_{\varrho_X d_X}"'] \\
\overline{X \times Y}^{d} \arrow[r, "\varphi_d"] & \overline{X}^{d_X} \times \overline{Y}^{d_Y} \arrow[r, "p_{d_X}"] & \overline{X}^{d_X}
\end{tikzcd}$$

where $\pi_{d\varrho}$, $\pi_{\varrho_X d_X}$ and $\pi_{\varrho_Y d_Y}$ are natural projections, and p_{ϱ_X} and p_{d_X} are standard projections. If we pass to the inverse limits and make use of the cofinality of $I(X \times Y)$ in $\mathrm{PM}(X \times Y)$ and $I(X)$ in $\mathrm{PM}(X)$ respectively, from the remark before Corollary 3.3 we obtain the limit maps

$$\beta(X \times Y) \xrightarrow{\varprojlim \varphi_\varrho} \varprojlim\{\overline{X}^{\varrho_X} \times \overline{Y}^{\varrho_Y}, \pi_{\varrho_X d_X} \times \pi_{\varrho_Y d_Y}\} \xrightarrow{P_X = \varprojlim p_{d_X}} \beta X.$$

Similarly we have the limit maps

$$\beta(X \times Y) \xrightarrow{\varprojlim \varphi_\varrho} \varprojlim\{\overline{X}^{\varrho_X} \times \overline{Y}^{\varrho_Y}, \pi_{\varrho_X d_X} \times \pi_{\varrho_Y d_X}\} \xrightarrow{P_Y = \varprojlim p_{\varrho_Y}} \beta Y.$$

Since φ_ϱ is a homeomorphism for each $\varrho \in I(X \times Y)$, $\varprojlim \varphi_\varrho$ is a homeomorphism as well. Also, it is easy to see that the diagonal map

$$P_X \triangle P_Y : \varprojlim\{\overline{X}^{\varrho_X} \times \overline{Y}^{\varrho_Y}, \pi_{\varrho_X d_X} \times \pi_{\varrho_Y d_Y}\} \to \beta X \times \beta Y$$

is injective and hence is a homeomorphism. Therefore we have a homeomorphism $(P_X \triangle P_Y) \circ \varprojlim \varphi_d : \beta(X \times Y) \to \beta X \times \beta Y$, which contradicts the Glicksberg theorem. ∎

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Institute of Mathematics
University of Tsukuba
Tsukuba-shi Ibaraki
305-8571, Japan
E-mail: kawamura@math.tsukuba.ac.jp

Miyakonojo National College of Technology
Miyakonojo-shi Miyazaki
885-8567, Japan
E-mail: tomoyasu@cc.miyakonojo-nct.ac.jp

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