COLLOQUIUM MATHEMATICUM

VOL. 89 2001 NO. 2

SOME ORTHOGONAL DECOMPOSITIONS OF SOBOLEV SPACES AND APPLICATIONS

BY

H. BEGEHR (Berlin) and Yu. DUBINSKIĬ (Moscow)

Abstract. Two kinds of orthogonal decompositions of the Sobolev space $\mathring{W}_2^1$ and hence also of W_2^{-1} for bounded domains are given. They originate from a decomposition of $\mathring{W}_2^1$ into the orthogonal sum of the subspace of the Δ^k-solenoidal functions, $k \geq 1$, and its explicitly given orthogonal complement. This decomposition is developed in the real as well as in the complex case. For the solenoidal subspace ($k = 0$) the decomposition appears in a little different form.

In the second kind decomposition the Δ^k-solenoidal function spaces are decomposed via subspaces of polyharmonic potentials. These decompositions can be used to solve boundary value problems of Stokes type and the Stokes problem itself in a new manner. Another kind of decomposition is given for the Sobolev spaces W_p^m. They are decomposed into the direct sum of a harmonic subspace and its direct complement which turns out to be $\Delta(W_p^{m+2} \cap \mathring{W}_p^2)$. The functions involved are all vector-valued.

1. Introduction. A series of decompositions of the Sobolev space $\mathring{W}_2^1$ and its conjugate W_2^{-1} are given which are connected with the solenoidal and potential subspaces. There are two well known such decompositions of the Lebesgue space L_2 (see [12]), namely if $G \subset \mathbb{R}^n$, $n \geq 1$, is a smooth enough domain then

$$L_2 = S_2 \oplus \nabla \mathring{W}_2^1,$$

with

$$S_2 = \{u_s \in L_2 : \operatorname{div} u_s = 0 \text{ in } G\},$$

and

$$L_2 = \mathring{S}_2 \oplus \nabla H_1 \oplus \nabla \mathring{W}_2^1$$

where $\mathring{S}_2$ is the closure in L_2 of the set of solenoidal functions in $C_0^\infty(G)$ and $H_1 \subset W_2^1$ is the subspace of harmonic potentials. Here the derivatives are understood in the distributional sense, i.e. in $\mathcal{D}'$.

2000 *Mathematics Subject Classification*: Primary 46E35, 46E30; Secondary 35Q30, 31B30, 32F99.

Paper prepared during the sojourn of the second-named author in spring 2000 at the Freie Universität Berlin on the basis of a DFG guest professorship.

The decompositions given in this paper can be viewed as generalizations of both these decompositions. They are connected with the orthogonality not only to "pure" solenoidal subspaces but also to so-called Δ^k-solenoidal subspaces. The latter consist of those elements $u \in \overset{\circ}{W}{}_2^1$ which satisfy $\operatorname{div} \Delta^k u = 0$ in G in the sense of $\mathcal{D}'$. Here k is any natural number. The particular case $k = 1$ was briefly discussed in [4, 5].

Each type of decomposition leads to a corresponding Stokes type boundary value problem or to the Stokes problem itself. In contrast to the traditional methods of finding the solenoidal vector function here at first the potential is determined and then the solenoidal part found as the solution to a Dirichlet problem. Such type of decompositions have also been developed in Clifford analysis (see [2, 3, 6–10, 14]), and applied to boundary value problems of mathematical physics. For decompositions in complex analysis compare [1, 13].

2. Orthogonality to the Δ^k-solenoidal subspaces. Let $G \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with Lipschitz boundary ∂G. Let

$$\overset{\circ}{W}{}_2^1 = \overset{\circ}{W}{}_2^1(G;\mathbb{C}^n) = \Big\{ u : G \to \mathbb{C}^n : \|u\|_1^2 = \int_G |\nabla u(x)|^2\, dx$$
$$= \sum_{\nu=1}^n \int_G (\nabla u_\nu(x), \nabla u_\nu(x))\, dx < \infty,\ u|_{\partial G} = 0 \Big\}$$

be a subspace of the well known Sobolev space W_2^1 of complex vector-valued functions $u = (u_1, \dots, u_n)$ in G, where $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ is the gradient operator. For fixed integer $k \geq 1$ let

$$\overset{\circ}{S}{}^1_{\Delta^k,2} = \{u \in \overset{\circ}{W}{}_2^1 : \operatorname{div} \Delta^k u = 0 \text{ in } G\}.$$

Here div and Δ are operators understood in the sense of distributions in $\mathcal{D}' = \mathcal{D}'(G)$. The set $\overset{\circ}{S}{}^1_{\Delta^k,2}$ is called the subspace of *Δ^k-solenoidal vector functions* of $\overset{\circ}{W}{}_2^1$. Since both operators div and Δ are closed in $\mathcal{D}'$ it follows that $\overset{\circ}{S}{}^1_{\Delta^k,2}$ are closed subspaces of $\overset{\circ}{W}{}_2^1$, and the inclusion

$$\overset{\circ}{S}{}^1_{\Delta^k,2} \subset \overset{\circ}{W}{}_2^1$$

is proper. Hence, $\overset{\circ}{W}{}_2^1$ is representable as the orthogonal sum

$$\overset{\circ}{W}{}_2^1 = \overset{\circ}{S}{}^1_{\Delta^k,2} \oplus (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$$

in the sense of the inner product in $\overset{\circ}{W}{}_2^1$,

$$\langle u, v\rangle_1 = \langle \nabla u, \nabla v\rangle_0 = \sum_{\nu=1}^n \int_G (\nabla u_\nu(x), \nabla v_\nu(x))\, dx.$$

First the subspace orthogonal to $\mathring{S}^1_{\Delta^k,2}$ will be described. To this end one introduces the following subspace of the scalar Sobolev space $W_2^{2k} = W_2^{2k}(G;\mathbb{C})$:

$$\mathring{W}^{2k}_{\Delta^k,2} = \{p_0 \in W_2^{2k} : \Delta^\kappa p_0|_{\partial G} = 0,\ \nabla\Delta^\kappa p_0|_{\partial G} = 0 \text{ for } 0 \le \kappa \le k-1\}.$$

It is the closure in the W_2^{2k} norm of all scalar functions $p \in C^\infty(\overline{G};\mathbb{C})$ satisfying the stated boundary conditions.

Remark 1. It is clear that

$$\mathring{W}^{2k}_{\Delta^k,2} = \{p_0 \in W_2^{2k} : \Delta^\kappa p_0 \in \mathring{W}^2_2 \text{ for } 0 \le \kappa \le k-1\}.$$

Remark 2. As the boundary conditions are selfadjoint the bilinear form

$$(1)\qquad \langle \Delta^{2k} p_1, p_0\rangle = \langle \Delta^k p_1, \Delta^k p_0\rangle_0 = \langle p_1, \Delta^{2k} p_0\rangle, \qquad p_1, p_0 \in \mathring{W}^{2k}_{\Delta^k,2},$$

is well-posed and selfadjoint. By the classical Lax–Milgram lemma (see e.g. [11]), the map

$$(2)\qquad \Delta^{2k} : \mathring{W}^{2k}_{\Delta^k,2} \leftrightarrow (\mathring{W}^{2k}_{\Delta^k,2})^*$$

is an isometric isomorphism.

Lemma 1. *Let $q \in (\mathring{S}^1_{\Delta^k,2})^\perp$. Then the boundary value problems*

$$(3)\qquad \nabla\Delta^{k-1} p_0 = q \quad \text{in } G, \qquad p_0 \in \mathring{W}^{2k}_{\Delta^k,2},$$

and

$$(4)\qquad \Delta^{2k} p_0 = \operatorname{div}\Delta^k q \quad \text{in } G, \qquad p_0 \in \mathring{W}^{2k}_{\Delta^k,2},$$

are equivalent.

Proof. (i) If p_0 is a solution to (3) then by differentation, because of $\operatorname{div}\Delta^k(\nabla\Delta^{k-1}) = \Delta^{2k}$, the function p_0 also solves (4).

(ii) Conversely, let p_0 be a generalized solution to (4) in the sense of the duality (2). Then from (4) follows

$$\nabla\Delta^{k-1} p_0 - q \in \ker\operatorname{div}\Delta^k,$$

i.e.

$$(5)\qquad \nabla\Delta^{k-1} p_0 - q \in \mathring{S}^1_{\Delta^k,2}.$$

On the other hand, for any $p_0 \in \mathring{W}^{2k}_{\Delta^k,2}$ and arbitrary $\varphi \in \mathring{S}^1_{\Delta^k,2}$ the relation

$$\langle \operatorname{div}\Delta^k\varphi, p_0\rangle = -\langle \varphi, \nabla\Delta^k p_0\rangle_0 = \langle \varphi, \nabla\Delta^{k-1} p_0\rangle_1$$

holds in the sense of the duality (2). The condition $\operatorname{div}\Delta^k\varphi = 0$ in the sense of (2) means that $\nabla\Delta^{k-1}p_0 \in (\mathring{S}^1_{\Delta^k,2})^\perp$. Thus as $q \in (\mathring{S}^1_{\Delta^k,2})^\perp$, also

$$(6)\qquad \nabla\Delta^{k-1} p_0 - q \in (\mathring{S}^1_{\Delta^k,2})^\perp.$$

Together with (5) this implies $\nabla\Delta^{k-1} p_0 - q = 0$, i.e. p_0 solves (3).

THEOREM 1. *The operator $\nabla\Delta^{k-1}$ establishes an elliptic isomorphism between $\overset{\circ}{W}{}^{2k}_{\Delta^k,2}$ and $(\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$,*

$$\nabla\Delta^{k-1} : \overset{\circ}{W}{}^{2k}_{\Delta^k,2} \leftrightarrow (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp.$$

This means that for any $p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2}$ the vector-valued function $\nabla\Delta^{k-1}p_0$ is orthogonal to the subspace $\overset{\circ}{S}{}^1_{\Delta^k,2}$ and conversely if $q \in (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$ then there exists a unique potential $p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2}$ such that $q = \nabla\Delta^{k-1}p_0$. Moreover, there exists some constant $M > 0$ such that

$$\|p_0\|_{\overset{\circ}{W}{}^{2k}_{\Delta^k,2}} \le M\|\operatorname{div}\Delta^k q\|_{(\overset{\circ}{W}{}^{2k}_{\Delta^k,2})^*}. \tag{7}$$

Proof. (i) Let $p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2}$. Then as shown in step (ii) of the preceding proof, $\nabla\Delta^{k-1}p_0 \in (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$.

(ii) Let $q \in (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$. Consider the boundary value problem (3). By Lemma 1 it is equivalent to problem (4). The latter problem is uniquely solvable. Its solution satisfies (3).

Finally, the estimate (7) follows from the isometric isomorphism (2).

As a corollary one has

THEOREM 2. *For any $k \ge 1$ the space $\overset{\circ}{W}{}^1_2$ can be decomposed as*

$$\overset{\circ}{W}{}^1_2 = \overset{\circ}{S}{}^1_{\Delta^k,2} \oplus \nabla\Delta^{k-1}(\overset{\circ}{W}{}^{2k}_{\Delta^k,2}). \tag{8}$$

EXAMPLE. For $k = 1$,

$$\overset{\circ}{W}{}^2_{\Delta,2} = \{p_0 \in W^2_2 : p_0|_{\partial G} = 0,\ \nabla p_0|_{\partial G} = 0\}$$

is the usual Sobolev space $\overset{\circ}{W}{}^2_2$ and formula (8) becomes

$$\overset{\circ}{W}{}^1_2 = \overset{\circ}{S}{}^1_{\Delta,2} \oplus \nabla(\overset{\circ}{W}{}^2_2).$$

This means that every function $u \in \overset{\circ}{W}{}^1_2$ can be represented as the sum

$$u = u_{\Delta,s} + \nabla p_0$$

where $(\operatorname{div}\Delta)u_{\Delta,s} = 0$ and $p_0 \in \overset{\circ}{W}{}^2_2$.

REMARK 3. From the representation

$$u = u_{\Delta^k,s} + \nabla\Delta^{k-1}p_0$$

corresponding to the decomposition (8) it is seen that p_0 is the solution to the boundary value problem

$$\Delta^{2k}p_0 = \operatorname{div}\Delta^k u \quad \text{in } G, \qquad \Delta^\kappa p_0 \in \overset{\circ}{W}{}^2_2 \quad \text{for } 0 \le \kappa \le k-1.$$

Therefore the map $u \mapsto \nabla\Delta^{k-1}p$ is the projector $\nabla\Delta^{k-1} : \overset{\circ}{W}{}^1_2 \to (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$.

3. Orthogonality to the Δ^k-solenoidal subspaces in the complex case. In this section the above results are extended to the case of several complex variables. Let G be a domain in $\mathbb{C}^n$, $n \geq 2$. It will be identified with $G \subset \mathbb{R}^{2n}$ of the real variables $\{x_\nu, y_\nu : 1 \leq \nu \leq n\}$ so that $z = (z_1, \ldots, z_n)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z_\nu = x_\nu + iy_\nu$, $1 \leq \nu \leq n$. The boundary ∂G of the domain $G \subset \mathbb{R}^{2n}$ is supposed to be a smooth hypersurface. Further, as usual, the basic complex differential operators are denoted by

$$\partial_{z_\nu} = \tfrac{1}{2}(\partial_{x_\nu} - i\partial_{y_\nu}), \qquad \partial_{\bar{z}_\nu} = \tfrac{1}{2}(\partial_{x_\nu} + i\partial_{y_\nu}), \qquad 1 \leq \nu \leq n.$$

With the same notations as in the real case, one defines the Sobolev function spaces of complex vector-valued functions $u = (u_1, \ldots, u_n)$ vanishing at the boundary ∂G:

$$\begin{aligned}\overset{\circ}{W}{}^1_2 = \Big\{u \in W^1_2(G; \mathbb{C}^n) : \|u\|_1^2 = \int_G |\nabla_z u(z)|^2 \, dx \, dy \\ = \sum_{\mu,\nu=1}^n \int_G |\partial_{z_\nu} u_\mu(z)|^2 \, dx \, dy < \infty, \ u|_{\partial G} = 0\Big\}.\end{aligned}$$

$\nabla_z = (\partial_{z_1}, \ldots, \partial_{z_n})$ is the complex gradient with respect to the variable z. Analogously $\nabla_{\bar{z}} = (\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$. The space $\overset{\circ}{W}{}^1_2$ endowed with the inner product

$$\begin{aligned}\langle u, v\rangle_1 &= \int_G (\nabla_z u(z), \overline{\nabla_z v(z)}) \, dx \, dy = \sum_{\mu,\nu=1}^n \int_G \partial_{z_\nu} u_\mu(z) \partial_{\bar{z}_\nu} \overline{v_\mu(z)} \, dx \, dy \\ &= \sum_{\nu=1}^n \int_G (\nabla_{z_\nu} u_\nu(z), \overline{\nabla_{z_\nu} v_\nu(z)}) \, dx \, dy\end{aligned}$$

becomes a Hilbert space. Let $\Delta = \frac{1}{4}\sum_{\nu=1}^n \partial_{z_\nu}\partial_{\bar{z}_\nu}$ denote the Laplace operator and $\operatorname{div}_z u = \sum_{\nu=1}^n \partial_{z_\nu} u_\nu$ the divergence operator with respect to the variable z, both understood in the sense of $\mathcal{D}'(G)$ or in the sense of any duality $\langle\cdot,\cdot\rangle$ which is an extension of the usual scalar product in L_2. The set

$$\overset{\circ}{S}{}^1_{\Delta^k,2} = \{u \in \overset{\circ}{W}{}^1_2 : \operatorname{div}_z \Delta^k u = 0\}$$

is a closed subspace of $\overset{\circ}{W}{}^1_2$ for any integer $k \geq 1$.

Remark 4. Obviously $\operatorname{div}_z \Delta^k w = 0$ if and only if

$$\operatorname{div}_x \Delta^k u - \operatorname{div}_y \Delta^k v = 0, \qquad \operatorname{div}_y \Delta^k u + \operatorname{div}_x \Delta^k v = 0$$

for $w = u + iv$. Hence, the subspaces $\overset{\circ}{S}{}^1_{\Delta^k,2}$ are not empty.

As usual for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ the notation $\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_n}^{\alpha_n}$

is used. Then

$$\overset{\circ}{W}{}^{2k}_{\Delta^k,2} = \Big\{ p_0 \in W_2^{2k}(G;\mathbb{C}) : \|p_0\|^2_{2k} = \sum_{|\alpha|\le 2k} \int_G |\partial_z^\alpha p_0(z)|^2 \, dx\, dy < \infty,$$
$$\Delta^\kappa p_0|_{\partial G} = 0,\ \nabla_z \Delta^\kappa p_0|_{\partial G} = 0,\ \text{for } 0 \le \kappa \le k-1 \Big\}$$

is the space of scalar "potentials".

THEOREM 2′. *For any integer $k \ge 1$ the space $\overset{\circ}{W}{}^1_2$ has the orthogonal decomposition*

$$\overset{\circ}{W}{}^1_2 = \overset{\circ}{S}{}^1_{\Delta^k,2} \oplus \nabla_{\bar z} \Delta^{k-1} \overset{\circ}{W}{}^{2k}_{\Delta^k,2}.$$

Proof. The proof is like the one of Theorem 2.

(i) If a function $q \in \overset{\circ}{W}{}^1_2$ has the form $q = \nabla_{\bar z}\Delta^{k-1}p_0$ for some scalar potential $p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2}$, then for any $\varphi \in \overset{\circ}{S}{}^1_{\Delta^k,2}$ one has

$$\langle \operatorname{div}_z \Delta^k \varphi, p_0 \rangle = -\langle \varphi, \nabla_{\bar z} \Delta^k p_0 \rangle_0 = \langle \varphi, \nabla_{\bar z}\Delta^{k-1} p_0\rangle_1 .$$

The condition $\operatorname{div}_z \Delta^k \varphi = 0$ then implies that $\nabla_{\bar z}\Delta^{k-1}p_0$ is orthogonal to $\overset{\circ}{S}{}^1_{\Delta^k,2}$.

(ii) Let $q \in (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$. Then there exists a unique potential $p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2}$ such that $q = \nabla_{\bar z}\Delta^{k-1}p_0$, i.e. the boundary value problem

$$\nabla_{\bar z}\Delta^{k-1} p_0 = q, \qquad p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2},$$

has a unique solution. In fact, repeating the proof of Lemma 1 one can establish that for $q \in (\overset{\circ}{S}{}^1_{\Delta^k,2})^\perp$ this problem is equivalent to the well-posed boundary value problem

$$\Delta^{2k} p_0 = \operatorname{div}_z \Delta^k q, \qquad p_0 \in \overset{\circ}{W}{}^{2k}_{\Delta^k,2}.$$

The unique solution p_0 of this problem is the desired potential p_0.

4. Orthogonality to the solenoidal subspace. In this section the orthogonal complement of the subspace of all solenoidal functions

$$\overset{\circ}{S}{}^1_2 = \{ u_s \in \overset{\circ}{W}{}^1_2 : \operatorname{div} u_s = 0 \text{ in } \mathcal{D}' \}$$

in $\overset{\circ}{W}{}^1_2$ is found. The subspace $\overset{\circ}{S}{}^1_2$ is closed in $\overset{\circ}{W}{}^1_2$. In order to describe $(\overset{\circ}{S}{}^1_2)^\perp$ one needs the operator

$$\Delta_0^{-1} : W_2^{-1} \to \overset{\circ}{W}{}^1_2$$

where $W_2^{-1} = (\overset{\circ}{W}{}^1_2)^*$ and Δ_0^{-1} is the inverse operator to $\Delta : \overset{\circ}{W}{}^1_2 \to W_2^{-1}$.

LEMMA 2. *The equations*

$$\Delta_0^{-1} \nabla p_1 = q_1 \tag{9}$$

and

$$\operatorname{div} \Delta_0^{-1} \nabla p_1 = \operatorname{div} q_1 \tag{10}$$

are equivalent in $\mathring{W}_2^1$ *if and only if* $q_1 \in (\mathring{S}_2^1)^\perp$.

Proof. (i) Applying the div operator to (9) shows that any solution to (9) is a solution of (10).

(ii) Let $p_1 \in L_2(G)$ be a solution to (10). Then

$$\Delta_0^{-1} \nabla p_1 - q_1 \in \ker(\operatorname{div}) = \mathring{S}_2^1.$$

On the other hand for any $p_1 \in L_2$ and any $\varphi \in \mathring{S}_2^1$,

$$\langle \Delta_0^{-1} \nabla p_1, \varphi \rangle_1 = -\langle \nabla p_1, \varphi \rangle_0 = \langle p_1, \operatorname{div} \varphi \rangle_0 = 0.$$

Hence, if $q_1 \in (\mathring{S}_2^1)^\perp$ then $\Delta_0^{-1} \nabla p_1 - q_1 \in (\mathring{S}_2^1)^\perp$. Thus $\Delta_0^{-1} \nabla p_1 = q_1$.

LEMMA 3. *Equation* (10) *is solvable for any* $q_1 \in \mathring{W}_2^1$. *More precisely, for any* $q_1 \in \mathring{W}_2^1$ *there exists a unique potential* $p_1 \in L_2/\mathbb{C}$ *such that* (10) *holds.*

Proof. Identifying as usual the factor space $L_2/\mathbb{C}$ with the subspace of L_2 of all functions orthogonal to the unity, a solution to (10) in this subspace is found by the Galerkin method. Let $\{v_\nu : \nu \geq 1\}$ be a basis of $L_2/\mathbb{C}$ with smooth functions v_ν. The approximate solutions p_1^N, $N \geq 1$, are defined as

$$p_1^N = \sum_{\nu=1}^{N} c_\nu^N v_\nu$$

where the unknown coefficients c_ν^N, $1 \leq \nu \leq N$, are defined from the Galerkin moment equations

$$\langle \operatorname{div} \Delta_0^{-1} \nabla p_1^N, v_\nu \rangle_0 = \langle \operatorname{div} q_1, v_\nu \rangle_0, \qquad \nu = 1, \ldots, N,$$

or, what is the same, from

$$\langle \Delta_0^{-1} \nabla p_1^N, \nabla v_\nu \rangle_0 = -\langle \operatorname{div} q_1, v_\nu \rangle_0, \qquad 1 \leq \nu \leq N. \tag{10_N}$$

The solvability of this algebraic linear system follows from the a priori estimate which will be deduced next. Multiplying (10_N) by c_ν^N and summing up gives

$$\langle \Delta_0^{-1} \nabla p_1^N, \nabla p_1^N \rangle_0 = -\langle \operatorname{div} q_1, p_1^N \rangle_0 \tag{11}$$

or

$$\|\nabla p_1^N\|_{-1}^2 = -\langle \operatorname{div} q_1, p_1^N \rangle_0.$$

Therefore, using the Cauchy–Schwarz–Bunyakovskiĭ inequality and the known inequality

$$\|p_1^N\|_0 \leq M \|\nabla p_1^N\|_{-1} \tag{12}$$

with some $M > 0$ for $p_1^N \in L_2/\mathbb{C}$ (Ladyzhenskaya–Babuška–Brezzi–Nečas, see [12], §1, p. 17) from (11) for $N \geq 1$ the estimate

$$\|\nabla p_1^N\|_{-1} \leq M\|\operatorname{div} q_1\|_0 \tag{13}$$

follows. By (12), for any $N \geq 1$,

$$\|p_1^N\|_0 \leq M^2\|\operatorname{div} q_1\|_0. \tag{14}$$

In particular the constant $M > 0$ does not depend on N. These estimates imply the weak compactness of the sequence (p_1^N) in $L_2/\mathbb{C}$. Assuming without loss of generality that the sequence itself converges weakly to some $p_1 \in L_2/\mathbb{C}$, then also (∇p_1^N) weakly converges in W_2^{-1} to ∇p_1. Obviously, p_1 is a solution of (10). The uniqueness is obvious too.

THEOREM 3. *The map*

$$\Delta_0^{-1}\nabla : L_2/\mathbb{C} \leftrightarrow (\mathring{S}_2^1)^\perp$$

is an elliptic isomorphism, *and*

$$\mathring{W}_2^1 = \mathring{S}_2^1 \oplus \Delta_0^{-1}\nabla(L_2/\mathbb{C}).$$

Proof. It has to be shown that for any $p_1 \in L_2/\mathbb{C}$ the image $q_1 \equiv \Delta_0^{-1}\nabla p_1$ is orthogonal to $\mathring{S}_2^1$ and that on the other hand for any $q_1 \in (\mathring{S}_2^1)^\perp$ there exists a (unique) potential $p_1 \in L_2/\mathbb{C}$ such that $\Delta_0^{-1}\nabla p_1 = q_1$. Moreover,

$$\|p_1\|_0 \leq M\|\operatorname{div} q_1\|_0 \tag{15}$$

for some constant $M \geq 0$.

(i) For any $p_1 \in L_2(G)$ and any $\varphi \in \mathring{S}_2^1$,

$$\langle \Delta_0^{-1}\nabla p_1, \varphi\rangle_1 = -\langle \nabla p_1, \varphi\rangle_0 = \langle p_1, \operatorname{div}\varphi\rangle_0 = 0.$$

Thus $\Delta_0^{-1}\nabla p_1 \in (\mathring{S}_2^1)^\perp$. This argument was already used in the proof of Lemma 2.

(ii) Let $q \in (\mathring{S}_2^1)^\perp$. It has to be shown that there exists a function $p_1 \in L_2/\mathbb{C}$ satisfying (9) in $\mathring{W}_2^1$. Indeed, from Lemma 3 combined with Lemma 2 a solution $p_1 \in L_2/\mathbb{C}$ can be obtained. Finally, if $\operatorname{div} q_1 = 0$ then

$$\operatorname{div} \Delta_0^{-1}\nabla p_1 = 0$$

immediately yields $\langle \nabla p_1, \nabla p_1\rangle_{-1} = 0$, so that $\nabla p_1 = 0$ and hence $p_1(x) = 0$. This could also be seen from the estimate (15).

The a priori estimate (15) follows from inequality (13) together with (14).

REMARK 5. If $u \in L_2(G)$ satisfies $\operatorname{div} u = \operatorname{div} q_1$ then the equation

$$\operatorname{div} \Delta_0^{-1}\nabla p_1 = \operatorname{div} u$$

has the same solution p_1 for all such u. This means that the mapping $u \mapsto p_1$ defines a projection $u \mapsto \Delta_0^{-1}\nabla p_1$ of $\mathring{W}_2^1$ onto $(\mathring{S}_2^1)^\perp$.

5. Orthogonal decomposition of Δ^k-solenoidal subspaces. Next a detailed representation of the subspace of the Δ-solenoidal functions and of the Δ^k-solenoidal functions for any positive integer k are given. According to Theorem 2 every function $u \in \mathring{S}^1_{\Delta,2} \ominus \mathring{S}^1_2$ can be written as $u = \Delta_0^{-1}\nabla p_1$ for some $p_1 \in L_2/\mathbb{C}$. However, in this case p_1 is a *harmonic potential*, i.e. $\Delta p_1 = 0$ in $\mathcal{D}'$.

THEOREM 4. *Denote the subspace of harmonic potentials in $L_2(G)$ by H_1. Then*

$$\mathring{S}^1_{\Delta,2} = \mathring{S}^1_2 \oplus \Delta_0^{-1}\nabla(H_1/\mathbb{C}).$$

Proof. (i) If $p_1 \in H_1 \subset L_2$ then of course $q_1 = \Delta_0^{-1}\nabla p_1$ is orthogonal to $\mathring{S}^1_2$ (see the proof of Theorem 2).

(ii) If $q_1 \in (\mathring{S}^1_2)^\perp$ and u is a solution of $\operatorname{div}\Delta u = 0$ satisfying $\operatorname{div} u = \operatorname{div} q_1$, then the corresponding potential p_1, satisfying $\Delta_0^{-1}\nabla p_1 = q_1$, is an L_2-solution to the equation

$$\operatorname{div}\Delta_0^{-1}\nabla p_1 = \operatorname{div} u$$

(see Remark 5). Applying the Laplace operator Δ to this equation leads to $\Delta p_1 = 0$. Hence, $p_1 \in H_1$.

DEFINITION 1. A potential p_0 is called *polyharmonic* of order $k \geq 1$ if it satisfies the equation $\Delta^k p_0 = 0$ in $\mathcal{D}'$. The set of all polyharmonic potentials of order k is denoted by H_k.

THEOREM 5. *For any positive integer k,*

$$\mathring{S}^1_{\Delta^{k+1},2} = \mathring{S}^1_{\Delta^k,2} \oplus \nabla\Delta^{k-1}(\mathring{W}^{2k}_{\Delta^k,2} \cap H_{2k+1}).$$

In other words, a function $u \in \mathring{W}^1_2$ belongs to the subspace $\mathring{S}^1_{\Delta^{k+1},2} \ominus \mathring{S}^1_{\Delta^k,2}$ if and only if it has the form $u = \nabla\Delta^{k-1}p_0$ with a potential $p_0 \in \mathring{W}^{2k}_{\Delta^k,2}$ which is polyharmonic of order $2k+1$, i.e. $\Delta^{2k+1}p_0 = 0$.

Proof. (i) For any $p_0 \in \mathring{W}^{2k}_{\Delta^k,2} \cap H_{2k+1}$ the function $\nabla\Delta^{k-1}p_0$ is obviously orthogonal to $\mathring{S}^1_{\Delta^k,2}$.

(ii) Let $u \in \mathring{W}^1_2$ be such that $\operatorname{div}\Delta^k u \neq 0$ but $\operatorname{div}\Delta^{k+1}u = 0$. Then in accordance with Theorem 2,

$$u = u_{\Delta^k,s} + q$$

with $u_{\Delta^k,s} \in \mathring{S}^1_{\Delta^k,2}$ and $q = \nabla\Delta^{k-1}p_0$ where the potential $p_0 \in \mathring{W}^{2k}_{\Delta^k,2}$ is the solution to the problem

$$(16) \qquad \Delta^{2k}p_0 = \operatorname{div}\Delta^k u, \qquad \Delta^\nu p_0 \in \mathring{W}^2_2, \qquad \nu = 0,1,\ldots,k-1.$$

Applying the Laplace operator Δ yields $\Delta^{2k+1}p_0 = 0$ in $\mathcal{D}'$.

COROLLARY 1. *Any function* $u \in \mathring{W}_2^1$ *can be written as*

$$u = u_s + \Delta_0^{-1} p_{-1} + \sum_{\nu=0}^{k-1} \nabla \Delta^\nu p_\nu + \nabla \Delta^k q_k$$

where p_{-1} *is a harmonic potential and* p_ν, $0 \le \nu \le k-1$, *are polyharmonic potentials of order* $2\nu+1$ *and* q_k *is the unique solution to* (15) *where* k *is replaced by* $k+1$.

6. Harmonic decompositions. For $p > 1$ and nonnegative integer m let

$$W_p^m = \Big\{ u : G \to \mathbb{C}^n : \|u\|_{m,p}^p = \sum_{|\alpha| \le m} \int_G |D^\alpha u(x)|^p \, dx < \infty \Big\}$$

be the Sobolev space for a bounded domain $G \subset \mathbb{R}^n$ with smooth boundary. Further let $H_p^m \subset W_p^m$ be the subspace of harmonic functions,

$$H_p^m = \{ u \in W_p^m : \Delta u = 0 \text{ in } \mathcal{D}' \}.$$

THEOREM 6. *The space* W_p^m *can be decomposed into the direct sum*

$$W_p^m = H_p^m \dot{+} \Delta(W_p^{m+2} \cap \mathring{W}_p^2),$$

i.e. $u = u_h + \Delta p_0$ *with* $u_h \in H_p^m$, $p_0 \in W_p^{m+2} \cap \mathring{W}_p^2$. *For* $p = 2$ *this is an orthogonal decomposition.*

Proof. For given $u \in W_p^m$ let $p_0 \in W_p^{m+2} \cap \mathring{W}_p^2$ be the solution of the boundary value problem

$$\Delta^2 p_0 = \Delta u \quad \text{in } G, \qquad p_0 = 0, \ \nabla p_0 = 0 \quad \text{on } \partial G.$$

This is a well-posed boundary value problem in the whole scale W_p^m: for any $u \in W_p^m$ with $m \ge 0$, $p > 1$ the problem has a unique solution $p_0 \in W_p^{m+2} \cap \mathring{W}_p^2$. Then

$$u = u_h + \Delta p_0$$

where u_h is defined by this relation. It is clear that $u_h \in H_p^m$ because

$$\Delta u_h = \Delta u - \Delta^2 p_0 = 0$$

due to the choice of p_0.

It remains to show that the decomposition is direct, that is, u_h and p_0 are uniquely determined. Let $0 = u_h + \Delta p_0$ with $u_h \in H_p^m$ and $p_0 \in W_p^{m+2} \cap \mathring{W}_p^2$. Then the function $v = \Delta p_0 = -u_h$ is harmonic as u_h is. Thus $\Delta v = \Delta^2 p_0 = 0$. As $p_0 \in \mathring{W}_p^2$, i.e. $p_0 = 0$ and $\nabla p_0 = 0$ on ∂G, it follows that $p_0 = 0$. Hence, also $u_h = 0$. This shows the sum is direct.

Remark 6. From the boundary value problem one has the a priori estimates

$$\|p_0\|_{W_p^{m+2}} \le M\|\Delta u\|_{W_p^{m-2}}.$$

Example. For $p = 2$ and $m = 0$ the last decomposition reads

$$L_2 = H_2 \oplus \Delta \mathring{W}_2^2. \tag{17}$$

This orthogonal decomposition will be applied to solving the following variational problem.

Variational Problem. *Let $0 < a_0 \le A_0$ be constants and a be measurable in G such that*

$$a_0 \le a(x) \le A, \qquad x \in G.$$

Moreover, let $h \in L_2$. Determine

$$\inf_{u \in H_2} \operatorname{Re} \int_G \left[\tfrac{1}{2} a(x)|u(x)|^2 - h(x)u(x)\right] dx.$$

If $u \in H_2$ is minimal then

$$\int_G [a(x)u(x) - h(x)]\overline{\varphi(x)}\, dx = 0 \tag{18}$$

for all $\varphi \in H_2$. In accordance with the decomposition (17) this leads to the next problem.

Problem. *Find $u \in H_2$ and $v \in \mathring{W}_2^2$ such that $au + \Delta v = h$ in the sense of L_2 and $\Delta u = 0$.*

Proposition 1. *For any $h \in L_2$ there exists a unique pair (u, v) solving the above Problem.*

Proof. (i) In order to find $u \in H_2$ such that

$$\langle au, \varphi\rangle_0 = \langle h, \varphi\rangle \qquad \text{for all } \varphi \in H_2$$

the standard Galerkin method is used. Let $\{v_k : k \in \mathbb{N}\}$ be a basis of H_2. An approximate solution

$$u^N = \sum_{j=1}^{N} c_j^N v_j, \qquad N = 1, 2, \ldots,$$

is determined from the Galerkin moment equations

$$\langle au^N, v_j\rangle_0 = \langle h, v_j\rangle_0, \qquad 1 \le j \le N, \tag{19}$$

fixing the coefficients c_j^N, $1 \le j \le N$. Multiplying (19) by c_j^N and summing up gives

$$\langle au^N, u^N\rangle_0 = \langle h, u^N\rangle_0,$$

leading to

$$a_0\|u^N\|_0^2 \le \langle h, u^N\rangle_0 \le \|h\|_0\|u^N\|_0$$

and hence to the a priori estimate

$$\|u^N\|_0 \le \frac{1}{a_0}\|h\|_0, \qquad N = 1, 2, \dots$$

Therefore the sequence (u^N) of approximate solutions has at least one weak limit point $u \in H_2$. This function u is a solution of the Problem.

(ii) The function $v \in \mathring{W}_2^2$ is found from the decomposition (17). Identity (18) means $au - h \in H_2^\perp$ in L_2. Then from (17) it is seen that there exists a unique $v \in \mathring{W}_2^2$ such that

$$au + \Delta v = h \quad \text{ in } L_2.$$

To prove the uniqueness of this solution to the Problem, let $h = 0$. Multiplying $au + \Delta v = 0$ by $u \in H_2$ and integrating over G gives $\langle au, u\rangle_0 = 0$ because Δv is orthogonal to u in L_2. Thus $u = 0$ as also follows directly from the a priori estimate. Hence, $v = 0$ follows.

7. Applications. The spaces $\mathring{W}_2^1$ and W_2^{-1} are connected by the one-to-one map

$$\Delta : \mathring{W}_2^1 \leftrightarrow W_2^{-1}.$$

Hence, in the sense of the inner product

$$\langle u, v\rangle_{-1} = \langle \Delta_0^{-1}u, v\rangle_0 = \langle u, \Delta_0^{-1}v\rangle_0$$

the orthogonal decompositions of the space W_2^{-1} are automatically obtained from those of $\mathring{W}_2^1$. In particular, according to Section 2, one has the chain of decompositions

$$W_2^{-1} = \Delta\mathring{S}^1_{\Delta^k,2} \oplus \nabla\Delta^k\mathring{W}^{2k}_{\Delta^k,2}, \qquad k = 1, 2, \dots$$

Thus, any function $h \in W_2^{-1}$ can be represented as

$$h = \Delta u_{\Delta^k,s} + \nabla\Delta^k p_k,$$

where $\operatorname{div}\Delta^k u_{\Delta^k,s} = 0$ in $\mathcal{D}'$ and $p_k \in \mathring{W}^{2k}_{\Delta^k,2}$. This suggests the following problem.

Boundary Value Problem 1. *Let $h \in W_2^{-1}$. Find functions $u_{\Delta^k,s}, p_k$ satisfying*

$$\Delta u_{\Delta^k,s} + \nabla\Delta^k p_k = h, \quad \operatorname{div}\Delta^k u_{\Delta^k,s} = 0 \quad \text{ in } G,$$
$$u_{\Delta^k,s}|_{\partial G} = 0, \quad \Delta^\nu p_k|_{\partial G} = 0, \quad \nabla\Delta^\nu p_k|_{\partial G} = 0, \quad \nu = 0, 1, \dots, k-1.$$

For any k this problem is well-posed and the solution $u_{\Delta^k,s}, p_k$ can be found from the decompositions given in Section 2.

In connection with Sections 4 and 5, one also has the chain of decompositions

$$W_2^{-1} = \Delta\mathring{S}_2^1 + \nabla[H_1/\mathbb{C} \oplus \Delta\mathring{W}_2^2], \tag{20}$$

$$W_2^{-1} = \Delta\mathring{S}_2^1 + \nabla[H_1/\mathbb{C} + \Delta(\mathring{W}_2^2 \cap H_3 \oplus \Delta\mathring{W}_{\Delta^2,2}^4)] \tag{21}$$

etc. Hence, any $h \in W_2^{-1}$ can be represented as

$$h = \Delta u_s + \nabla(p_{-1} + \Delta p_0), \tag{20$'$}$$

$$h = \Delta u_s + \nabla[p_{-1} + \Delta(p_0 + \Delta p_1)] \tag{21$'$}$$

etc. Any of these representations again corresponds to a boundary value problem.

Boundary Value Problem 2. *For given $h \in W_2^{-1}$ find three functions u_s, p_{-1}, p_0 such that*

$$\begin{aligned} &\Delta u_s + \nabla(p_{-1} + \Delta p_0) = h, \quad \operatorname{div} u_s = 0, \quad \Delta p_{-1} = 0 \quad \text{in } G,\\ &u_s|_{\partial G} = 0, \quad p_0|_{\partial G} = 0, \quad \nabla p_0|_{\partial G} = 0. \end{aligned} \tag{20$''$}$$

Boundary Value Problem 3. *For given $h \in W_2^{-1}$ find four functions u_s, p_{-1}, p_0, p_1 satisfying*

$$\begin{aligned} &\Delta u_s + \nabla(p_{-1} + \Delta p_0 + \Delta^2 p_1) = h,\\ &\operatorname{div} u_s = 0, \quad \Delta p_{-1} = 0, \quad \Delta^3 p_0 = 0 \quad \text{in } G,\\ &u_s|_{\partial G} = 0, \quad p_0|_{\partial G} = 0, \quad \nabla p_0|_{\partial G} = 0,\\ &p_1|_{\partial G} = 0, \quad \nabla p_1|_{\partial G} = 0, \quad \Delta p_1|_{\partial G} = 0, \quad \nabla\Delta p_1|_{\partial G} = 0. \end{aligned} \tag{21$''$}$$

All these problems are well-posed. The solutions can be found by using the decompositions of W_2^{-1} given in (20), (21). Obviously, these problems are some versions of the classical Stokes problem

$$\begin{aligned} &\Delta u_s + \nabla p = h, \quad \operatorname{div} u_s = 0 \quad \text{in } G,\\ &u_s|_{\partial G} = 0, \end{aligned}$$

revealing the structure of the potential $p \in L_2/\mathbb{C}$ in more detail. Contrary to the classical procedure of finding the divergence part u_s, here the potential is found first e.g. in the form $p = p_{-1} + \Delta p_0$ by solving the problems

$$\Delta^2 p_0 = \operatorname{div} h \quad \text{in } G, \qquad p_0 \in \mathring{W}_2^2,$$

and

$$\operatorname{div}(\Delta_0^{-1} p_{-1}) = \operatorname{div}(\Delta_0^{-1} h) \quad \text{in } G, \qquad p_1 \in L_2/\mathbb{C}.$$

In the second step the divergence part u_s can be found as the solution to the Dirichlet problem

$$\Delta u_s = h - \nabla p \quad \text{in } G, \qquad u_s|_{\partial G} = 0$$

(compare [4, 5]).

REFERENCES

[1] H. Begehr, *Orthogonal decompositions of the function space* $L_2(\overline{D};\mathbb{C})$, preprint, FU Berlin, 2000.

[2] H. Begehr and Yu. Dubinskiĭ, *Orthogonal decompositions of Sobolev spaces in Clifford analysis*, Ann. Mat. Pura Appl., to appear.

[3] S. Bernstein, *Elliptic boundary value problems in unbounded domains*, in: Clifford Algebras Appl. in Math. Phys., F. Brackx *et al.* (eds.), Kluwer, Dordrecht, 1993, 45–53.

[4] Yu. Dubinskiĭ, *Some decompositions of Sobolev spaces and Stokes type problems*, Dokl. Ross. Akad. Nauk 373 (2000), 727–730 (in Russian).

[5] —, *Some orthogonal decompositions of the Sobolev spaces* $\overset{\circ}{W}{}_2^1$ *and* W_2^{-1} *and their applications to the Stokes problem*, ibid. 374 (2000), 13–16 (in Russian).

[6] K. Gürlebeck, U. Kähler, J. Ryan, and W. Sprößig, *Clifford analysis over unbounded domains*, Adv. in Appl. Math. 19 (1997), 216–239.

[7] K. Gürlebeck and W. Sprößig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Akademie-Verlag, Berlin, 1989; Birkhäuser, Basel, 1990.

[8] —, —, *Quaternionic and Clifford Calculus for Engineers and Physicists*, Wiley, Chichester, 1997.

[9] U. Kähler, *Elliptic boundary value problems in bounded and unbounded domains*, in: Dirac Operators in Analysis, J. Ryan and D. Stuppa (eds.), Addison-Wesley Longman, Harlow, 1998, 122–140.

[10] —, *On a direct decomposition of the space* $L_p(\Omega)$, Z. Anal. Anwendungen 18 (1999), 839–848.

[11] M. Renardy and R. Rogers, *An Introduction to Partial Differential Equations*, 2nd corr. printing, Springer, New York, 1996.

[12] R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.

[13] J. Witte, *A reflection principle and its applications*, in: Complex Methods for Partial Differential Equations, H. Begehr *et al.* (eds.), Kluwer, Dordrecht, 1999, 1–19.

[14] —, *A reflection principle and an orthogonal decomposition in Clifford algebras*, preprint, FU Berlin, 1998.

I. Math. Institut
Freie Universität Berlin
Arnimallee 3
D-14195 Berlin, Germany
E-mail: begehr@math.fu-berlin.de

Moscow Power Engineering Institute
Krasnokazarmennaja 14
Moscow 111250, Russia
E-mail: dubinskii@mm.mpei.ac.ru

Received 27 November 2000 (4000)