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QUASI-EINSTEIN HYPERSURFACES IN SEMI-RIEMANNIAN SPACE FORMS

BY

RYSZARD DESZCZ (Wrocław), MARIAN HOTLOŚ (Wrocław) and
ZERRIN ŞENTÜRK (Istanbul)

Dedicated to Professor Dr. Radu Rosca on his 90th birthday

Abstract. We investigate curvature properties of hypersurfaces of a semi-Riemannian space form satisfying $R \cdot C = LQ(S, C)$, which is a curvature condition of pseudosymmetry type. We prove that under some additional assumptions the ambient space of such hypersurfaces must be semi-Euclidean and that they are quasi-Einstein Ricci-semisymmetric manifolds.

1. Introduction. A semi-Riemannian manifold (M, g), $n = \dim M \geq 3$, is said to be an *Einstein manifold* if $S = (\kappa/n)g$ on M, where S and κ denote the Ricci tensor and the scalar curvature of (M, g), respectively. The manifold (M, g), $n \geq 3$, is called a *quasi-Einstein manifold* if at every point x of M its Ricci tensor S has the form

$$S = \alpha g + \beta w \otimes w, \qquad w \in T_x^* M,\ \alpha, \beta \in \mathbb{R}. \tag{1}$$

We refer to [11] for a review of recent results on quasi-Einstein hypersurfaces.

Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, with signature $(s, n+1-s)$. We denote by U_H the subset of M consisting of all points x at which the transformation $\mathcal{A}^2$ is not a linear combination of the shape operator $\mathcal{A}$ and the identity transformation Id at x. If (1) is satisfied at a point $x \in M - U_H$ then, at x, the Weyl tensor C of M vanishes or the Ricci tensor S is proportional to the metric tensor ([10], Lemma 4.1(iii); see also Proposition 3.3(iii) of the present paper). Therefore we restrict our considerations to the subset $U_H \subset M$.

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Quasi-Einstein hypersurfaces in semi-Euclidean spaces $\mathbb{E}_s^{n+1}$, $n \geq 4$, were investigated in [10]. We have the following

THEOREM 1.1. *Let M be a quasi-Einstein hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$, and let* (1) *be satisfied on $U_H \subset M$.*

(i) ([10], Theorem 5.1) *On U_H any of the following three conditions is equivalent to each other*:

$$\begin{aligned} &\text{(a) } R \cdot S = 0, \qquad \text{(b) } \mathcal{A}^3 = \operatorname{tr}(\mathcal{A})\mathcal{A}^2 - \frac{\varepsilon\kappa}{n-1}\mathcal{A}, \quad \varepsilon = \pm 1, \\ &\text{(c) } \mathcal{A}(W) = 0, \end{aligned} \tag{2}$$

where the vector W is related to w by $g(W,X) = w(X)$ for all $X \in T_xM$ and w and α are defined by (1).

(ii) ([10], Corollary 5.2) *If at every point $x \in U_H$ one of the conditions* (2)(a), (2)(b) *or* (2)(c) *is satisfied then the following relations hold on U_H*:

$$\begin{aligned} &\text{(a) } \operatorname{rank}\left(S - \frac{\kappa}{n-1}g\right) = 1, \qquad \text{(b) } R \cdot C = Q(S,C), \\ &\text{(c) } C \cdot S = 0. \end{aligned} \tag{3}$$

It is obvious that every semi-Riemannian semisymmetric as well as conformally flat manifold (M,g), $n \geq 4$, satisfies the following condition of pseudosymmetry type ([8]) at every point of M:

$(*)$ the tensors $R \cdot C$ and $Q(S,C)$ are linearly dependent.

Semi-Riemannian manifolds satisfying $(*)$ were recently investigated in [8] and [9]. The condition $(*)$ is equivalent to

$$R \cdot C = LQ(S,C) \tag{4}$$

on the set $U = \{x \in M \mid Q(S,C) \neq 0 \text{ at } x\}$, where L is some function on U. Evidently, (3)(b) is (4) with $L = \text{const} = 1$. Examples of nonsemisymmetric manifolds satisfying $(*)$ are given in [8]. We denote by U_L the set of all points of U at which L is nonzero.

In this paper we consider hypersurfaces M isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, satisfying $(*)$. In Section 2 we fix notations and review the curvature conditions of pseudosymmetry type. In Section 3 we present preliminary results. Among other things we prove (Proposition 3.12) that if $(*)$ holds on a hypersurface M of $N_s^{n+1}(c)$, $n \geq 4$, and $U_H \cap U_L$ is nonempty then the scalar curvature $\widetilde{\kappa}$ of $N_s^{n+1}(c)$ vanishes, i.e. the ambient space is a semi-Euclidean space. Finally, in the last section we present our main results (Theorem 4.3).

In [5] it was shown that if at a point $x \in U_H$ of a quasi-Einstein hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$, the scalar curvature κ of M is nonzero, (1) holds and either (2)(a), (2)(b) or (2)(c) is satisfied then the tensor $R \cdot R$ is nonzero at x.

In our opinion, the last result and Theorems 1.1 and 4.3 of the present paper play an important role in the problem of equivalence of Ricci-semisymmetry ($R\cdot S = 0$) and semisymmetry ($R\cdot R = 0$) on hypersurfaces of semi-Euclidean spaces (see [9] and references therein).

2. Preliminaries. Let (M, g), $n \geq 3$, be a connected semi-Riemannian manifold of class C^∞. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Next, we define the endomorphisms $\mathcal{R}(X, Y)$, $\mathcal{C}(X, Y)$ and $X \wedge_A Y$ of $\Xi(M)$ by

$$\begin{aligned}
(X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y,\\
\mathcal{R}(X, Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,\\
\mathcal{C}(X, Y)Z &= \mathcal{R}(X, Y)Z\\
&\qquad - \frac{1}{n-2}\Big(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\Big)Z,
\end{aligned}$$

where A is a symmetric $(0, 2)$-tensor and $X, Y, Z \in \Xi(M)$. The Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0, 4)$-tensor G of (M, g) are defined by

$$\begin{aligned}
R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4),\\
C(X_1, X_2, X_3, X_4) &= g(\mathcal{C}(X_1, X_2)X_3, X_4),\\
G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4).
\end{aligned}$$

For a $(0, k)$-tensor field T, $k \geq 1$, and a $(0, 2)$-tensor field A on (M, g) we define the tensors $R \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}
(R \cdot T)(X_1, \ldots, X_k; X, Y) = &- T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k)\\
&- \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),\\
Q(A, T)(X_1, \ldots, X_k; X, Y) = &- T((X \wedge_A Y)X_1, X_2, \ldots, X_k)\\
&- \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).
\end{aligned}$$

In the same manner as $R\cdot S$ we define the $(0, 4)$-tensor $C\cdot S$. For $(0, 2)$-tensors A and B we define their Kulkarni–Nomizu product $A \wedge B$ by

$$\begin{aligned}
(A \wedge B)(X_1, X_2; X, Y) = A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y)\\
- A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).
\end{aligned}$$

We note that if $A = B$ then $\overline{A} = \frac{1}{2} A \wedge A$, where the $(0, 4)$-tensor $\overline{A}$ is defined by

$$\overline{A}(X_1, X_2, X_3, X_4) = A(X_1, X_4)A(X_2, X_3) - A(X_1, X_3)A(X_2, X_4).$$

The Weyl tensor C can also be represented in the form

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G. \tag{5}$$

Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{W; x^k\}$. We denote by $g_{ij},\ R_{hijk},\ S_{ij},\ S_i^j = g^{jk} S_{ik},\ S^2_{ij} = S_i{}^p S_{pj},\ G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik}$ and

$$\begin{aligned} C_{hijk} = R_{hijk} &- \frac{1}{n-2} (g_{hk} S_{ij} - g_{hj} S_{ik} + g_{ij} S_{hk} - g_{ik} S_{hj}) \\ &+ \frac{\kappa}{(n-2)(n-1)} G_{hijk} \end{aligned} \tag{6}$$

the local components of the tensors g, R, S, $\mathcal{S}$, S^2, G and C, respectively. In particular, for (4) we have $(R \cdot C)_{hijklm} = LQ(S, C)_{hijklm}$, i.e.

$$\begin{aligned} g^{pq}(C_{pijk} R_{qhlm} + C_{hpjk} R_{qilm} + C_{hipk} R_{qjlm} + C_{hijp} R_{qklm}) \\ = L(S_{hl} C_{mijk} + S_{il} C_{hmjk} + S_{jl} C_{himk} + S_{kl} C_{hijm} \\ - S_{hm} C_{lijk} - S_{im} C_{hljk} - S_{jm} C_{hilk} - S_{km} C_{hijl}). \end{aligned} \tag{7}$$

A profound investigation of properties of semisymmetric manifolds (with $R \cdot R = 0$) gave rise to another generalization: the pseudosymmetric manifolds. A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* ([2], [15]) if

$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent

at every point of M. This is equivalent to $R \cdot R = L_R Q(g, R)$ on the set $U_R = \left\{x \in M \,\middle|\, R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\right\}$, where L_R is some function on U_R. Evidently, every semi-Riemannian semisymmetric manifold is pseudosymmetric.

It is easy to see that if $(*)_1$ holds on a semi-Riemannian manifold (M, g), then

$(*)_2$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent

at every point of M. The converse is not true ([2], [15]). A semi-Riemannian manifold (M, g) is called *Ricci-pseudosymmetric* if $(*)_2$ holds at every point of M.

The condition $(*)_2$ is equivalent to $R \cdot S = L_S Q(g, S)$ on the set $U_S = \{x \in M \,|\, S \neq (\kappa/n) g \text{ at } x\}$, where L_S is some function on U_S. A semi-Riemannian manifold (M, g) satisfying $R \cdot S = 0$ is called *Ricci-semisymmetric*. In general, Ricci-semisymmetric manifolds are not semisymmetric. However, under some additional assumptions the conditions $R \cdot S = 0$ and $R \cdot R = 0$ are equivalent (see e.g. [9] and references therein).

As shown in [12] (Proposition 3.1), at every point of a hypersurface M in $N_s^{n+1}(c)$ the following condition is fulfilled:

$(*)_3$ the tensors $R \cdot R - Q(S,R)$ and $Q(g,C)$ are linearly dependent.

More precisely,

$$R \cdot R - Q(S,R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g,C) \tag{8}$$

on M, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space. Evidently, if the ambient space is a semi-Euclidean space $\mathbb{E}_s^{n+1}$ then (8) reduces to

$$R \cdot R = Q(S,R). \tag{9}$$

In [1] (Theorem 3.2) it was shown that every quasi-Einstein conformally flat manifold is pseudosymmetric and satisfies (9). Note also that every pseudosymmetric Einstein manifold satisfies $(*)_3$. Pseudosymmetric manifolds satisfying $(*)_3$ were investigated in [7].

Semi-Riemannian manifolds fulfilling $(*)_1$, $(*)_2$, $(*)_3$, $(*)$ or other conditions of this kind are called *manifolds of pseudosymmetry type* ([2], [15]). Hypersurfaces satisfying curvature conditions of pseudosymmetry type (pseudosymmetry type hypersurfaces) were studied in many papers (see e.g. [3], [6], [12] and [13]).

Using the above definitions we can prove the following

Proposition 2.1 ([10], Lemma 3.1). *Let A and B be symmetric $(0,2)$-tensors on a semi-Riemannian manifold (M,g), $n \geq 3$. Then $Q(A, A \wedge B) = -Q(B, \overline{A})$ on M. In particular, $Q(g, g \wedge S) = -Q(S,G)$ and $Q(S, g \wedge S) = Q(S, S \wedge g) = -Q(g, \overline{S})$.*

As an immediate consequence of the above result and (5) we obtain the following identity which holds on every semi-Riemannian manifold:

$$Q(g,C) = Q(g,R) + \frac{1}{n-2} Q(S,G). \tag{10}$$

Proposition 2.2 ([6], Proposition 3.1(iii)). *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold satisfying the following three equalities at a point $x \in U_S \subset M$:*

$$\text{(a) } R \cdot S = L_S Q(g,S), \qquad \text{(b) } R \cdot R = Q(S,R) + L Q(g,C), \tag{11}$$

$$S = \frac{\kappa}{n-1} g + \beta w \otimes w, \qquad w \in T_x^*(M),\ \beta \in \mathbb{R}. \tag{12}$$

Then at x we have

$$R \cdot C = Q(S,C) + L Q(g,R) + \frac{1}{n-2}(L_S + L) Q(S,G). \tag{13}$$

As a consequence we have

Proposition 2.3 ([10], Corollary 3.1). *Let (M,g), $n \geq 4$, be a semi-Riemannian Ricci-semisymmetric manifold satisfying the following three equalities at every point of M: $\kappa = 0$, $\mathrm{rank}(S) = 1$ and $R \cdot R = Q(S,R)$. Then $R \cdot C = Q(S,C)$ on M.*

We also have the following identity on every quasi-Einstein manifold.

Proposition 2.4 ([10], Proposition 3.1). *On every semi-Riemannian quasi-Einstein manifold (M,g), $n \geq 4$, the following identity is satisfied:*

$$C \cdot S = R \cdot S + \beta\Big(\alpha - \frac{\kappa}{n-1}\Big)Q(g, w \otimes w). \tag{14}$$

To end this section we present a result related to semi-Riemannian manifolds satisfying $(*)$.

Proposition 2.5 ([8], Theorem 3.1). *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold satisfying $Q(S,C) = 0$ at a point $x \in M$. If $S \neq 0$ and $C \neq 0$ at x, then $R \cdot R = \frac{\kappa}{n-1} Q(g,R)$ at x.*

It can be shown that on every semi-Riemannian manifold (M,g), $n \geq 4$, we have

$$\begin{aligned}(n-2)(R \cdot C - C \cdot R)_{hijklm} - Q\Big(S - \frac{\kappa}{n-1} g, R\Big)_{hijklm} &= g_{hl}A_{mijk} \\ - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} + g_{jl}A_{mkhi} &- g_{jm}A_{lkhi} - g_{kl}A_{mjhi} \\ + g_{km}A_{ljhi} - g_{ij}(A_{hklm} + A_{khlm}) &- g_{hk}(A_{ijlm} + A_{jilm}) \\ + g_{ik}(A_{hjlm} + A_{jhlm}) + g_{hj}(A_{iklm} + A_{kilm}),&\end{aligned}$$

where the $(0,4)$-tensor A is defined by $A_{hijk} = S_h{}^s R_{sijk}$. As a consequence of this and the identity $(R \cdot S)_{hijk} = S_h{}^s R_{sijk} + S_i{}^s R_{shjk}$ we have the following

Proposition 2.6. *On every Ricci-semisymmetric semi-Riemannian manifold (M,g), $n \geq 4$, the following identity is satisfied:*

$$\begin{aligned}(n-2)(R \cdot C - C \cdot R)_{hijklm} &- Q\Big(S - \frac{\kappa}{n-1} g, R\Big)_{hijklm} \\ &= g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}.\end{aligned} \tag{15}$$

3. Hypersurfaces. Let M, $n = \dim M \geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, \widetilde{g})$. We denote by g the metric tensor of M, induced from the metric tensor $\widetilde{g}$. Further, we denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections of $\widetilde{g}$ and g, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = \widetilde{g}(\xi,\xi) = \pm 1$. We can write the *Gauss formula* and the *Weingarten* *formula* of M in N in the following form:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X,Y)\xi, \qquad \widetilde{\nabla}_X \xi = -\mathcal{A}(X),$$

respectively, where X, Y are vector fields tangent to M, H is the *second fundamental tensor* of M in N, $\mathcal{A}$ is the *shape operator* of M in N and $H^k(X,Y) = g(\mathcal{A}^k(X), Y)$, $\mathrm{tr}(H^k) = \mathrm{tr}(\mathcal{A}^k)$, $k \geq 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and $\widetilde{R}$ the Riemann–Christoffel curvature tensors of M and N, respectively. We denote by U_H the set of all points $x \in M$ at which $\mathcal{A}^2$ is not a linear combination of $\mathcal{A}$ and Id. Note that $U_H \subset U_S$. The *Gauss equation* of M in N has the form

$$R(X_1, X_2, X_3, X_4) = \widetilde{R}(X_1, X_2, X_3, X_4) + \varepsilon \overline{H}(X_1, X_2, X_3, X_4), \tag{16}$$

where $X_1, \ldots, X_4$ are vector fields tangent to M and $\overline{H} = \frac{1}{2} H \wedge H$. Let $x^r = x^r(y^h)$ be the local parametric expression of M in $(N, \widetilde{g})$, where y^h and x^r are local coordinates of M and N, respectively, and $h, i, j, k, l, m, p, q \in \{1, \ldots, n\}$ and $r, s, t, u \in \{1, \ldots, n+1\}$. Now we can write (16) in the form

$$R_{hijk} = \widetilde{R}_{rstu} B_h^{\ r} B_i^{\ s} B_j^{\ t} B_k^{\ u} + \varepsilon \overline{H}_{hijk}, \qquad B_h^{\ r} = \frac{\partial x^r}{\partial y^k}, \tag{17}$$

where $\widetilde{R}_{rstu}$, R_{hijk}, $\overline{H}_{hijk} = H_{hk} H_{ij} - H_{hj} H_{ik}$ and H_{hk} are the local components of the tensors $\widetilde{R}, R, \overline{H}$ and H, respectively.

If the ambient space $(N, \widetilde{g})$ is conformally flat then the Weyl conformal curvature tensor of M satisfies (cf. [12])

$$C = \mu G + \varepsilon \overline{H} - \varepsilon \frac{\mathrm{tr}(H)}{n-2} g \wedge H + \varepsilon \frac{1}{n-2} g \wedge H^2, \tag{18}$$

$$\mu = \frac{1}{(n-2)(n-1)} (\kappa - 2\widetilde{S}_{rs} B_e^r B_f^s g^{ef}) + \frac{2\widetilde{\kappa}}{n(n-2)}. \tag{19}$$

Using (18) we can easily check that on every hypersurface M in a conformally flat manifold $(N, \widetilde{g})$ we have:

$$\begin{aligned} C \cdot H = \frac{\varepsilon}{n-2} (Q(g, H^3) &+ (n-3) Q(H, H^2) \\ &- \mathrm{tr}(H) Q(g, H^2)) + \mu Q(g, H), \end{aligned} \tag{20}$$

$$\begin{aligned} C \cdot H^2 = \mu Q(g, H^2) + \varepsilon \Big(Q(H, H^3) &+ \frac{1}{n-2} (-\mathrm{tr}(H) Q(g, H^3) \\ &+ Q(g, H^4) - \mathrm{tr}(H) Q(H, H^2)) \Big). \end{aligned} \tag{21}$$

From now on we will assume that M is a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$. Then (17) turns into

$$R_{hijk} = \varepsilon \overline{H}_{hijk} + \frac{\widetilde{\kappa}}{n(n+1)} G_{hijk}, \tag{22}$$

from which, by contraction with g^{ij} and transvection with H^r_p, we easily get

$$S_{hk} = \varepsilon(\mathrm{tr}(H)H_{hk} - H^2_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} g_{hk}, \tag{23}$$

$$H_{hr}S_k{}^r = \varepsilon(\mathrm{tr}(H)H^2_{hk} - H^3_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} H_{hk}, \tag{24}$$

$$H^2_{hr}S_k{}^r = \varepsilon(\mathrm{tr}(H)H^3_{hk} - H^4_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} H^2_{hk}. \tag{25}$$

Moreover, contracting (23) with g^{hk} we obtain

$$\kappa = \varepsilon((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1}. \tag{26}$$

We also note that the following identity holds on M ([3], eq. (22)):

$$R \cdot R - \frac{\widetilde{\kappa}}{n(n+1)} Q(g, R) = -Q(H^2, \overline{H}). \tag{27}$$

We quote the following statements.

Proposition 3.1. *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 3$.*

(i) ([3], Theorem 3.1) *If at a point x of M the tensor H has the form*

$$H = \beta v \otimes v + \gamma w \otimes w, \qquad v, w \in T^*_x(M),\ \beta, \gamma \in \mathbb{R}, \tag{28}$$

then at x we have

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, R). \tag{29}$$

(ii) ([13], Lemma 2.1) *If at a point x of M the tensor H satisfies*

$$H^2 = \alpha H + \beta g, \qquad \alpha, \beta \in \mathbb{R}, \tag{30}$$

then at x we have

$$R \cdot R = \left(\frac{\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta\right) Q(g, R). \tag{31}$$

Proposition 3.2 ([4], Theorem 5.1). *A hypersurface M in $N^{n+1}_s(c)$, $n \geq 4$, is pseudosymmetric if and only if at every point of M either* (28) *or* (30) *is satisfied.*

Proposition 3.3 ([10], Lemma 4.1). *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$.*

(i) *If $S = (\kappa/n)g$ at $x \in M$ then $x \in M - U_H$.*
(ii) *If $C = 0$ at $x \in M$ then $x \in M - U_H$.*
(iii) *If* (1) *is satisfied at $x \in M - U_H$ then $S = (\kappa/n)g$ or $C = 0$ at x.*
(iv) *If $H = \widetilde{\alpha} g + \widetilde{\beta} w \otimes w$ at $x \in M$ then* (1) *holds at x, where $w \in T^*_x M$, $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}$.*

Proposition 3.3(iv) and Theorem 4.1 of [12] yield

COROLLARY 3.1. *On every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have $U_H \subset U_C$.*

PROPOSITION 3.4 ([10], Lemma 4.2). *If M is a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying* (2)(b) *then $C \cdot S = 0$ on M.*

PROPOSITION 3.5 ([10], Proposition 4.1). *If M is a Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then on $U_H \subset M$ we have*

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S) \quad \text{and} \quad H^3 = \operatorname{tr}(H) H^2 + \lambda H,$$

where λ is some function on U_H.

LEMMA 3.1. *If M is a Ricci-semisymmetric hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$, then on $U_H \subset M$ we have*

$$(n-2)(R \cdot C - C \cdot R) = Q\left(S - \left(\varepsilon\lambda + \frac{\kappa}{n-1}\right) g, R\right), \tag{32}$$

where λ is defined in Proposition 3.5.

Proof. By making use of Proposition 3.5, (24) reduces to

$$H_{hr} S_k{}^r = -\varepsilon\lambda H_{hk}. \tag{33}$$

Transvecting now (22) with $S_l{}^h$ and using (33) and (22) we obtain

$$A_{lijk} = -\varepsilon\lambda R_{lijk}. \tag{34}$$

Applying this in (15) we obtain (32). Our lemma is thus proved.

We now present some applications of Proposition 2.2.

PROPOSITION 3.6 ([6], Proposition 5.1). *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If*

$$R \cdot S = L_S Q(g, S), \qquad S = \frac{\kappa}{n-1} g + \beta w \otimes w, \qquad \beta \in \mathbb{R},\ w \in T_x^*(M),$$

at a point $x \in U_S \subset M$, then at x we have

$$R \cdot C = Q(S, C) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, R) + \frac{1}{n-2}\left(L_S - \frac{(n-2)\widetilde{\kappa}}{n(n+1)}\right) Q(S, G). \tag{35}$$

In particular, when $x \in U_H$, Proposition 3.5 and (35) imply

PROPOSITION 3.7 ([10], Theorem 4.2). *Let M be a Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. If $S = \frac{\kappa}{n-1} g + \beta w \otimes w$, $\beta \in \mathbb{R}$, $w \in T_x^* M$, at every point x of U_H then on U_H we have*

$$R \cdot C = Q(S, C) - \frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, R) - \frac{(n-3)\widetilde{\kappa}}{(n-2)n(n+1)} Q(S, G). \tag{36}$$

The last result, together with Proposition 2.4, leads to

PROPOSITION 3.8 ([10], Corollary 4.1). *Let M be a Ricci-semisymmetric hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$. If $S = \frac{\kappa}{n-1} g + \beta w \otimes w$, $\beta \in \mathbb{R}$, $w \in T^*_x M$, at every point x of U_H then on U_H we have $R \cdot C = Q(S, C)$ and $C \cdot S = 0$.*

Next, we prove the following four propositions which will be used later.

PROPOSITION 3.9. *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$, satisfying $(*)$. Then on $U_L \subset M$ we have*

$$C \cdot S = 0, \tag{37}$$

$$R \cdot S = \frac{1}{n-2} Q\left(g, S^2 - \frac{\kappa}{n-1} S\right), \tag{38}$$

$$\frac{n}{n-2} H^4 = \frac{n+2}{n-2} \operatorname{tr}(H) H^3 + \widetilde{\alpha}_2 H^2 + \widetilde{\alpha}_1 H + \widetilde{\alpha}_0 g, \tag{39}$$

where

$$\begin{aligned} \widetilde{\alpha}_2 = &-\frac{2}{n-2} (\operatorname{tr}(H))^2 - \frac{\varepsilon n \kappa}{(n-2)(n-1)} + \frac{\varepsilon n \widetilde{\kappa}}{(n-2)(n+1)}, \\ \widetilde{\alpha}_1 = &-\frac{\varepsilon n \widetilde{\kappa}}{(n-2)(n+1)} \operatorname{tr}(H) - \operatorname{tr}(H) \operatorname{tr}(H^2) + \operatorname{tr}(H^3) \\ &+ \frac{\varepsilon n \kappa}{(n-2)(n-1)} \operatorname{tr}(H), \\ \widetilde{\alpha}_0 = &\frac{\varepsilon \widetilde{\kappa}}{(n-2)(n+1)} (\operatorname{tr}(H))^2 - \frac{\varepsilon \widetilde{\kappa}}{(n-2)(n+1)} \operatorname{tr}(H^2) \\ &- \frac{\kappa^2}{(n-2)(n-1)} + \frac{1}{n-2} \operatorname{tr}(H^4) - \frac{2}{n-2} \operatorname{tr}(H) \operatorname{tr}(H^3) \\ &+ \frac{1}{n-2} \operatorname{tr}(H^2)(\operatorname{tr}(H))^2 + \frac{\kappa \widetilde{\kappa}}{(n-2)(n+1)}. \end{aligned} \tag{40}$$

Proof. Let W be the $(0,4)$-tensor with local components W_{hijk} defined by

$$W_{hijk} = S^p_h C_{pijk} + S^p_j C_{pikh} + S^p_k C_{pihj}. \tag{41}$$

It is easy to verify that on every semi-Riemannian manifold we have

$$W_{hijk} = S^p_h R_{pijk} + S^p_j R_{pikh} + S^p_k R_{pihj}.$$

Applying the Gauss equation (22) we get

$$\begin{aligned} \varepsilon W_{hijk} = &S^p_h H_{pk} H_{ij} - S^p_h H_{pj} H_{ik} + S^p_j H_{ph} H_{ik} - S^p_j H_{pk} H_{ih} \\ &+ S^p_k H_{pj} H_{ih} - S^p_k H_{ph} H_{ij}. \end{aligned} \tag{42}$$

Further, (24) implies $S^p_h H_{pk} = S^p_k H_{ph}$, which means that (42) reduces to

$$S^p_h C_{pijk} + S^p_j C_{pikh} + S^p_k C_{pihj} = 0. \tag{43}$$

On the other hand, contracting (7) with g^{ij} we get $Lg^{ij}Q(S,C)_{hijklm} = 0$, and since L is nonzero at every point of U_L, we obtain

$$S^p_m C_{pklh} + S^p_l C_{pkhm} + S^p_m C_{phlk} + S^p_l C_{phkm} = 0.$$

Applying (43) we find

$$(C \cdot S)_{hklm} = S^p_h C_{pklm} + S^p_k C_{phlm} = 0, \tag{44}$$

i.e. the equality (37). Hence, applying (6) we get (38). Further, contracting (44) with g^{hm} we obtain $S^{hk}C_{hijk} = 0$, which, by (6), turns into

$$\frac{2}{n-2}S^2_{ij} + S^{hk}R_{hijk} - \frac{n\kappa}{(n-2)(n-1)}S_{ij} + \frac{1}{n-2}\left(\frac{\kappa^2}{n-1} - \operatorname{tr}(S^2)\right)g_{ij} = 0.$$

Applying now (22)–(25) we find (39), completing the proof.

Proposition 3.10. *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$, satisfying $(*)$. Then on $U_L \subset M$ we have*

$$\frac{n}{n-2}H^4 = \frac{n+2}{n-2}\operatorname{tr}(H)H^3 + \alpha_2 H^2 + \alpha_1 H + \alpha_0 g, \tag{45}$$

$$\operatorname{tr}(H)\operatorname{tr}(H^3) = 0, \tag{46}$$

where

$$\begin{aligned} \alpha_2 &= -\frac{2}{n-2}(\operatorname{tr}(H))^2 - \varepsilon n\mu, \\ \alpha_1 &= \varepsilon n\mu \operatorname{tr}(H) - \operatorname{tr}(H)\operatorname{tr}(H^2) + \operatorname{tr}(H^3), \\ \alpha_0 &= \varepsilon\mu(\operatorname{tr}(H^2) - (\operatorname{tr}(H))^2) + \frac{1}{n-2}((\operatorname{tr}(H))^2\operatorname{tr}(H^2) \\ &\qquad - \operatorname{tr}(H)\operatorname{tr}(H^3) + \operatorname{tr}(H^4)) \end{aligned} \tag{47}$$

and μ, defined by (19), *is expressed by*

$$\mu = \frac{\kappa}{(n-2)(n-1)} - \frac{2\widetilde{\kappa}}{(n-2)(n-1)n(n+1)}. \tag{48}$$

Proof. Applying in (44) the identity (23) we obtain

$$\operatorname{tr}(H)(C \cdot H)_{hklm} - (C \cdot H^2)_{hklm} = 0, \tag{49}$$

which, by making use of (20) and (21), turns into

$$\begin{aligned} \frac{1}{n-2}Q(g,H^4)_{hklm} &= \frac{2}{n-2}\operatorname{tr}(H)Q(g,H^3)_{hklm} \\ &\quad - \left(\varepsilon\mu + \frac{(\operatorname{tr}(H))^2}{n-2}\right)Q(g,H^2)_{hklm} + \varepsilon\mu\operatorname{tr}(H)Q(g,H)_{hklm} \\ &\quad + \operatorname{tr}(H)Q(H,H^2)_{hklm} - Q(H,H^3)_{hklm}. \end{aligned} \tag{50}$$

Contracting this with g^{hm} we obtain (45). Further, from (45) we get

$$\frac{n}{n-2}\operatorname{tr}(H^4) = \frac{n+2}{n-2}\operatorname{tr}(H)\operatorname{tr}(H^3) + \alpha_2\operatorname{tr}(H^2) + \alpha_1\operatorname{tr}(H) + n\alpha_0.$$

This, by (47), reduces to (46), which completes the proof.

PROPOSITION 3.11. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* $(*)$. *If* (30) *is satisfied at a point $x \in U_L \subset M$ then at x we have*

$$(\operatorname{tr}(H) - \alpha)\widetilde{\kappa} = 0. \tag{51}$$

Proof. Comparing (45) with (39) we obtain

$$(\alpha_2 - \widetilde{\alpha}_2)H^2 + (\alpha_1 - \widetilde{\alpha}_1)H + (\alpha_0 - \widetilde{\alpha}_0)g = 0, \tag{52}$$

which by (30) yields

$$(\alpha(\alpha_2 - \widetilde{\alpha}_2) + \alpha_1 - \widetilde{\alpha}_1)H + (\beta(\alpha_2 - \widetilde{\alpha}_2) + \alpha_0 - \widetilde{\alpha}_0)g = 0. \tag{53}$$

Using now (40) and (47) we find

$$\alpha_2 - \widetilde{\alpha}_2 = -\frac{\varepsilon\widetilde{\kappa}}{n-1}, \tag{54}$$

$$\alpha_1 - \widetilde{\alpha}_1 = \frac{\varepsilon\widetilde{\kappa}}{n-1}\operatorname{tr}(H). \tag{55}$$

From (53), by our assumptions, it follows that $\alpha(\alpha_2 - \widetilde{\alpha}_2) + \alpha_1 - \widetilde{\alpha}_1 = 0$. Applying (55) we hence obtain (51), which completes the proof.

We now restrict our considerations to the subset $U_H \cap U_L \subset U$ consisting of all points of U at which the tensor H^2 is not a linear combination of H and g and the associated function L is nonzero.

PROPOSITION 3.12. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying* $(*)$. *Then on $U_H \cap U_L$ we have*

$$\widetilde{\kappa} = 0, \tag{56}$$

$$\frac{n}{n-2}H^4 = \frac{n+2}{n-2}\operatorname{tr}(H)H^3 + \widetilde{\beta}_2 H^2 + \widetilde{\beta}_1 H + \widetilde{\beta}_0 g, \tag{57}$$

$$H^3 = \operatorname{tr}(H)H^2 + \lambda H + \beta_0 g, \qquad \lambda \in \mathbb{R}, \tag{58}$$

where

$$\begin{aligned} \widetilde{\beta}_2 &= -\frac{3n-2}{(n-2)(n-1)}(\operatorname{tr}(H))^2 + \frac{n}{(n-2)(n-1)}\operatorname{tr}(H^2), \\ \widetilde{\beta}_1 &= \frac{n}{(n-2)(n-1)}(\operatorname{tr}(H))^3 - \frac{n^2-2n+2}{(n-2)(n-1)}\operatorname{tr}(H)\operatorname{tr}(H^2) \\ &\quad + \operatorname{tr}(H^3), \\ \widetilde{\beta}_0 &= -\frac{1}{(n-2)(n-1)}(\operatorname{tr}(H))^4 + \frac{n+1}{(n-2)(n-1)}(\operatorname{tr}(H))^2\operatorname{tr}(H^2) \end{aligned} \tag{59}$$

$$- \frac{1}{(n-2)(n-1)}(\operatorname{tr}(H^2))^2 - \frac{2}{n-2}\operatorname{tr}(H)\operatorname{tr}(H^3)$$
$$+ \frac{1}{n-2}\operatorname{tr}(H^4),$$

$$\beta_0 = \frac{1}{n}(-\operatorname{tr}(H)\operatorname{tr}(H^2) - \lambda \operatorname{tr}(H) + \operatorname{tr}(H^3)), \tag{60}$$

$$\begin{aligned} &\text{(a) } \lambda = \frac{1}{n-1}(\operatorname{tr}(H^2) - (\operatorname{tr}(H))^2), \qquad \text{(b) } \lambda = -\frac{1}{n-1}\varepsilon\kappa, \\ &\text{(c) } \mu + \frac{1}{n-2}\varepsilon\lambda = 0. \end{aligned} \tag{61}$$

Proof. Let $x \in U_H \cap U_L$. From (52) it follows that $\alpha_2 = \widetilde{\alpha}_2$ at x. Applying (40), (47) and (48) we get $\widetilde{\kappa} = 0$. Now (39) and (45) reduce to (57). Next, applying (45) in (50) and using (47) we obtain

$$\begin{aligned} &\frac{1}{n}(\operatorname{tr}(H^3) - \operatorname{tr}(H)\operatorname{tr}(H^2))Q(g,H) + \frac{1}{n}(\operatorname{tr}(H))^2 Q(g,H^2) \\ &\qquad - \frac{1}{n}\operatorname{tr}(H)Q(g,H^3) - \operatorname{tr}(H)Q(H,H^2) + Q(H,H^3) = 0, \end{aligned} \tag{62}$$

which can be written in the form

$$Q\left(H - \frac{1}{n}\operatorname{tr}(H)g, H^3 - \operatorname{tr}(H)H^2 + \frac{1}{n}(\operatorname{tr}(H)\operatorname{tr}(H^2) - \operatorname{tr}(H^3))g\right) = 0.$$

But the last relation, in view of Lemma 3.4 of [1], implies (58), where β_0 is defined by (60). Finally, using (57)–(59) and the fact that at every point of U_L the tensor H^2 is not a linear combination of H and g, we obtain (61)(a). (61)(b) and (61)(c) are immediate consequences of (26), (56) and (19). Our proposition is thus proved.

4. Main results

PROPOSITION 4.1. *Let M be a hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$, satisfying* $(*)$. *Then on $U_H \cap U_L$ we have*

$$\beta_0 = 0, \tag{63}$$

$$R \cdot S = 0, \tag{64}$$

$$S^2 = \frac{\kappa}{n-1}S, \tag{65}$$

$$\kappa(L-1) = 0. \tag{66}$$

Moreover, if κ vanishes at a point $x \in U_H \cap U_L$ then at x we have

$$\operatorname{rank}(S) = 1. \tag{67}$$

Proof. First of all, we note that (20), by making use of (58) and (61)(c), reduces to

$$C \cdot H = \frac{n-3}{n-2}\varepsilon Q(H, H^2). \tag{68}$$

Transvecting (44) with H_q^m and symmetrizing the resulting equality in q, l we obtain

$$S_h^p(H_q^m C_{mlkp} + H_l^m C_{mqkp}) + S_k^p(H_q^m C_{mlhp} + H_l^m C_{mqhp}) = 0,$$

which, by (24), (25), (56), (58) and (68), reduces to

$$\beta_0(-2(H_{jk}H_{hl} + H_{lk}H_{hj}) + g_{hl}H^2_{kj} + g_{hj}H^2_{kl} + g_{kl}H^2_{hj} + g_{kj}H^2_{hl}) = 0.$$

Contracting this with g^{kl} and using the fact that at every point of U_L the tensor H^2 is not a linear combination of H and g, we get (63). Now (58) reduces to $H^3 - \mathrm{tr}(H)H^2 = \lambda H$. Applying this and (56) in (24) we obtain $H_{hr}S^r{}_k = 0$. Transvecting now (22) with S_l^h and using the last relation we easily obtain (64). Further, (38), by (64), reduces to $Q\big(g, S^2 - \frac{\kappa}{n-1}S\big) = 0$, which, by an application of Lemma 2.4(i) of [12], shows that $S^2 - \frac{\kappa}{n-1}S = \tau g$, $\tau \in \mathbb{R}$, at every $x \in U_L$. From the last relation, by making use of (23), (56), (58), (61)(a), (61)(b) and (63), we find $\tau = 0$, which means that (65) holds on U_L.

We now prove that (66) holds on U_L. First of all we note that (4), in view of (64), reduces to $R \cdot R = LQ(S, C)$, which, by (27) and (56), turns into

$$-Q(H^2, \overline{H})_{hijklm} = LQ(S, C)_{hijklm}.$$

Contracting this with g^{hm} and using (43) we obtain

$$\begin{aligned}(\mathrm{tr}(H^2) - \lambda)(H_{lk}H_{ij} - H_{lj}H_{ik}) + \mathrm{tr}(H)(H_{ik}H^2_{lj} - H_{ij}H^2_{kl} - H_{kl}H^2_{ij} \\ + H_{lj}H^2_{ik}) - (H^2_{lj}H^2_{ik} - H^2_{lk}H^2_{ij}) \\ = L(\kappa C_{iljk} - \varepsilon(\mathrm{tr}(H)H_i^p - H_i^{2p})C_{pljk}).\end{aligned} \tag{69}$$

Transvecting this with H_q^i and using (58), (61)(a) and (63) we find

$$\begin{aligned}(n-1)\lambda(H_{lk}H^2_{qj} - H_{lj}H^2_{qk}) + \lambda\,\mathrm{tr}(H)(H_{qk}H_{lj} - H_{qj}H_{kl}) \\ + \lambda(H_{lj}H^2_{qk} - H_{lk}H^2_{qj} - H_{qk}H^2_{lj} + H_{qj}H^2_{lk}) = \frac{n-2}{n-1}L\kappa H_q^p C_{pljk}.\end{aligned} \tag{70}$$

Symmetrizing this in l, j and using (68) we obtain (66).

We now assume that κ vanishes at $x \in U_H \cap U_L$. Thus (58) and (65) reduce to

$$H^3 = \mathrm{tr}(H)H^2, \tag{71}$$

$$S^2 = 0, \tag{72}$$

respectively. Transvecting (22) and (6) with S_l^h and using (24), (56), (71) and (72) we find

$$S_l^p R_{pijk} = 0, \tag{73}$$

$$S^p_l C_{pijk} = -\frac{1}{n-2}(S_{lk}S_{ij} - S_{lj}S_{ik}). \tag{74}$$

Next, transvecting (7) with S^m_p and using (73) and (74) we get

$$\begin{aligned} S_{li}(S_{hk}S_{pj} - S_{hj}S_{pk}) &+ S_{lh}(S_{pk}S_{ij} - S_{pj}S_{ik}) \\ &+ S_{lj}(S_{hk}S_{ip} - S_{hp}S_{ik}) + S_{lk}(S_{hp}S_{ij} - S_{hj}S_{ip}) = 0. \end{aligned} \tag{75}$$

Let V, with local components V^p, be a vector at x such that the covector w with local components $W_k = V^p S_{pk}$ is nonzero at x. Transvecting now (75) with V^l we obtain

$$\begin{aligned} W_i(S_{hk}S_{pj} - S_{hj}S_{pk}) &+ W_h(S_{pk}S_{ij} - S_{pj}S_{ik}) \\ &+ W_j(S_{hk}S_{ip} - S_{hp}S_{ik}) + W_k(S_{hp}S_{ij} - S_{hj}S_{ip}) = 0, \end{aligned}$$

which, in view of Lemma 4 of [14], implies (67). Our proposition is thus proved.

THEOREM 4.1. *Let M be a hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$, satisfying $(*)$. Then $R \cdot C = Q(S, C)$ on $U_H \cap U_L \subset M$.*

Proof. First of all we note that (9) holds on M. Further, Proposition 3.11 states that $\kappa(L-1) = 0$ on U_L. In the case when κ vanishes at a point $x \in U_H \cap U_L$, our assertion is a consequence of Propositions 2.3 and 4.1.

THEOREM 4.2. *Let M be a hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$, satisfying $(*)$. Then at every point x of $U_H \cap U_L$ the Ricci tensor S has the form*

$$S = \frac{\kappa}{n-1} g + \beta w \otimes w, \qquad \beta \in \mathbb{R},\ w \in T^*_x M,\ \mathcal{A}(W) = 0, \tag{76}$$

where the vector W is related to the covector w by $w(X) = g(W, X)$ for all $X \in T_x(M)$.

Proof. From Theorem 4.1 it follows that $R \cdot C = Q(S, C)$ on $U_H \cap U_L$. This by (64) turns into $R \cdot R = Q(S, C)$. Applying (5) we get

$$R \cdot R = Q(S, R) - \frac{1}{n-2} Q(S, g \wedge S) + \frac{\kappa}{(n-1)(n-2)} Q(S, G),$$

which, by (9), reduces to $Q(S, g \wedge S) = \frac{\kappa}{n-1} Q(S, G)$. Applying now Proposition 2.1 we find $Q\big(g, \overline{S} - \frac{\kappa}{n-1} g \wedge S\big) = 0$, whence it follows that ([2], Section 2.3)

$$\overline{S} - \frac{\kappa}{n-1} g \wedge S = \widetilde{\psi} G \tag{77}$$

on $U_H \cap U_L$, where $\widetilde{\psi}$ is some function on $U_H \cap U_L$. Note that (77) can be represented in the form

$$\overline{A} = \psi G, \tag{78}$$

where $A = S - \frac{\kappa}{n-1}g$ and $\psi = \widetilde{\psi} + \frac{\kappa^2}{(n-1)^2}$. Further, (78) implies

$$Q(A, \overline{A}) = \psi Q(A, G). \tag{79}$$

Evidently, $Q(A, \overline{A}) = 0$. Thus from (79) we easily get

$$\psi\left(A - \frac{1}{n}\operatorname{tr}(A)g\right) = 0.$$

If $A = (1/n)\operatorname{tr}(A)g$ at a point $x \in U_H \cap U_L$, then $S = (\kappa/n)g$, a contradiction. Thus ψ vanishes on $U_H \cap U_L$ and, in consequence, at every point of $U_H \cap U_L$ we have (76), which completes the proof.

Proposition 4.2. *Let M be a Ricci-semisymmetric hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$, satisfying $(*)$. Then on $U_H \cap U_L$ we have*

$$C \cdot R = \frac{n-3}{n-2} Q(S, R). \tag{80}$$

Proof. From Theorems 1.1(i) and 4.2 it follows that (2)(b) holds on $U_H \cap U_L$. Now Lemma 3.1 implies

$$(n-2)(R \cdot C - C \cdot R)_{hijklm} = Q(S, R)_{hijklm}. \tag{81}$$

This, by making use of (9), leads to (80), which completes the proof.

Propositions 3.8, 3.12 and 4.2 and Theorems 4.1 and 4.2 lead to our main result.

Theorem 4.3. *Let M be a hypersurface in $N^{n+1}_s(c)$, $n \geq 4$, satisfying $(*)$. If $U_H \cap U_L \neq \emptyset$ then the ambient space is semi-Euclidean and on $U_H \cap U_L$ we have*

$$\begin{gathered} R \cdot S = 0, \qquad C \cdot S = 0, \qquad R \cdot C = Q(S, C), \\ C \cdot R = \frac{n-3}{n-2} Q(S, R), \qquad \mathcal{A}^3 = \operatorname{tr}(\mathcal{A})\mathcal{A}^2 - \frac{\varepsilon\kappa}{n-1}\mathcal{A}, \qquad \varepsilon = \pm 1, \\ \mathcal{A}(W) = 0, \qquad S = \frac{\kappa}{n-1} g + \beta w \otimes w, \qquad w \in T^*_x M,\ \beta \in \mathbb{R}, \end{gathered} \tag{82}$$

where $g(W, X) = w(X)$ for all $X \in T_x M$.

Examples of hypersurfaces satisfying (82), with $U_H \cap U_L$ nonempty, were found in [5].

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Ryszard Deszcz
Department of Mathematics
Agricultural University of Wrocław
Grunwaldzka 53
50-357 Wrocław, Poland
E-mail: rysz@ozi.ar.wroc.pl

Marian Hotloś
Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: hotlos@im.pwr.wroc.pl

Zerrin Şentürk
Department of Mathematics
Technical University of Istanbul
80626 Maslak, Istanbul, Turkey
E-mail: senturk@fariyer.cc.itu.edu.tr

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