COLLOQUIUM MATHEMATICUM

VOL. 88 2001 NO. 1

AN ADDENDUM AND CORRIGENDUM TO "ALMOST FREE SPLITTERS"

(COLLOQ. MATH. 81 (1999), 193–221)

BY

RÜDIGER GÖBEL (Essen)
and SAHARON SHELAH (Jerusalem and New Brunswick, NJ)

Abstract. Let R be a subring of the rational numbers $\mathbb{Q}$. We recall from [3] that an R-module G is a splitter if $\mathrm{Ext}^1_R(G,G)=0$. In this note we correct the statement of Main Theorem 1.5 in [3] and discuss the existence of non-free splitters of cardinality $\aleph_1$ under the negation of the special continuum hypothesis CH.

1. Introduction. We refer to [3] for definitions and all details. Recall that an R-module G is a *splitter* if $\mathrm{Ext}^1_R(G,G)=0$. We may also assume that splitters are torsion-free abelian groups; see [3, p. 194]. Hence the *nucleus* R of a torsion-free abelian group $G\neq 0$ is defined to be the (now fixed) subring R of the field of rational numbers $\mathbb{Q}$ generated by all $1/p$ (p any prime) for which G is *p-divisible*, i.e. $pG=G$. Recall that G is an *$\aleph_1$-free R-module* if any countably generated R-submodule is free.

Under the special continuum hypothesis CH, any $\aleph_1$-free splitter of cardinality $\aleph_1$ is free over its nucleus as shown in [3]. Generally these modules are very close to being free but may not be free in particular models of set theory as explained below. This modification of a statement from [3] is due to an incomplete proof (noticed thanks to Paul Eklof) in [3, first paragraph on p. 207]. Assuming the negation of CH, it is shown in [6] that under Martin's axiom (MA) these splitters are free indeed. However there are models of set theory having non-free $\aleph_1$-free splitters of cardinality $\aleph_1$.

2. Reductions from [3] and the existence of non-free splitters. One of the main results in [3] needs CH and now should read as follows.

THEOREM 2.1. *Under the assumption of the special continuum hypothesis CH any $\aleph_1$-free splitter of cardinality $\aleph_1$ is free over its nucleus.*

2000 *Mathematics Subject Classification*: Primary 13C05, 18E40, 18G05, 20K20, 20K35, 20K40; Secondary 13D30, 18G25, 20K25, 20K30, 13C10.

Key words and phrases: self-splitting modules, criteria for freeness of modules.

We must recall that G is of *type* I if there is an $\aleph_1$-filtration $G = \bigcup_{\alpha<\omega_1} G_\alpha$ of pure, free R-submodules such that $G_{\alpha+1}/G_\alpha$ are minimal non-free for all $\alpha > 0$. Also recall that a non-free, torsion-free R-module of finite rank is *minimal non-free* if all submodules of smaller rank are free. Modules of type II and III are defined in [3]. The following statements are proved in [3, Sections 3, 5, 6 and 7]

(i) *Any $\aleph_1$-free R-module G of cardinality $\aleph_1$ is either of type I, II or III.*

(ii) *Modules of type II or III are splitters if and only if they are free over the nucleus R (hence of type II).*

(iii) *Modules of type I are not splitters if we assume CH.*

So Theorem 2.1 follows from these statements. In order to characterize $\aleph_1$-free splitters it remains to assume the negation of the special continuum hypothesis, hence $\aleph_1 < 2^{\aleph_0}$ and to consider modules G of type I. In this case it is not needed to assume $\aleph_1$-freeness. In fact this is a consequence of an easy extension of a result of Hausen [4] (see also [2]). It also remains to consider splitters satisfying the following hypothesis:

- *Let G be a splitter of type I with an $\aleph_1$-filtration $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ of pure and free R-submodules G_α such that* $\operatorname{nuc} G_\alpha = R$ *for all $\alpha \in \omega_1$.*

(See [3, p. 203].) For the remaining arguments let us assume that G is such a fixed R-module which is not free.

The next Proposition 2.3 depends on a condition about solving linear equations, which is closely related to the answer to the Whitehead problem.

Definition 2.2. If X is an R-submodule of G, then we consider the set $\mathfrak{W} = \mathfrak{W}(X)$ of all finite sequences $\overline{a} = (a_0, a_1, \dots, a_n)$ such that:

(i) $a_i \in G$ $(i \le n)$.

(ii) $\bigoplus_{i<n}(a_i + X)R$ is pure in G/X.

(iii) $\langle (a_i + X)R : i \le n\rangle_*$ is not a free R-module in G/X.

If $G_{\bar{a}}$ is the pure submodule of G (purely) generated by $\{X,\ a_iR : i \le n\}$, that is to say,

$$G_{\bar{a}} = \langle X,\ a_iR : i \le n\rangle_* \subseteq G,$$

then it is clear that $G_{\bar{a}}/X$ is a minimal non-free R-module of rank n. Hence there are natural numbers $p_{\bar{a}m}$ which are not units of R and elements $k_{\bar{a}im} \in R$ $(i < n)$ and $g_{\bar{a}m} \in G_{\bar{a}}$ such that

$$y_{\bar{a}m+1}p_{\bar{a}m} = y_{\bar{a}m} + \sum_{i<n} a_i k_{\bar{a}im} + g_{\bar{a}m} \qquad (m \in \omega). \tag{2.1}$$

If we choose a sequence $\overline{z} = (z_m : m \in \omega) \subset G$, then the $\overline{z}$-inhomogeneous counterpart of (2.1) is by definition the system of equations

$$Y_{m+1} p_{\bar{a}m} \equiv Y_m + \sum_{i<n} X_i k_{\bar{a}im} + z_m \bmod X \qquad (m \in \omega). \tag{2.2}$$

We say that $\overline{a} \in \mathfrak{W}$ is *contra-Whitehead* if (2.2) has no solutions y_m $(m \in \omega)$ in G (hence in $G_{\bar{a}}$) for some $\overline{z}$ and $X_i = a_i$. Otherwise we say that $\overline{a}$ is *pro-Whitehead*. In this terminology, the following was shown in [3, Proposition 4.4].

PROPOSITION 2.3. *If* $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ *and*

$$S = \{\alpha \in \omega_1 : \exists \overline{a} \in \mathfrak{W}(G_\alpha),\ \overline{a} \textit{ is contra-Whitehead}\}$$

is stationary in ω_1, *then* G *is not a splitter.*

By the above assumption on G, the set S is not stationary in ω_1 and hence we may assume that all modules G_α are pro-Whitehead in G.

This case is covered by our next result, which needs the extra assumption that $\operatorname{nuc}(G/X) = R$.

THEOREM 2.4. *Let* G *be a splitter of cardinality* $< 2^{\aleph_0}$ *with* $\operatorname{nuc} G = R$. *If* X *is a pure*, *countable* R*-submodule of* G *with* $\operatorname{nuc}(G/X) = R$ *which is also pro-Whitehead in* G, *then* G/X *is an* $\aleph_1$*-free* R*-module.*

The proof given in [3, p. 206 (first case)] applies.

Let $C = \{\alpha \in \omega_1 : \operatorname{nuc}(G/G_\alpha) = R\}$. If $C = \emptyset$, then $G_{\alpha+1}/G_\alpha$ is free by Theorem 2.4 and the last assumption on G, hence G is a free R-module. But we assumed above that G is not free, hence $C \neq \emptyset$ and there is an ordinal $\alpha_0 < \omega_1$ such that $C = (\alpha_0, \omega_1)$ is an interval, a final segment of ω_1. We get the following

COROLLARY 2.5. *Any non-free splitter of type I and cardinality at most* $\aleph_1 < 2^{\aleph_0}$ *has a countable* R*-submodule* X *such that* $\operatorname{nuc}(G/X)$ *is strictly larger than* R.

If R is a local ring then by Corollary 2.5 the module G is free-by-free, an extension of a countable free R-module by a divisible module, that is, a free module over the field $\mathbb{Q}$ of rational numbers.

It remains to consider splitters as in Corollary 2.5 under $\aleph_1 < 2^{\aleph_0}$:

If we assume now, in addition (to negation of CH), Martin's axiom, then $\aleph_1$-free splitters of cardinality $\aleph_1$ are free, as shown by Shelah [6]. On the other hand there is a model of set theory with non-free $\aleph_1$-free splitters of cardinality $\aleph_1$. Hence freeness of ($\aleph_1$-free) splitters (of type I) cannot be decided in ordinary set theory ZFC, even under Martin's axiom MA.

REFERENCES

[1] P. Eklof and A. Mekler, *Almost Free Modules*, *Set-Theoretic Methods*, North-Holland, Amsterdam, 1990.

[2] R. Göbel and S. Shelah, *Cotorsion theories and splitters*, Trans. Amer. Math. Soc. 352 (2000), 5337–5379.

[3] —, —, *Almost free splitters*, Colloq. Math. 81 (1999), 193–221.

[4] J. Hausen, *Automorphismen gesättigte Klassen abzählbaren abelscher Gruppen*, in: Studies on Abelian Groups, Springer, Berlin, 1968, 147–181.

[5] P. Schultz, *Self-splitting groups*, preprint, Univ. of Western Australia at Perth, 1980.

[6] S. Shelah, *On uniformization and splitters*, in preparation.

Fachbereich 6
Mathematik und Informatik
Universität Essen
45117 Essen, Germany
E-mail: r.goebel@uni-essen.de

Department of Mathematics
Hebrew University
91904 Jerusalem, Israel
E-mail: shelah@math.huji.ac.il

Department of Mathematics
Rutgers University
New Brunswick, NJ 08854, U.S.A.

Received 4 July 2000 (3952)