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SOME REMARKS ON QUASI-COHEN SETS

BY

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Abstract. We are interested in Banach space geometry characterizations of quasi-Cohen sets. For example, it turns out that they are exactly the subsets E of the dual of an abelian compact group G such that the canonical injection $C(G)/C_{E^c}(G) \hookrightarrow L^2_E(G)$ is a 2-summing operator. This easily yields an extension of a result due to S. Kwapień and A. Pełczyński. We also investigate some properties of translation invariant quotients of L^1 which are isomorphic to subspaces of L^1.

0. Introduction. Let G be an infinite metrizable compact abelian group, equipped with its normalized Haar measure dx, and Γ its dual group (discrete and countable).

It is well known that subsets Λ of Γ for which $C_\Lambda(G)$ is complemented in $C(G)$ are those for which there exists a measure μ such that $\widehat{\mu} = 1$ on Λ and $\widehat{\mu} = 0$ on $\Gamma \setminus \Lambda$. Due to the characterization of P. Cohen [C] of these sets, S. Kwapień and A. Pełczyński [K-P] called such sets Cohen sets, and introduced quasi-Cohen sets as the subsets Λ of Γ for which there exists a measure μ such that $|\widehat{\mu}| \geq 1$ on Λ and $\widehat{\mu} = 0$ on $\Gamma \setminus \Lambda$. Every Cohen set is then a quasi-Cohen set, but S. Drury's construction [D] shows that the complement of any Sidon set is a quasi-Cohen set, though it is not a Cohen set (if this Sidon set is infinite). S. Kwapień and A. Pełczyński characterized the quasi-Cohen sets Λ by properties of operators acting on the spaces $C_\Lambda(G)$ or $L^p_\Lambda(G)$, $p = 1$ or 2 ([K-P], Th. 2.1, 2.2) and showed that Λ is a quasi-Cohen set whenever $C_\Lambda(G)$ is a quotient of an $\mathcal{L}^\infty$-space ([K-P], Prop. 2.2). It seems that these sets have not been investigated since then (see [H-M-P], Chap. III, however).

The purpose of this note is to give some new characterizations of quasi-Cohen sets Λ in terms of factorization properties of the canonical injection from $C_\Lambda(G)$ into $L^1_\Lambda(G)$ and of 2-summing properties of the canonical injection from $C(G)/C_{\Lambda^c}(G)$ to $L^2_\Lambda(G)$.

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1. Notations and definitions. In this paper, G will be an infinite metrizable compact abelian group and Γ its dual group (discrete and countable). In the case of the circle group $G = \mathbb{T}$, Γ is identified with $\mathbb{Z}$ by the map $p \mapsto e_p$ with $e_p(x) = e^{2i\pi px}$.

$M(G)$ will denote the space of complex regular Borel measures over G, equipped with the total variation norm. If $\mu \in M(G)$, its Fourier transform at the point γ is defined by

$$\widehat{\mu}(\gamma) = \int_G \gamma(-x)\, d\mu(x).$$

As usual, the space $C(G)$ and the Lebesgue spaces $L^p(G)$, $1 \le p \le \infty$, related to the Haar measure, are identified with linear subspaces of $M(G)$ by the map $f \mapsto f dx$.

For $B \subset M(G)$ and $\Lambda \subset \Gamma$, we set

$$B_\Lambda = \{\mu \in B \mid \forall \gamma \notin \Lambda,\ \widehat{\mu}(\gamma) = 0\}.$$

B_Λ is the set of elements of B whose spectrum is contained in Λ.

The complement $\Gamma \setminus E$ of any subset of Γ will be denoted by E^{c}. If $x \in X$ and $Y \subset X$, we denote by $\dot{x}$ the class of x in the quotient X/Y.

We recall that a subset Λ of Γ is said to be a *Sidon set* if there exists $C > 0$ such that $\sum_{\gamma\in\Lambda} |\widehat{f}(\gamma)| \le C\|f\|_\infty$ for all $f \in C_\Lambda(G)$.

Definition 1.1. A subset Λ of Γ is said to be a *Cohen set* if there exists a measure $\mu \in M_\Lambda(G)$ such that

$$\widehat{\mu}(\gamma) = 1 \qquad \text{for every } \gamma \in \Lambda;$$

or, what is the same, if $C_\Lambda(G)$ is complemented in $C(G)$.

Definition 1.2. A subset Λ of Γ is a *quasi-Cohen set* if there exists a measure $\mu \in M_\Lambda(G)$ such that

$$|\widehat{\mu}(\gamma)| \ge 1 \qquad \text{for every } \gamma \in \Lambda.$$

It should be noticed that we can actually assume that $\widehat{\mu}(\gamma) \ge 1$ for all $\gamma \in E$, by replacing μ by $\mu * \widetilde{\mu}$ (where $\widetilde{\mu}(A) = \overline{\mu(-A)}$).

It is clear that every Cohen set is a quasi-Cohen set. The converse is false: it has been observed by I. Glicksberg [G] that S. Drury's [D] construction shows that the complement S^{c} of every Sidon set S is a quasi-Cohen set; however, it is not a Cohen set (at least if S is infinite), since ℓ^1 is not isomorphic to any complemented subspace of $C(G)$. In fact, it has been observed by M. Déchamps-Gondim that, as a consequence of the paper of B. Host and F. Parreau [H-P], any subset Λ of Γ for which both Λ and Λ^{c} are quasi-Cohen sets is actually Cohen (see [K-P], p. 317 or [H-M-P], Chap. III). S. Kwapień and A. Pełczyński also proved that: E is a Sidon set if and only if every subset of $E \subset \Gamma$ is the complement of a quasi-Cohen set ([K-P], Th. 3.2).

We also recall that a bounded operator T from a Banach space X to a Banach space Y is said to be *p-summing* if there is a constant $C > 0$ such that for any finite family of vectors (x_n) in X,

$$\Big(\sum_n \|T(x_n)\|^p\Big)^{1/p} \le C \sup_{\substack{\chi\in X^* \\ \|\chi\|=1}} \Big(\sum_n |\chi(x_n)|^p\Big)^{1/p}.$$

We denote by $\pi_p(T)$ the smallest such constant C.

Definition 1.3. A Banach space X is said to be a GT-*space* if it satisfies the Grothendieck theorem: every bounded operator from X into a Hilbert space is 1-summing.

Remark 1.4. It is known (see [P], Prop. 6.2) that X is a GT-space if and only if every bounded operator from X^* into a cotype 2 space is 2-summing.

Definition 1.5. A Banach space X is said to be a GL-*space* (or to have the GL *property*) if it has the Gordon–Lewis property: every 1-summing operator from X into an arbitrary Banach space factorizes through an L^1-space.

The reader has to watch out for the different terminology on GL-spaces that can be found in the literature. Here, we adopt the terminology that can be found in the book of G. Pisier ([P], Def. 8.13).

We introduce the following

Definition 1.6. For $\Lambda \subset \Gamma$, the space $C_\Lambda(G)$ will said to be a $\mathrm{GL}^{\mathrm{inv}}$-space if the canonical injection from $C_\Lambda(G)$ to $L^1_\Lambda(G)$ factorizes through an L^1-space.

This definition is different from [K-P], Def. 5.1. Notice that a GL-space is clearly a $\mathrm{GL}^{\mathrm{inv}}$-space.

Finally, we introduce the following notion:

Definition 1.7. Let $E \subset \Gamma$ and X be a Banach space. Let $\varphi : C(G)/C_{E^c}(G) \to X$ be a bounded operator. We say that the pair (E, X) is φ-*admissible* if there exists a constant $\delta > 0$ such that for all $\gamma \in E$, $\|\varphi(\dot\gamma)\| \ge \delta$, where $\dot\gamma$ is the class of γ in $C(G)/C_{E^c}(G)$.

Notation 1.8. We denote by $i_{2,E}$ the projection

$$C(G)/C_{E^c}(G) \to L^2(G), \qquad \dot f \mapsto \sum_{\gamma\in E} \widehat f(\gamma)\gamma.$$

Remark 1.9. For any $E \subset \Gamma$, the pair $(E, L^2(G))$ is $i_{2,E}$-admissible.

2. Quasi-Cohen sets. The main theorem of this section is the following

THEOREM 2.1. *Let $E \subset \Gamma$. The following assertions are equivalent*:

(i) *E is a quasi-Cohen set.*

(ii) *The canonical injection from $C_E(G)$ to $L^1_E(G)$ factorizes through the canonical injection from $L^2(G)$ to $L^1(G)$.*

(iii) *The canonical injection from $C_E(G)$ to $L^1_E(G)$ factorizes through an operator $T : Y \to Z$, where Z is a* GT*-space and Y^* has cotype* 2.

(iv) *There exists a Banach space X such that the pair (E, X) is φ-admissible*, *where φ is a 2-summing operator.*

As L^2 has cotype 2, every operator which is p-summing for some $p \geq 2$, with range in L^2, is actually 2-summing; hence an immediate corollary is the following:

THEOREM 2.2. *Let $E \subset \Gamma$. The operator $i_{2,E} : C(G)/C_{E^c}(G) \to L^2(G)$ is p-summing for some $p \geq 2$ if and only if E is a quasi-Cohen set.*

Proof of Theorem 2.1. (i)⇒(ii). There exists a measure $\mu \in M_E(G)$ satisfying $|\widehat{\mu}(\gamma)| \geq 1$ for every $\gamma \in E$. Setting $m_\gamma = \widehat{\mu}(\gamma)^{-1}$ for every $\gamma \in E$, we have $m = (m_\gamma)_{\gamma \in E} \in \ell^\infty(E)$ with $\|m\|_\infty \leq 1$. Thus m defines a bounded operator $T_m : L^2_E(G) \to L^2_E(G)$ with $T_m(f) = \sum_{\gamma \in E} m_\gamma \widehat{f}(\gamma)\gamma$. Now the result follows from the factorization

$$C_E(G) \hookrightarrow L^2_E(G) \xrightarrow{T_m} L^2_E(G) \hookrightarrow L^2(G) \hookrightarrow L^1(G) \xrightarrow{*\mu} L^1_E(G)$$

where $*\mu$ is convolution by μ and the unspecified maps are the natural injections.

(ii)⇒(iii) is trivial since L^1 is a GT-space and L^2 has cotype 2.

(iii)⇒(iv). By assumption, we have the following factorization for the canonical injection of $C_E(G)$ into $L^1_E(G)$:

$$C_E(G) \xrightarrow{\alpha} Y \xrightarrow{T} Z \xrightarrow{\beta} L^1_E(G)$$

where Y^* has cotype 2, Z is a GT-space and T, α, β are bounded operators.

By duality, we get the following factorization for the canonical injection $L^\infty(G)/L^\infty_{E^c}(G)$ into $M(G)/M_{E^c}(G)$:

$$L^\infty(G)/L^\infty_{E^c}(G) \xrightarrow{\beta^*} Z^* \xrightarrow{T^*} Y^* \xrightarrow{\alpha^*} M(G)/M_{E^c}(G).$$

Thanks to Remark 1.4, the operator T^* is 2-summing. Hence, the canonical injection from $L^\infty(G)/L^\infty_{E^c}(G)$ to $M(G)/M_{E^c}(G)$ is also 2-summing. A fortiori, the canonical injection from $C(G)/C_{E^c}(G)$ to $M(G)/M_{E^c}(G)$ is 2-summing. As $\|\dot{\gamma}\|_{M(G)/M_{E^c}(G)} = 1$ for any $\gamma \in E$, we have proved (iv) with $X = M(G)/M_{E^c}(G)$.

(iv)⇒(i). The argument with the Pietsch domination theorem which simplifies the original one was suggested to us by G. Pisier. There exists a probability measure ν on the unit ball of the dual of $C(G)/C_{E^c}(G)$, i.e. on the unit ball of $M_E(G)$, such that for any $h \in C(G)/C_{E^c}(G)$,

$$\|\varphi(h)\| \le \pi_2(\varphi)\Big(\int\limits_{B_{M_E(G)}} |\langle\zeta, h\rangle|^2 d\nu(\zeta)\Big)^{1/2}$$

where $\langle\,,\rangle$ denotes the duality bracket. It should be noted that, for convenience, we actually use $\langle\zeta, f\rangle = f * \zeta(0)$ for the duality between $M_E(G)$ and $C(G)/C_{E^c}(G)$.

Testing the previous inequality at $\dot\gamma$, with $\gamma \in E$, we obtain

$$0 < \delta^2 \le \|\varphi(\dot\gamma)\|^2 \le \pi_2(\varphi)^2 \int\limits_{B_{M_E(G)}} |\zeta * \gamma(0)|^2\, d\nu(\zeta)$$
$$= \pi_2(\varphi)^2 \int\limits_{B_{M_E(G)}} |\widehat{\zeta}(\gamma)|^2\, d\nu(\zeta).$$

We then define the measure μ as the integral (in the weak star sense)

$$\mu = \int\limits_{B_{M_E(G)}} (\zeta * \widetilde{\zeta})\, d\nu(\zeta)$$

where, as usual, $\widetilde{\zeta}(\Omega) = \overline{\zeta(-\Omega)}$ for any Borel set $\Omega \subset G$. Thus, we have $\widetilde{\zeta} \in M(G)$, $\|\widetilde{\zeta}\| = \|\zeta\|$ and $\widehat{\widetilde{\zeta}} = \overline{\widehat{\zeta}}$ for any $\zeta \in M(G)$.

The measure μ is in $M_E(G)$. Moreover, for any $\gamma \in E$ we have

$$\widehat{\mu}(\gamma) = \mu * \gamma(0) = \int\limits_{B_{M_E(G)}} \zeta * \widetilde{\zeta} * \gamma(0)\, d\nu(\zeta) = \int\limits_{B_{M_E(G)}} \widehat{\zeta}(\gamma) \cdot \widehat{\widetilde{\zeta}}(\gamma)\, d\nu(\zeta)$$
$$= \int\limits_{B_{M_E(G)}} |\widehat{\zeta}(\gamma)|^2\, d\nu(\zeta).$$

This leads to $\pi_2(\varphi)^2\widehat{\mu}(\gamma) \ge \delta^2$. As the measure $\pi_2(\varphi)^2\mu$ is in $M_E(G)$, this exactly means that E is a quasi-Cohen set. ■

3. GL-spaces and quotients of $\mathcal{L}^\infty$. Theorem 2.1 allows us to extend a result on quotients of $\mathcal{L}^\infty$ contained in [H-M-P] and [K-P], which is linked to a problem raised by S. Kwapień and A. Pełczyński:

Let $E \subset \Gamma$ be a quasi-Cohen set. Is $C_E(G)$ isomorphic to a quotient of an $\mathcal{L}^\infty$-space?

Concerning the converse, S. Kwapień and A. Pełczyński noticed that if there exists a translation invariant surjection from $C(G)$ to $C_E(G)$ then E is a Cohen set [K-P]. This is based on a result of B. Host and F. Parreau on closed ideals of $L^1(G)$. Without the translation invariance assumption, they proved that E is a quasi-Cohen set if $C_E(G)$ is isomorphic to a quotient of a $C(K)$-space.

The following result is a corollary of Theorem 2.1.

THEOREM 3.1. *Let $E\subset\Gamma$. If $C_E(G)$ is a* $\mathrm{GL}^{\mathrm{inv}}$*-space and $M(G)/M_{E^c}(G)$ has cotype* 2 *then E is a quasi-Cohen set.*

Proof. Consider the canonical injection from $C_E(G)$ to $L^1_E(G)$. As $C_E(G)$ is a $\mathrm{GL}^{\mathrm{inv}}$-space, it factorizes through an L^1-space by an operator $A : C_E(G)\to L^1$. But $C_E(G)^*$ and L^1 have cotype 2 and L^1 is a GL-space, so by a result of G. Pisier ([P], Th. 8.17), A factorizes through an L^2-space. Hence the canonical injection from $C_E(G)$ to $L^1_E(G)$ factorizes through an operator from L^2 to L^1. By Theorem 2.1(iii), E is a quasi-Cohen set. ∎

Theorem 3.1 leads to recovering some known results:

COROLLARY 3.2 ([H-M-P], [K-P]). *Let $E\subset\Gamma$. If $C_E(G)$ is isomorphic to a quotient of a $C(K)$-space then E is a quasi-Cohen set.*

Proof. If $C_E(G)$ is isomorphic to a quotient of a $C(K)$-space then $C_E(G)$ is a GL-space and $C_E(G)^*$ is isomorphic to a subspace of an L^1-space, hence has cotype 2. Theorem 3.1 gives the result. ∎

The second part of the following corollary is well known but usually proved using the Paley inequality.

COROLLARY 3.3. *The disk algebra is not a* $\mathrm{GL}^{\mathrm{inv}}$*-space*, *hence is not a* GL*-space.*

Proof. As the quotient $M(\mathbb{T})/H^1$ has cotype 2 (this is due to J. Bourgain, see [P], Th. 6.17), if the disk algebra were a $\mathrm{GL}^{\mathrm{inv}}$-space, this would imply that $\mathbb{N}$ is a quasi-Cohen set. This is clearly false by the classical Riesz theorem: every measure with spectrum contained in $\mathbb{N}$ is absolutely continuous with respect to the Haar–Lebesgue measure, hence its Fourier coefficients tend to zero at infinity ($\mathbb{N}$ is a so-called Rajchman set). ∎

More generally, we notice the following consequence of Theorem 3.1, which produces examples of spaces without the GL-property:

COROLLARY 3.4. *For any $E\subset\Gamma$, which is not a quasi-Cohen set, such that $L^1(G)/L^1_{E^c}(G)$ has cotype* 2, *the space $C_E(G)$ does not have the* GL-*property.*

For example, this includes the case of the disk algebra.

REMARK 3.5. This also leads to the following examples: if A is a $\Lambda(1)$-set (i.e. L^1_A is reflexive) but not Sidon, then there exists $B\subset A$ such that B^c is not quasi-Cohen (else A would be Sidon by the result of S. Kwapień and A. Pełczyński quoted in the introduction). Then L^1/L^1_B has cotype 2 ([P], p. 78) and fails the GL-property. Moreover, if A is chosen $\Lambda(2)$ and still not Sidon then L^1_B is even isomorphic to a Hilbert space.

4. Some remarks on quotients of L^1 isomorphic to subspaces of L^1-spaces. We are interested in this section in results in the spirit of 3.2 in terms of quotients of L^1. There is a characterization (with summing operators) of the sets $\Lambda \subset \Gamma$ such that $L^1(G)/L^1_{\Lambda^c}(G)$ is isomorphic to a subspace of an L^1-space:

THEOREM 4.1. *Let $\Lambda \subset \Gamma$. The following assertions are equivalent*:

(i) *$L^1(G)/L^1_\Lambda(G)$ is isomorphic to a subspace of an L^1-space.*

(ii) *The canonical injection from $C(G)/C_\Lambda(G)$ to $L^1(G)/L^1_\Lambda(G)$ is 1-summing.*

(iii) *There is a probability measure ν on the unit ball B of $M_{\Lambda^c}(G)$ such that $L^1(G)/L^1_\Lambda(G)$ is isomorphic to a subspace of $L^1(B\times G, \nu\otimes dx)$. Moreover the isomorphism can be taken as $T(f)=F$ with $F(\zeta,\cdot)=f*\zeta$ for all $f\in L^1(G)/L^1_\Lambda(G)$, $\zeta\in B$.*

Proof. (i)⇒(ii). Thanks to Theorem 9.12.b of [D-J-T] and as the canonical injection from $C(G)/C_\Lambda(G)$ to $L^1(G)/L^1_\Lambda(G)$ has a 1-summing adjoint, it is itself 1-summing.

(ii)⇒(iii). We use the Pietsch domination theorem: there exists a probability measure ν on B and a constant C such that for any $h\in C(G)/C_\Lambda(G)$,

$$\|h\|_{L^1/L^1_\Lambda} \le C\int_B |\zeta * h(0)|\,d\nu(\zeta).$$

Applying this inequality to h_x for every $x\in G$, where $h_x(t)=h(x-t)$ with additive notation of the group operation on G (notice that $\|h_x\|_{L^1/L^1_\Lambda} = \|h\|_{L^1/L^1_\Lambda}$), and integrating over G with respect to the Haar measure, we obtain

$$\|h\|_{L^1/L^1_\Lambda} \le C\int_G\int_B |\zeta * h(x)|\,d\nu(\zeta)\,dx = C\|H\|_{L^1(\nu\otimes dx)}$$

where $H(\zeta,x)=h*\zeta(x)$.

As obviously $\|H\|_{L^1(\nu\otimes dx)} \le \|h\|_{L^1/L^1_\Lambda}$, the quotient $L^1(G)/L^1_\Lambda(G)$ is then isomorphic to the space $Z=\{H\in L^1(\nu\otimes dx) \mid \exists h\in L^1(G)/L^1_\Lambda(G), H(\zeta,\cdot)=h*\zeta,\ \zeta\in B\}$.

(iii)⇒(i) is trivial. ∎

REMARK 4.2. Suppose that we are in the situation of the preceding theorem. Then, by duality, $L^\infty_{\Lambda^c}(G)$ is isomorphic to the quotient $L^\infty(B\times G)/Z^\perp$ by the map $F(\zeta,x)\in L^\infty(B\times G)\mapsto \int_B \zeta * F_\zeta\,d\nu(\zeta)\in L^\infty_{\Lambda^c}(G)$ where $F_\zeta(x)=F(\zeta,x)$. Hence, by approximation, $C_{\Lambda^c}(G)$ is isomorphic to a quotient of $C(B\times G)$.

The following corollary shows the link between this section and quasi-Cohen sets.

COROLLARY 4.3. *Suppose that $L^1(G)/L^1_\Lambda(G)$ is isomorphic to a subspace of an L^1-space. Then Λ^c is a quasi-Cohen set.*

Proof. The preceding remark and Corollary 3.2 suffice to prove the claim.

Another argument is: the preceding theorem asserts that the canonical injection from $C(G)/C_\Lambda(G)$ to $L^1(G)/L^1_\Lambda(G)$ is 1-summing, hence 2-summing. Theorem 2.1(iv) then gives the result. ∎

We now state some properties of such sets.

THEOREM 4.4. *Let $\Lambda \subset \Gamma$ be such that $L^1(G)/L^1_\Lambda(G)$ is isomorphic to a subspace of an L^1-space. Then every $f \in C_{\Lambda^c}(G)$ has a decomposition $f = \sum_j \mu_j * y_j$, where $\mu_j \in L^1_{\Lambda^c}(G)$, $y_j \in C(G)$ and $\sum_j \|\mu_j\| \cdot \|y_j\| < \infty$. Hence, any Fourier multiplier $m = (m_\gamma)_{\gamma\in\Gamma}$ from $C(G)$ to $C(G)/C_\Lambda(G)$ is induced by a measure: there exists a measure $\mu \in M(G)$ such that $\widehat{\mu}(\gamma) = m_\gamma$ for all $\gamma \notin \Lambda$.*

Proof. First, we can factorize f as $f_1 * f_2$ where $f_1 \in L^1(G)$ and $f_2 \in C_{\Lambda^c}(G)$. By Theorem 4.1(ii), the canonical injection from $C(G)/C_\Lambda(G)$ to $L^1(G)/L^1_\Lambda(G)$ is 1-summing, hence its composition T_2 with the operator from $L^1(G)/L^1_\Lambda(G)$ to $C(G)$ of convolution by f_2 is also 1-summing. By [D-J-T], Th. 5.7, T_2 is 1-integral. As the operator T_1 from $C(G)$ to $C(G)$ of convolution by f_1 is compact, the composition $T = T_1 \circ T_2$ is nuclear. Notice that T is in fact convolution by f from $C(G)/C_\Lambda(G)$ to $C(G)$.

Therefore, there exists some measure $\mu_j \in M_{\Lambda^c}(G)$ (the dual space of $C(G)/C_\Lambda(G)$) and $y_j \in C(G)$ such that for every $h \in C(G)/C_\Lambda(G)$, $T(h) = \sum_j \mu_j * h(0) y_j$, where $\sum_j \|\mu_j\| \cdot \|y_j\| < \infty$. The last condition implies that $\sigma = \sum_j \mu_j * y_j \in C_{\Lambda^c}(G)$. Thus, $\widehat{T(\gamma)}(\gamma) = \sum_j \widehat{\mu}_j(\gamma)\widehat{y}_j(\gamma) = \widehat{\sigma}(\gamma)$ for all $\gamma \notin \Lambda$. Hence $f = \sigma$.

The second part is standard: for all Fourier multipliers m from $C(G)$ to $C(G)/C_\Lambda(G)$, $f * m = \sum_j \mu_j * y_j * m$ does define a function in $C(G)$ and even in $C_{\Lambda^c}(G)$. Hence, by duality, m belongs to the dual $M(G)/M_\Lambda(G)$ of $C_{\Lambda^c}(G)$. So it is induced by a measure. ∎

REMARK 4.5. By duality, the same conclusion holds in the second part of the theorem for every Fourier multiplier from $M_{\Lambda^c}(G)$ into itself.

Therefore, the Paley projection viewed from H^1 (i.e. $M_{\mathbb{N}}$) into itself produces an immediate example of a multiplier (by the characteristic function of the set $\{2^n\}$) which is surely not induced by a measure. This shows the known fact that L^1/H^1 is not isomorphic to a subspace of L^1.

REMARK 4.6. If one showed that L^1/L^1_S is not isomorphic to a subspace of L^1 when S is a Sidon set, it "would suffice", together with the previous results, to produce a Fourier multiplier from $M_{S^c}(G)$ into itself which is not induced by a measure.

5. Descriptive point of view. The difficulty of the study of classes of subsets of Γ can be viewed through descriptive set theory (see [K-L], [T], [Go]). We have

PROPOSITION 5.1. *The set $\mathcal{QC}$ of all quasi-Cohen subsets of Γ is analytic in the set $\mathcal{P}(\Gamma)$ of all subsets of Γ, equipped with the product topology on $\{0,1\}^{\Gamma}$.*

Proof. Let $\Lambda \in \mathcal{QC}$. This means that there exists $\mu \in M_{\Lambda}(G)$ such that $\widehat{\mu}(\gamma) \geq 1$ for every $\gamma \in \Lambda$. Let us introduce the sets

$$D_K = \{(\Lambda, \mu) \in \mathcal{P}(\Gamma) \times M_{\Lambda}(G) \mid \|\mu\| \leq K;\ \forall \gamma \in \Lambda,\ \widehat{\mu}(\gamma) \geq 1\} \subset \mathcal{P}(\Gamma) \times M(G).$$

Bounded subsets of $M(G)$ are w^*-metrizable. We will show that D_K is closed. Indeed, if (Λ_n, μ_n) converges to (Λ, μ), then for every $\gamma \in \Gamma$, there exists some n_γ such that for all $n \geq n_\gamma$, $\gamma \in \Lambda \Leftrightarrow \gamma \in \Lambda_n$. Moreover $\widehat{\mu}_n(\gamma) \to \widehat{\mu}(\gamma)$ as $n \to \infty$; so we conclude that $\mu \in M_{\Lambda}(G)$.

On the other hand, if $\gamma \in \Lambda$, then $\gamma \in \Lambda_n$ for all $n \geq n_\gamma$, hence $\widehat{\mu}_n(\gamma) \geq 1$. Letting n tend to infinity gives $\widehat{\mu}(\gamma) \geq 1$.

We conclude that $\mathcal{QC}$ is a projection of the F_σ set $\bigcup_{K \geq 1} D_K$, hence it is analytic. ■

Of course, it would be interesting to know whether or not $\mathcal{QC}$ is a Borel set.

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