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NONCOERCIVE DIFFERENTIAL OPERATORS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE AND THEIR GREEN FUNCTIONS

BY

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Abstract. We obtain upper and lower estimates for the Green function for a second order noncoercive differential operator on a homogeneous manifold of negative curvature.

1. Introduction and the main result. In this paper we study the Green function for a second order noncoercive differential operator $\mathcal{L}$ on a connected, simply connected homogeneous manifold of negative curvature. Such a manifold is a solvable Lie group $S = NA$, a semidirect product of a nilpotent Lie group N and an abelian group $A = \mathbb{R}^+$. Moreover, for an H belonging to the Lie algebra $\mathcal{A}$ of A, the eigenvalues of $\mathrm{Ad}_{\exp H}|_N$ are all greater than 0. Conversely, every such group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature (see [H]).

On S we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y.$$

We assume that $Y_0, Y_1, \ldots, Y_m$ generate the Lie algebra $\mathcal{S}$ of S. Moreover, we can choose $Y_0, Y_1, \ldots, Y_m$ so that $Y_1(e), \ldots, Y_m(e)$ belong to the Lie algebra $\mathcal{N}$ of N. Let $\pi : S \to A = S/N$ be the canonical homomorphism. Then the image of $\mathcal{L}$ under π is a second order left-invariant operator on $\mathbb{R}^+$,

$$(a\partial_a)^2 - \gamma a\partial_a,$$

where $\gamma \in \mathbb{R}$. The operator $\mathcal{L} = \mathcal{L}_\gamma$ is noncoercive (there is no $\varepsilon > 0$ such that $\mathcal{L} + \varepsilon I$ admits the Green function) if and only if $\gamma = 0$.

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Finally, the operator we are interested in can be written in the form

$$\mathcal{L} = \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + a^2\partial_a^2 + a\partial_a, \tag{1.1}$$

where $X, X_1, \ldots, X_m$ are left-invariant vector fields on N and $X_1, \ldots, X_m$ generate $\mathcal{N}$, $\Phi_a = \mathrm{Ad}_{\exp(\log a)Y_0} = e^{(\log a)\,\mathrm{ad}_{Y_0}} = e^{(\log a)D}$ and $D = \mathrm{ad}_{Y_0}$ is a derivation of the Lie algebra $\mathcal{N}$ of the Lie group N such that the real parts d_j of the eigenvalues λ_j of D are positive. By multiplying $\mathcal{L}$ by a constant we can make d_j arbitrarily large (see [DHU]).

Let $\mathcal{G}(xa, yb)$ be the *Green function* for $\mathcal{L}$. It is (uniquely) defined by two conditions:

(i) $\mathcal{L}\mathcal{G}(\cdot, yb) = -\delta_{yb}$ as distributions (functions are identified with distributions via the right Haar measure),

(ii) for every $yb \in S$, $\mathcal{G}(\cdot, yb)$ is a potential for $\mathcal{L}$.

Let

$$\mathcal{G}(x, a) = \mathcal{G}(xa, e), \tag{1.2}$$

where e is the identity element of the group S. In this paper we call $\mathcal{G}(x, a)$ the Green function for $\mathcal{L}$.

For a positive δ less than $1/2$ define

$$T_\delta = \{(x, a) \in N \times \mathbb{R}^+ : 1 - \delta < a < 1 + \delta,\ |x| < \delta\}, \tag{1.3}$$

where $|\cdot|$ denotes the "homogeneous norm" (see Preliminaries).

Our aim is to prove the following result:

THEOREM 1.4. *For a given* $0 < \delta < 1/2$ *there exists a positive constant* C *such that for* $(x, a) \notin T_\delta$ *we have the following estimate for the Green function* $\mathcal{G}$ *defined in* (1.2):

$$C^{-1}w(x, a) \le \mathcal{G}(x, a) \le Cw(x, a), \tag{1.5}$$

where the function w *is defined by*

$$w(x, a) = \begin{cases} 1 & \text{if } |x| \le 1,\ a \le 1, \\ |x|^{-Q} & \text{if } |x| \ge 1,\ |x| \ge a, \\ a^{-Q} & \text{if } a \ge 1,\ a \ge |x|, \end{cases} \tag{1.6}$$

and $Q = \sum d_j = \sum \mathrm{Re}\,\lambda_j$.

The above result looks like the limit case (as γ tends to 0) of the estimate of the Green function for the operator $\mathcal{L}_\gamma$ with positive γ (i.e. for a coercive operator). This has been proved by E. Damek [D] by means of Ancona's theory. However, (1.5) cannot be obtained from Damek's estimate by taking the limit and so requires essentially new methods. In this paper we make use of a probabilistic method introduced in [DH] and then developed e.g. in [DHZ], [DHU].

The structure of this paper is as follows. In Section 2 we state precisely notation and all necessary definitions.

In Section 3 we recall the basic properties of the Bessel process which appears as the "vertical" component of the diffusion generated by $a^{-2}\mathcal{L}$ on $N \times \mathbb{R}^+$ (cf. [DHU]).

In Section 4 we state the estimate of the transition probabilities of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N \times \mathbb{R}^+$ mentioned above.

In Section 5 we prove the main lemmas, which are a crucial point in the proof of Theorem 1.4 given in Section 6.

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2. Preliminaries. Some of the notions which appear in this section have been introduced in the previous one. However, for the sake of completeness we state them precisely once again.

Let N be a connected and simply connected nilpotent Lie group. Let D be a derivation of the Lie algebra $\mathcal{N}$ of N. For every $a \in \mathbb{R}^+$ we define an automorphism Φ_a of $\mathcal{N}$ by

$$\Phi_a = e^{(\log a)D}.$$

Writing $x = \exp X$ we have

$$\Phi_a(x) := \exp \Phi_a(X).$$

We assume that the real parts d_j of the eigenvalues λ_j of the matrix D are strictly greater than 0 and we define the number

$$Q = \sum_j \operatorname{Re} \lambda_j = \sum_j d_j.$$

In this paper $D = \mathrm{ad}_{Y_0}$ (see Introduction). We consider a group S which is a *semidirect* product of N and the multiplicative group $A = \mathbb{R}^+ = \{\exp tY_0 : t \in \mathbb{R}\}$:

$$S = NA = \{xa : x \in N,\ a \in A\}$$

with multiplication given by

$$(xa)(yb) = (x\Phi_a(y)ab).$$

In N we define the *homogeneous norm* $|\cdot|$ ([DHZ], [DHU]). Let $(\cdot,\cdot)$ be a fixed inner product in $\mathcal{N}$. We define a new inner product

$$\langle X, Y\rangle = \int_0^1 (\Phi_a(X), \Phi_a(Y))\frac{da}{a}$$

and the corresponding norm

$$\|X\| = \langle X, X\rangle^{1/2}.$$

We put

$$|X| = (\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\})^{-1}.$$

One can easily show that for every $Y \neq 0$ there exists precisely one $a > 0$ such that $Y = \Phi_a(X)$ with $|X| = 1$. Then we have $|Y| = a$.

Finally, we define a homogeneous norm on N. For $x = \exp X$ we put

$$|x| = |X|.$$

Notice that if the action of $A = \mathbb{R}^+$ on N (given by Φ_a) is diagonal, the norm we have just defined is the usual homogeneous norm on N (see [FS]).

And a final remark about notation: The letter C occurs in inequalities as a positive constant and may vary from statement to statement, even in the same calculation.

3. Bessel process. Let b_t denote the *Bessel process* with a parameter $\alpha \geq 0$ (cf. [RY]), i.e. a continuous Markov process with state space $[0, \infty)$ generated by

$$\Delta = \partial_a^2 + \frac{2\alpha + 1}{a}\partial_a.$$

The transition function with respect to the measure $y^{2\alpha+1}\,dy$ is given by (cf. [RY] again)

$$p_t(x, y) = \begin{cases} c_\alpha \dfrac{1}{2t} \exp\left(\dfrac{-x^2 - y^2}{4t}\right) I_\alpha\left(\dfrac{xy}{2t}\right)\dfrac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\ c_\alpha \dfrac{1}{(2t)^{\alpha+1}} \exp\left(\dfrac{-y^2}{4t}\right) & \text{for } x = 0, y > 0, \end{cases} \tag{3.1}$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\alpha}}{k!\Gamma(k + \alpha + 1)}$$

is the *Bessel function* (see [L]). Therefore for $x \geq 0$ and a measurable set $B \subset (0, \infty)$,

$$\mathbf{P}_x(b_t \in B) = \int_B p_t(x, y) y^{2\alpha+1}\,dy.$$

The following lemmas concerning some properties of the Bessel process are very well known and their proofs are rather standard. Sketches of those proofs can be found in [DHU] or [U].

LEMMA 3.2. *Let $D,\gamma,a \geq 0$. There exists a positive constant C such that for every $t > 0$,*

$$\sup_{a>0} \mathbf{E}_a \Big(\int_0^1 b_s^\gamma \, ds\Big)^{-D/2} < \infty.$$

Moreover,

$$\mathbf{E}_a \Big(\int_0^t b_s^\gamma \, ds\Big)^{-D} \leq C t^{-D(1+\gamma/2)}.$$

LEMMA 3.3. *There exist constants c_1, c_2 such that for every $x \geq 0$, for every $\lambda > 0$ and for every $t > 0$,*

$$\mathbf{P}_x(\sup_{s\in[0,t]} b_s > x + \lambda) \leq c_1 e^{-c_2\lambda^2/t}.$$

LEMMA 3.4. *Let $0 < \eta < 1$. There exist constants c_1, c_2 such that for every $t > 0$,*

$$\mathbf{P}_1(\inf_{s\in[0,t]} b_s \leq 1 - \eta) \leq c_1 e^{-c_2/t}.$$

Proof. It is enough to rewrite the proof of Lemma 2.4 in [DHU]. ∎

By a straightforward computation, using the definition of the transition function $p_t(x,y)$ of the Bessel process (3.1) and the asymptotic behaviour of the Bessel function (see [L]):

$$I_\alpha(x) \asymp \begin{cases} \dfrac{x^\alpha}{2^\alpha \Gamma(1+\alpha)}, & x \to 0, \\ \dfrac{\exp(x)}{(2\pi x)^{1/2}}, & x \to \infty, \end{cases}$$

we get

LEMMA 3.5. *There exists a constant C independent of x such that*

$$\mathbf{P}_x(a - \eta \leq b_t \leq a + \eta) \leq C t^{-(\alpha+1)} m([a-\eta, a+\eta]),$$

where $m(B) = \int_B y^{2\alpha+1} \, dy$.

4. Evolutions. For a multiindex $I = (i_1, \ldots, i_n)$, $i_j \in \mathbb{Z}^+$ and a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathcal{N}$ of N we write $X^I = X_1^{i_1} \ldots X_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. For $k = 0, 1, \ldots, \infty$ we define

$$C^k = \{f : X^I f \in C(N) \text{ for } |I| < k+1\}$$

and

$$C^k_\infty = \{f \in C^k : \lim_{x\to\infty} X^I f(x) \text{ exists for } |I| < k+1\}.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I|\le k} \|X^I f\|_{C(N)}.$$

Let

$$L_{\sigma(t)} = \sigma(t)^{-2}\Big(\sum \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X)\Big).$$

For a continuous function $\sigma : [0,\infty) \to [0,\infty)$ let $\{U^\sigma(s,t) : 0 \le s \le t\}$ be the unique family of bounded operators on $C_\infty = C^0_\infty$ which satisfy

(i) $U^\sigma(s,s) = I$,
(ii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $s < r < t$,
(iii) $\partial_s U^\sigma(s,t)f = -L_{\sigma(s)}U^\sigma(s,t)f$ for every $f \in C_\infty$,
(iv) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L_{\sigma(t)}f$ for every $f \in C_\infty$,
(v) $U^\sigma(s,t) : C^2_\infty \to C^2_\infty$.

$U^\sigma(s,t)$ is a convolution operator. Namely, $U^\sigma(s,t)f = f * p^\sigma(t,s)$, where $p^\sigma(t,s)$ is a smooth density of a probability measure. By (ii) we have $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. Existence of the family $U^\sigma(s,t)$ follows from [T].

In [DHU], using the Nash inequality, the following estimate of the evolution kernels $p^\sigma(t,0)$ has been proved.

THEOREM 4.1. *For every compact set $K \subset N$ which does not contain the identity e of N, there exist positive constants C, ξ, β_1, β_2 and $D \le Q$ such that for every $x \in K$ and for every $t > 0$,*

$$p^\sigma(t,0)(x) \le C\Big(\int_0^t \sigma^{-2(1-Q/D)}(u)\,du\Big)^{-D/2} \exp\Big(-\frac{\xi}{A(0,t)}\Big),$$

where $A(s,t) = \int_s^t (\sigma^{\beta_1}(u) + \sigma^{\beta_2}(u))\,du$.

In the proof of the above theorem the following estimate of the norm $\|p^\sigma(t,s)\|_{L^\infty(N)}$ has been obtained:

THEOREM 4.2. *There exist positive constants C and $D \le Q$ such that for every $s < t$,*

$$\|p^\sigma(t,s)\|_{L^\infty(N)} \le C\Big(\int_s^t \sigma^{-2(1-Q/D)}(u)\,du\Big)^{-D/2}.$$

5. Main lemmas. From now on we consider the Bessel process b_t with a parameter $\alpha = 0$. In this case $b_t = \|w_t\|$, where w_t is a Brownian motion on $\mathbb{R}^2$.

In this section we prove some lemmas, which are our main tools in writing estimates for the Green function.

LEMMA 5.2. *Let $D,\gamma > 0$ and $dm(a) = a da$. For every $\delta > 0$ there exists a constant C such that for every $a \le 1-\delta$,*

$$\sup_{0<\eta<\delta/2} \int_0^\infty \mathbf{E}_1\Big(\int_0^t b_s^\gamma\, ds\Big)^{-D/2} m([a-\eta,a+\eta])^{-1} 1_{[a-\eta,a+\eta]}(b_t)\, dt \le C.$$

Proof. In order to simplify notation let $I_{a,\eta} = [a-\eta, a+\eta]$.

First we consider large time ($t \ge 1$):

$$\int_1^\infty \mathbf{E}_1\Big(\int_0^t b_s^\gamma\, ds\Big)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_t)\, dt \le \int_1^\infty \mathbf{E}_1\Big(\int_0^{t/2} b_s^\gamma\, ds\Big)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\theta_{t/2} b_{t/2})\, dt,$$

where θ_s is the shift operator. Using the Markov property and Lemma 3.2 we get

$$\begin{aligned}(5.3)\qquad &\int_1^\infty \mathbf{E}_1\Big(\int_0^{t/2} b_s^\gamma\, ds\Big)^{-D/2} \mathbf{E}_{b_{t/2}} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_{t/2})\, dt \\ &= \int_1^\infty \mathbf{E}_1\Big(\int_0^{t/2} b_s^\gamma\, ds\Big)^{-D/2} m(I_{a,\eta})^{-1} \mathbf{P}_{b_{t/2}}(\sigma_{t/2} \in I_{a,\eta})\, dt \\ &\le C\int_1^\infty t^{-(D/2)(1+\gamma/2)} m(I_{a,\eta})^{-1} \mathbf{P}_{b_{t/2}}(\sigma_{t/2} \in I_{a,\eta})\, dt.\end{aligned}$$

By Lemma 3.5,

$$(5.4)\qquad \mathbf{P}_x(\sigma_t \in I_{a,\eta}) \le Ct^{-1} m(I_{a,\eta})$$

with C independent of the starting point x. Hence by (5.3) we get

$$(5.5)\qquad \sup_{\eta>0} \int_1^\infty \mathbf{E}_1\Big(\int_0^t b_s^\gamma\, ds\Big)^{-D/2} m(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(b_t)\, dt \le C\int_1^\infty t^{-(D/2)(1+\gamma/2)-1}\, dt \le C_1.$$

Now we consider $t \le 1$. We divide the set of all trajectories of the Bessel process b_t (with parameter 0) starting from 1 into two subsets:

$$A = \{b : \sup_{s\in[0,t]} b_s > 2\}, \qquad B = \{b : \sup_{s\in[0,t]} b_s \le 2\}.$$

Consider the set A. Let $T=\inf\{s: b_s=2\}$. For $n\geq 1$, let

$$A_n=\{b: t/2^n<T\leq t/2^{n-1}\}.$$

Then the Markov property gives

$$\begin{aligned}
(5.6)\qquad &\int_0^1 \mathbf{E}_1\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_A(b)\,dt\\
&=\int_0^1\sum_{n=1}^\infty \mathbf{E}_1\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{A_n}(b)\,dt\\
&\leq\int_0^1\sum_{n=1}^\infty \mathbf{E}_1\Big(\int_0^T b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{\{T\leq t/2^{n-1}\}}(b)\,dt\\
&\leq\int_0^1\sum_{n=1}^\infty \mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{\{T\leq t/2^{n-1}\}}(b)\,dt\\
&=\int_0^1\sum_{n=1}^\infty \mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{\{T\leq t/2^{n-1}\}}(b)\\
&\qquad\times \mathbf{E}_{b_{t/2^{n-1}}}1_{\{\sigma:\sigma_{t-t/2^{n-1}}\in I_{a,\eta}\}}(\sigma)\,dt\\
&\leq\int_0^1\sum_{n=1}^\infty \mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{\{b:\sup_{s\in[0,t/2^{n-1}]}b_s\geq 2\}}(b)\\
&\qquad\times \mathbf{E}_{b_{t/2^{n-1}}}1_{\{\sigma:\sigma_{t-t/2^{n-1}}\in I_{a,\eta}\}}(\sigma)\,dt.
\end{aligned}$$

By (5.4) it follows that for $n\geq 2$,

$$\begin{aligned}
(5.7)\qquad \mathbf{E}_{b_{t/2^{n-1}}}1_{\{\sigma:\sigma_{t-t/2^{n-1}}\in I_{a,\eta}\}}(\sigma)&\leq C(t-t/2^{n-1})^{-1}m(I_{a,\eta})\\
&\leq C(t/2)^{-1}m(I_{a,\eta}).
\end{aligned}$$

For $n=1$ the expectation in (5.7) is equal to

$$\mathbf{P}_{b_t}(\sigma_0\in I_{a,\eta})=\mathbf{P}_1(b_t\in I_{a,\eta})$$

and by (5.4) we get (5.7) for $n=1$.

Therefore using (5.7), Lemma 3.2, Lemma 3.3 and the Schwarz inequality we get

$$\begin{aligned}
(5.8)\qquad &\int_0^1\sum_{n=1}^\infty \mathbf{E}_1\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{A_n}(b)\,dt\\
&\qquad\leq C\int_0^1\sum_{n=1}^\infty t^{-1}\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2}1_{\{b:\sup_{s\in[0,t/2^{n-1}]}b_s\geq 2\}}(b)\,dt
\end{aligned}$$

$$\leq C\int_0^1 t^{-1}\sum_{n=1}^{\infty}\Big[\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D}\Big]^{1/2}$$
$$\times\,[\mathbf{E}_1 1_{\{b:\sup_{s\in[0,t/2^{n-1}]} b_s\geq 2\}}(b)]^{1/2}\,dt$$
$$\leq C\int_0^1\sum_{n=1}^{\infty} t^{-1}(t/2^n)^{-(D/2)(1+\gamma/2)}e^{-c2^{n-1}/t}\,dt\leq C_2.$$

Now we consider the set B. Let $T=\inf\{s: b_s=1-\delta/2\}$. For $n\geq 1$, let

$$A_n=\{b: t/2^n<T\leq t/2^{n-1}\}.$$

Notice that

$$T\leq t/2^{n-1}\quad\text{implies}\quad \inf_{s\in[0,t/2^{n-1}]} b_s\leq 1-\delta/2.$$

Moreover, by Lemma 3.4,

$$\mathbf{P}_1\big(\inf_{s\in[0,t]} b_s\leq 1-\delta/2\big)\leq c_1e^{-c_2/t}. \tag{5.9}$$

Then

$$\int_0^1\mathbf{E}_1\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_B(b)\,dt$$
$$=\int_0^1\sum_{n=1}^{\infty}\mathbf{E}_1\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{A_n}(b)\,dt$$
$$\leq\int_0^1\sum_{n=1}^{\infty}\mathbf{E}_1\Big(\int_0^T b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{\{T\leq t/2^{n-1}\}}(b)\,dt$$
$$\leq\int_0^1\sum_{n=1}^{\infty}\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_{\{T\leq t/2^{n-1}\}}(b)\,dt$$
$$=\int_0^1\sum_{n=1}^{\infty}\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{\{T\leq t/2^{n-1}\}}(b)$$
$$\times\,\mathbf{E}_{b_{t/2^{n-1}}}1_{\{\sigma:\sigma_{t-t/2^{n-1}}\in I_{a,\eta}\}}(\sigma)\,dt$$
$$\leq\int_0^1\sum_{n=1}^{\infty}\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{\{b:\inf_{s\in[0,t/2^{n-1}]} b_s\leq 1-\delta/2\}}(b)$$
$$\times\,\mathbf{E}_{b_{t/2^{n-1}}}1_{\{\sigma:\sigma_{t-t/2^{n-1}}\in I_{a,\eta}\}}(\sigma)\,dt$$
$$\leq\int_0^1\sum_{n=1}^{\infty}\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D/2} 1_{\{b:\inf_{s\in[0,t/2^{n-1}]} b_s\leq 1-\delta/2\}}(b)t^{-1}\,dt,$$

where in the last inequality we have used (5.7) for $n \geq 1$ (see the remark after (5.7)). Now, as before, in order to estimate the expectation we use the Schwarz inequality. By Lemma 3.2 and (5.9) we have

$$(5.10)\quad \int_0^1 \mathbf{E}_1\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m(I_{a,\eta})^{-1}1_{I_{a,\eta}}(b_t)1_B(b)\,dt$$
$$\leq C\int_0^1 t^{-1}\sum_{n=1}^\infty \Big[\mathbf{E}_1\Big(\int_0^{t/2^n} b_s^\gamma\,ds\Big)^{-D}\Big]^{1/2}$$
$$\times\,[\mathbf{E}_1 1_{\{b:\inf_{s\in[0,t/2^{n-1}]} b_s\leq 1-\delta/2\}}(b)]^{1/2}\,dt$$
$$\leq C\int_0^1 t^{-1}\sum_{n=1}^\infty (t/2^n)^{-(D/2)(1+\gamma/2)}[\mathbf{E}_1 1_{\{b:\inf_{s\in[0,t/2^{n-1}]} b_s\leq 1-\delta/2\}}(b)]^{1/2}\,dt$$
$$\leq C\int_0^1 t^{-1}\sum_{n=1}^\infty (t/2^n)^{-(D/2)(1+\gamma/2)}[\mathbf{P}_1(\inf_{s\in[0,t/2^{n-1}]} b_s\leq 1-\delta/2)]^{1/2}\,dt$$
$$\leq C\int_0^1\sum_{n=1}^\infty t^{-1}(t/2^n)^{-(D/2)(1+\gamma/2)}e^{-c2^{n-1}/t}\,dt\leq C_3.$$

Now (5.5), (5.8) and (5.10) complete the proof. ■

LEMMA 5.11. *Let $D,\gamma>0$ and $dm(a)=a\,da$. For every $0<\delta<1/2$ there exists a constant C such that for every $x\leq 1/2-\delta$ and every $(1-\delta)/2\leq a\leq 1/2$,*

$$\sup_{0<\eta<\delta/4}\int_0^\infty \mathbf{E}_x\Big(\int_0^t b_s^\gamma\,ds\Big)^{-D/2} m([a-\eta,a+\eta])^{-1}1_{[a-\eta,a+\eta]}(b_t)\,dt\leq C.$$

Proof. For large time ($t\geq 1$) it is enough to rewrite the proof of the previous lemma.

Let $t\leq 1$. We define $T=\inf\{s: b_s=1/2-3\delta/4\}$. For $n\geq 1$, let

$$A_n=\{b: t/2^n<T\leq t/2^{n-1}\}.$$

Notice that

$$T\leq t/2^{n-1}\quad\text{implies}\quad \sup_{s\in[0,t/2^{n-1}]} b_s\geq 1/2-3\delta/4.$$

Then, since $x\leq 1/2-\delta$, by Lemma 3.3,

$$(5.12)\quad \mathbf{P}_x(\sup_{s\in[0,t/2^{n-1}]} b_s\geq 1/2-3\delta/4)$$

$$= \mathbf{P}_x\big(\sup_{s\in[0,t/2^{n-1}]} b_s \geq (1/2 - 3\delta/4 - x) + x\big)$$

$$\leq c_1 e^{-c_2(1/2-3\delta/4-x)^2 2^{n-1}/t} \leq c_1 e^{-c_2(\delta/4)^2 2^{n-1}/t}.$$

Now, because of (5.12) it is enough to rewrite the end of the proof of Lemma 5.2 starting after (5.9). Namely, we have to change the starting point to x and instead of $\{b : \inf_{[0,t/2^{n-1}]} b_s \leq 1-\delta/2\}$ put $\{b : \sup_{s\in[0,t/2^{n-1}]} b_s \geq 1/2 - 3\delta/4\}$. ∎

The next lemma is taken from [DHU] (Lemma 5.18):

LEMMA 5.13. *Let* $D, \xi, \gamma > 0$, $dm(a) = a\,da$. *For every* $a_1 > 0$ *there is a constant* C *such that for every* $x \leq a_1$, $0 < a < 1$,

$$\sup_{0<\eta<1} \int_0^\infty \mathbf{E}_x\Big(\int_0^t b_s^\gamma\, ds\Big)^{-D/2} e^{-\xi/A(0,t)} m([a-\eta, a+\eta])^{-1} 1_{[a-\eta,a+\eta]}(b_t)\, dt \leq C,$$

where $A(0,t)$ *is defined in Theorem* 4.1.

6. Proof of Theorem 1.4. It turns out that it is very convenient to consider along with the operator $\mathcal{L}$ defined in (1.1) the corresponding operator L,

$$L = a^{-2}\mathcal{L} = a^{-2}\sum_j \Phi_a(X_j)^2 + \Phi_a(X) + \partial_a^2 + \frac{1}{a}\partial_a. \tag{6.1}$$

The Green function G for L is given by

$$G(x,a;y,b) = \int_0^\infty p_t(x,a;y,b)\, dt, \tag{6.2}$$

where $T_t f(x,a) = \int f(y,b) p_t(x,a;y,b)\, dy\, b\, db$ is the heat semigroup on $L^2(N\times\mathbb{R}^+, dybdb)$ with infinitesimal generator L.

In (6.2) we allow (x,a) to be $(e,0)$ since $\lim_{(x,a)\to(e,0)} G(x,a;y,b)$ exists (see [DHU]).

On $N\times\mathbb{R}^+$ we define *dilations*

$$D_t(x,a) = (\Phi_t(x), ta), \qquad t > 0.$$

It is not difficult to check that although the operator L is not left-invariant it has some homogeneity with respect to the family of dilations introduced above:

$$L(f\circ D_t) = t^2 Lf\circ D_t.$$

This implies that

$$G(x,a;y,b) = t^{-Q} G(D_{t^{-1}}(x,a); D_{t^{-1}}(y,b)). \tag{6.3}$$

It turns out (see (1.17) in [DHU]) that

$$\mathcal{G}(x,a) = G(x,a;e,1) = G^*(e,1;x,a),$$

where G^* is the Green function for the operator

$$L^* = a^{-2}\sum \Phi_a(X_j)^2 - a^{-2}\Phi_a(X) + \partial_a^2 + a^{-1}\partial_a,$$

conjugate to L with respect to the measure $adxda$. Moreover,

$$G^*(e,1;x,a) = \lim_{\eta\to 0}\int_0^\infty \mathbf{E}_1 p^\sigma(t,0)(x)\frac{1}{m([a-\eta,a+\eta])}1_{[a-\eta,a+\eta]}(\sigma_t)\,dt, \tag{6.4}$$

where the expectation is taken with respect to the distribution of the Bessel process starting from 1 on the space $C([0,\infty),(0,\infty))$. All the above facts are proved in [DHU].

Now we are ready to give

Proof of Theorem 1.4. For $r \geq 0$, define

$$V_r = \{(x,a) \in N \times \mathbb{R}^+ : |(x,a)| = r\},$$

where $|(x,a)| = |x| + a$. Let $0 < \delta < 1/2$ be fixed.

CASE 1. We consider the set

$$S_1 = \{(x,a) \notin T_\delta : |x| \leq 1,\ a \leq 1\}.$$

We have to show that there exists a positive constant C such that

$$C^{-1} \leq \mathcal{G}(x,a) = G^*(e,1;x,a) \leq C \tag{6.5}$$

for every $(x,a) \in S_1$.

It follows immediately from (6.4), Theorem 4.2, and Lemma 5.2 that we have the upper bound in (6.5) on $\widetilde{S}_1 = S_1 \cap \{(x,a) \in N \times \mathbb{R}^+ : a \leq 1-\delta\}$. Therefore we are left with $(x,a) \in S_1 \backslash \widetilde{S}_1$. But

$$S_1 \backslash \operatorname{Int} \widetilde{S}_1 = \{(x,a) : N \times \mathbb{R}^+ : \delta \leq |x| \leq 1,\ 1-\delta \leq a \leq 1\}$$

is a compact set. Since G^* is a continuous function we get the upper bound on S_1. The lower bound in (6.5) is a consequence of Lemma 5.21 of [DHU].

CASE 2. We consider the set

$$S_2 = \{(x,a) \in N \times \mathbb{R}^+ : |x| \geq 1,\ |x| \geq a\}.$$

(Of course, $S_2 \cap T_\delta = \emptyset$.)

Every element $(x,a) \in N \times \mathbb{R}^+$ can be written as

$$(x,a) = D_t(y,b), \qquad \text{where } (y,b) \in V_1 \text{ and } t = |(x,a)| = |x| + a.$$

(Recall that $D_t(x,a) = (\Phi_t(x), ta)$.) By homogeneity of G (see (6.3)), we get

$$\begin{aligned} G^*(e,1;x,a) &= G^*(D_t(e,t^{-1}); D_t(y,b)) = t^{-Q}G^*(e,t^{-1};y,b) \\ &= |(x,a)|^{-Q}G^*(e,|(x,a)|^{-1};y,b) \end{aligned} \tag{6.6}$$

$$= (|x| + a)^{-Q} G^*(e, (|x| + a)^{-1}; y, b).$$

If $(x, a) \in S_2$ then the corresponding $(y, b) \in V_1$ has the property $|y| \geq b$. Indeed, $x = \Phi_t(y)$ and $a = tb$, thus $t|y| = |x| \geq a = tb$. The above property and $|y| + b = 1$ imply that $b \leq 1/2$. Therefore

$$(y, b) \in V_1 \cap \{(x, a) \in N \times \mathbb{R}^+ : a \leq 1/2\} \subset V_1.$$

Let $\beta = |(x, a)|^{-1}$. For $(x, a) \in S_2$ we have $\beta \leq 1$. Thus by (6.4), Theorem 4.1 and Lemma 5.13 we get

$$G^*(e, \beta; x, a) \leq C \qquad \text{for } (x, a) \in S_2.$$

Once again, Lemma 5.21 in [DHU] gives the lower bound

$$G^*(e, \beta; x, a) \geq C^{-1}.$$

Thus by (6.6) we get

$$C^{-1}(|x| + a)^{-Q} \leq \mathcal{G}(x, a) \leq C(|x| + a)^{-Q}.$$

Since $|x| \leq |x| + a \leq 2|x|$ for $(x, a) \in S_2$, the proof of the second case is complete.

Case 3. Finally we consider the set

$$S_3 = \{(x, a) \notin T_\delta : a \geq |x|,\ a \geq 1\}.$$

Because $V_1 \cap T_\delta \neq \emptyset$ we write every element $(x, a) \in N \times \mathbb{R}^+$ as a dilation of some element from $V_{1/2}$:

$$(x, a) = D_t(y, b), \qquad \text{where } (y, b) \in V_{1/2} \text{ and } t = 2|(x, a)| = 2|x| + 2a.$$

By homogeneity, we can write, analogously to (6.6),

$$G^*(e, 1; x, a) = 2^{-Q}(|x| + a)^{-Q} G(e, \widetilde{\beta}; y, b), \tag{6.7}$$

where $\widetilde{\beta} = 2^{-1}(|x| + a)^{-1}$. If $(x, a) \in S_3$ then the corresponding $(y, b) \in V_{1/2}$ has the property $|y| \leq b$. Indeed, $|x| = t|y| \leq a = tb$. This, together with $|y| + b = 1/2$, implies that $b \in [1/4, 1/2]$.

For $(x, a) \in S_3$ we have $\widetilde{\beta} \leq (2 + 2\delta)^{-1} := 1/2 - \widetilde{\delta}$. Indeed, this is clear if $a \geq 1 + \delta$. But if $a < 1 + \delta$ then $|x| \geq \delta$. Thus by (6.4), using Theorem 4.2 and Lemma 5.11 if $b \geq (1 - \widetilde{\delta})/2$, or Theorem 4.1 and Lemma 5.13 if $b \leq (1 - \widetilde{\delta})/2$ (then $|y| \geq \widetilde{\delta}/2$), we find that there exists a constant C such that $G^*(e, \widetilde{\beta}; x, a)$ in (6.7) is less than or equal to C. By Lemma 5.21 of [DHU], $G^*(e, \widetilde{\beta}; x, a)$ is also greater than or equal to C^{-1}. Thus by (6.7),

$$C^{-1} 2^{-Q}(|x| + a)^{-Q} \leq \mathcal{G}(x, a) \leq C 2^{-Q}(|x| + a)^{-Q}, \qquad (x, a) \in S_3.$$

Since $a \leq |x| + a \leq 2a$ for $(x, a) \in S_3$, the proof is complete. ∎

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