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ON THE SET REPRESENTATION OF AN ORTHOMODULAR POSET

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Abstract. Let P be an orthomodular poset and let B be a Boolean subalgebra of P. A mapping $s : P \to \langle 0,1\rangle$ is said to be a centrally additive B-state if it is order preserving, satisfies $s(a') = 1 - s(a)$, is additive on couples that contain a central element, and restricts to a state on B. It is shown that, for any Boolean subalgebra B of P, P has an abundance of two-valued centrally additive B-states. This answers positively a question raised in [13, Open question, p. 13]. As a consequence one obtains a somewhat better set representation of orthomodular posets and a better extension theorem than in [2, 12, 13]. Further improvement in the Boolean vein is hardly possible as the concluding example shows.

Our notation is standard. We use OMP to abbreviate orthomodular poset, OML to abbreviate orthomodular lattice, Z to denote the centre of an orthomodular poset, $\subset$ for set inclusion, and $\langle 0,1\rangle$ for the real unit interval. We remind the reader that a subset B of an orthomodular poset P is called a *Boolean subalgebra* of P if B is closed under orthocomplementation and finite orthogonal joins and B forms a Boolean algebra under these inherited operations. It is well known that any two elements of B also have a join (resp., a meet) in P and that the join (resp., the meet) taken in B coincides with the join (resp., the meet) taken in P. For general background on orthomodular posets the reader should consult [11], on orthomodular lattices [1, 6], and for various results related to set representations of orthomodular posets [2, 5, 7, 8, 9, 13, 14].

Definition 1. Let P be an OMP and $s : P \to \langle 0,1\rangle$ be a map that satisfies

(1) $s(0) = 0$,
(2) $s(a') = 1 - s(a)$ for all $a \in P$,
(3) if $a \leq b$ then $s(a) \leq s(b)$.

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We say s is a *state* if it satisfies

(4) if $a \leq b'$, then $s(a \vee b) = s(a) + s(b)$.

We say s is a *centrally additive state* if it satisfies

(4′) if $a \leq b'$ and $b \in Z$, then $s(a \vee b) = s(a) + s(b)$.

If B is a Boolean subalgebra of P we say s is a B-*state* if it satisfies

(4″) if $a \leq b'$ and $a, b \in B$, then $s(a \vee b) = s(a) + s(b)$.

Centrally additive states are obtained by weakening the additivity requirement for states to those orthogonal pairs where at least one element belongs to the centre, and B-additive states are obtained by weakening the additivity requirement for states to those orthogonal pairs where both elements belong to the subalgebra B. Note that a centrally additive state is more than just a B-additive state for B being the Boolean algebra Z. We shall call a state *two-valued* if its range is $\{0, 1\}$. The following notion is key to the study of two-valued centrally additive states.

Definition 2. Let P be an OMP. We say $I \subset P$ is a *central ideal* if

(1) $b \in I$ and $a \leq b$ imply $a \in I$,
(2) if $a \in I$ then $a' \notin I$ for every $a \in P$,
(3) if $a \leq b'$, $a, b \in I$, and $b \in Z$ then $a \vee b \in I$,
(4) I contains a prime ideal of Z.

Lemma 3. *Let I be a central ideal of P and $a' \notin I$. Then*

$$J = \{x \in L \mid x \leq m \vee a \text{ for some } m \in I \cap Z\} \cup I$$

is a central ideal of P containing I and the element a.

Proof. Let $Q = I \cap Z$. By assumption (4), Q contains a prime ideal of the centre, hence by assumption (2), Q is a prime ideal of the centre.

As J is the union of two order ideals, it is an order ideal. Hence J satisfies the first condition.

For the second condition suppose $x, x' \in J$. Obviously not both $x, x' \in I$. If $x \leq m_1 \vee a$ and $x' \leq m_2 \vee a$, then $1 = (m_1 \vee m_2) \vee a$, giving $a' \leq m_1 \vee m_2$. As $m_1 \vee m_2$ belongs to Q, we have the contradiction $a' \in I$. We are left with the possibility that $x \in I$ and $x' \leq m \vee a$ for some $m \in Q$. This implies $m' \wedge a' \leq x$, hence $m' \wedge a' \in I$. As $m \in Q$ and I is a central ideal we see that $m \vee (m' \wedge a') = m \vee a' \in I$, yielding the contradiction $a' \in I$. Note that the second condition implies $J \cap Z = I \cap Z$ since $I \cap Z$ is a prime ideal of Z.

For the third condition suppose $x, y \in J$, $x \leq y'$ and $y \in Z$. Then $y \in Q$. If $x \in I$, then as I is a central ideal we have $x \vee y \in I$. Otherwise $x \leq m \vee a$ for some $m \in Q$. Then $x \vee y \leq m \vee a \vee y = (m \vee y) \vee a$ and since both $m, y \in Q$ it follows that $x \vee y \in J$.

Finally, the fourth condition follows trivially as I contains a prime ideal of Z. ∎

Corollary 1. *For I a central ideal of P these are equivalent*:

(1) *I is a maximal central ideal.*
(2) *For each $a \in P$ exactly one of a, a' belongs to I.*

The connection between maximal central ideals and two-valued centrally additive states can now be made clear.

Proposition 4. *Let P be an OMP and B be a Boolean subalgebra of P. For $s : P \to \{0, 1\}$ these are equivalent*:

(1) *s is a centrally additive B-state.*
(2) *$s^{-1}(0)$ is a maximal central ideal which contains a prime ideal of B.*

For $I \subset P$ these are equivalent:

(3) *I is a maximal central ideal which contains a prime ideal of B.*
(4) *$I = s^{-1}(0)$ for some two-valued centrally additive B-state s.*

Proof. (1)⇒(2). Set $I = s^{-1}(0)$. As s restricts to a state on Z, $I \cap Z$ is a prime ideal of Z. Similarly, as s restricts to a state on B, $I \cap B$ is a prime ideal of B. Obviously, I is a downset and for each $a \in P$ exactly one of a, a' belongs to I. Finally, if $x, y \in I$, $x \leq y'$ and $y \in Z$, then as s is centrally additive, $s(x \vee y) = s(x) + s(y) = 0$, yielding $x \vee y \in I$.

(2)⇒(1). Set $I = s^{-1}(0)$. As $0 \in I$ we have $s(0) = 0$, and as I is a downset, s is order preserving. As I is maximal, exactly one of a, a' belongs to I for each $a \in P$, so $s(a') = 1 - s(a)$. Assume $x \leq y'$ with either $x, y \in B$ or $y \in Z$. To show $s(x \vee y) = s(x) + s(y)$ it suffices to show this under the assumption that $x, y \in I$. The result follows from the assumptions that $I \cap B$ is a prime ideal of B and that I is a central ideal.

(3)⇒(4). Define $s : P \to \{0, 1\}$ by setting $s(x) = 0$ if $x \in I$ and $s(x) = 1$ if $x \notin I$. Then $I = s^{-1}(0)$. That s is a centrally additive B-state then follows from the equivalence of (1) and (2).

(4)⇒(3). This follows directly from the equivalence of (1) and (2). ∎

The following result is crucial for the representation theorem.

Lemma 5. *Let L be an OMP. Let B be a Boolean subalgebra of L containing Z and let $a, b \in L$ with $a \not\leq b$. Then there is a central ideal I with $a', b \in I$ such that $I \cap B$ is a prime ideal of B.*

Proof. Set $X = \{x \in B \mid a \leq x\} \cup \{y \in B \mid b' \leq y\} \cup \{z \in Z \mid a \leq z \vee b\}$. We first claim that X generates a proper filter of B. As each of the three sets involved in the definition of X is closed under finite meets, it suffices to show that for $x, y \in B$ and $z \in Z$ with $a \leq x$, $b' \leq y$, $a \leq z \vee b$ we have $x \wedge y \wedge z \neq 0$. Assume to the contrary that $x \wedge y \wedge z = 0$. We want to derive the contradiction $a \le b$. Certainly, $a \le z \vee b$ implies by the centrality of z that $a \wedge z' \le b \wedge z'$. Also, $x \wedge y \wedge z = 0$ implies $z \le x' \vee y'$. As $a \le x$ and $z \le x' \vee y'$ we have $a \wedge z \le x \wedge (x' \vee y') = x \wedge y' \le y' \le b$, so $a \wedge z \le b \wedge z$. As $a \wedge z \le b \wedge z$ and $a \wedge z' \le b \wedge z'$, the centrality of z yields $a \le b$.

Since X generates a proper filter, there is a prime ideal Q of B which is disjoint from X. Let $I_0 = \{x \in L \mid x \le p \text{ for some } p \in Q\}$. We claim that I_0 is a central ideal. The first condition is trivial from the definition. The second follows as I_0 is the downset generated by a proper ideal of B. The third condition also follows: I_0 is closed under all finite joins. The fourth follows as I_0 contains a prime ideal of B and the centre is contained in B. We next want to show that $a, b' \notin I_0$. Indeed, if $a \in I_0$ then $a \le x$ for some $x \in Q$. But then $x \in X \cap Q$, a contradiction. Similarly, if $b' \in I_0$ then $b' \le y$ for some $y \in Q$ and $y \in X \cap Q$, a contradiction. Let us set

$$I_1 = \{x \in L \mid x \le m \vee b \text{ for some } m \in I_0 \cap Z\} \cup I_0.$$

By Lemma 3, I_1 is a central ideal of L. We claim that $a \notin I_1$. Indeed, $a \in I_1$ would imply that $a \le z \vee b$ for some $z \in I_0 \cap Z$. But this z would then belong to $X \cap Q$, which is absurd. As $a \notin I_1$, we apply Lemma 3 again to extend I_1 to a central ideal containing both a', b. ■

THEOREM 6. *Let P be an OMP, B be a Boolean subalgebra of P, and $a \not\le b$ be elements of P. Then there is a centrally additive B-state $s : P \to \{0,1\}$ such that $s(a) = 1$ and $s(b) = 0$.*

Proof. Taking the subalgebra generated by $B \cup Z$ if necessary, we may assume without loss of generality that B contains the centre of P. Use Lemma 5 to produce a central ideal I with $a', b \in I$ such that $I \cap Z$ is a prime ideal of B. By a standard Zorn's lemma argument extend I to a maximal central ideal M. By Proposition 4 there is a centrally additive B-state $s : P \to \{0,1\}$ with $M = s^{-1}(0)$. Then $a', b \in M$ yield $s(a) = 1$ and $s(b) = 0$. ■

THEOREM 7. *Let P be an OMP and let B be a Boolean subalgebra of P. Then there is a set S and a mapping $\sigma : P \to \exp S$ into the power set of S such that, for any $a, b \in L$,*

(1) *$a \le b$ if and only if $\sigma(a) \subset \sigma(b)$,*
(2) *$\sigma(a') = S - \sigma(a)$,*
(3) *if $a, b \in B$ then $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$ and $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$,*
(4) *if $a \in Z$, then $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$ and $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$.*

Proof. The proof closely follows the Boolean patterns and we therefore omit the details. Let S be the set of all two-valued centrally additive B-states on P. Define $\sigma : P \to \exp S$ by setting $\sigma(a) = \{s \in S \mid s(a) = 1\}$. ■

The "topological" version of the above representation theorem is also in force. Again, the technique is similar to the Boolean case. The resulting Stone space will however be a closure space only (see [13] for details; recall that a closure space (see [3]) differs from a topological space in that the union of two closed sets need not be closed).

THEOREM 8. *Let P be an OMP and let B be a Boolean subalgebra of P. Then there exists a compact Hausdorff closure space C and a mapping $\sigma : L \to \mathrm{Clop}(C)$ to the collection $\mathrm{Clop}(C)$ of all clopen subspaces of C such that*

(1) $a \leq b$ *if and only if* $\sigma(a) \subset \sigma(b)$,
(2) $\sigma(a') = S - \sigma(a)$,
(3) *if* $a, b \in B$ *then* $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$ *and* $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$,
(4) *if* $a \in Z$, *then* $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$ *and* $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$.

Further, if P is an OML then the map σ is onto $\mathrm{Clop}(C)$.

Proof. Let S and σ be as in the previous theorem. Let C be the closure space whose underlying set is S and whose basic closed sets are $\{\sigma(a) \mid a \in P\}$. As each $\sigma(a)$ and its complement are closed, each $\sigma(a)$ is clopen. For distinct states $s, t \in S$ there is $a \in P$ with $s(a) \neq t(a)$ hence $\sigma(a)$ is a clopen set separating these points. Therefore C is Hausdorff. As the state space S is compact under the subspace topology inherited from $\langle 0, 1 \rangle^P$, and each $\sigma(a)$ is closed in this subspace topology, the collection $\{\sigma(a) \mid a \in P\}$ has the finite intersection property, and it follows that C is also compact. Conditions (1) through (4) of the theorem are established in the previous result.

For the further remark assume P is an OML. Let $A \subset S$ be a clopen set of C. Using the compactness of C and the fact that A is open, we have $A = \sigma(a_1) \cup \ldots \cup \sigma(a_n)$ for some $a_1, \ldots, a_n \in P$. But A is closed so for some $T \subset P$ we have $A = \bigcap\{\sigma(a) \mid a \in T\}$. It follows from (1) that $A \subset \sigma(a_1 \vee \ldots \vee a_n) \subset \bigcap\{\sigma(a) \mid a \in T\}$ hence equality. This shows σ is onto. ■

Our next theorem generalizes the extension property for Boolean states.

THEOREM 9. *Let P be an OMP and B_1, B_2 be Boolean subalgebras of P. Let $s : B_1 \to \langle 0, 1 \rangle$ be a (Boolean) state on B_1. Then there is a centrally additive B_2-state $t : P \to \langle 0, 1 \rangle$ that restricts to s on B_1.*

Proof. Assume first s is two-valued. From well known properties of states on Boolean algebras, s can be extended to a two-valued state on the Boolean subalgebra of P generated by $B_1 \cup Z$, so we may assume without loss of generality that B_1 contains Z. Also, from the form of the problem, we may assume that B_2 contains Z. Let $J = s^{-1}(0)$, a prime ideal of B_1. Note that

J contains a prime ideal of Z. By the prime ideal theorem, there is a prime ideal K of B_2 containing $\{x \in B_2 \mid x \leq j \text{ for some } j \in J\}$. Then K contains $J \cap B_2$. Hence $K \cap Z$ contains $J \cap B_2 \cap Z = J \cap Z$ and as both are prime ideals of Z we have $K \cap Z = J \cap Z$. Let

$$I = \{x \in P \mid x \leq y \text{ for some } y \in J \cup K\}.$$

We claim I is a central ideal. Obviously, I is a downset. Suppose $x, x' \in I$. Then as both J, K are closed under finite joins and neither contains 1 we deduce that $x \leq j$ for some $j \in J$ and $x' \leq k$ for some $k \in K$. Then $k' \leq j$. But this would imply $k' \in K$, contrary to K being a prime ideal. Since $I \supset J, K$ it follows that I contains $J \cap Z = K \cap Z$, a prime ideal of Z, and as we have shown that I never contains an element and its orthocomplement, $I \cap Z = J \cap Z = K \cap Z$. Suppose $x, y \in I$ with $x \leq y'$ and $y \in Z$. If $x \leq j$ for some $j \in J$, then as $y \in I \cap Z = J \cap Z$ we have $j, y \in J$ hence $j \vee y \in J$, and as $x \vee y \leq j \vee y$ we have $x \vee y \in I$. If $x \leq k$ for some $k \in K$ the argument is similar. Therefore I is a central ideal of P.

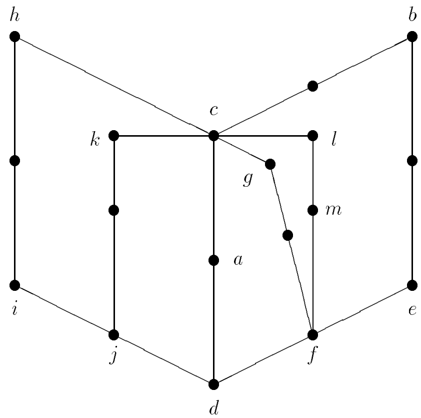
Taking the two-valued centrally additive state $t : P \to \{0, 1\}$ associated with I we see that t extends s since $I \supset J$ and t is a B_2 state since I contains a prime ideal of B_2. We have proved every two-valued state s on B_1 can be extended to a two-valued centrally additive B_2-state on P. The general result then follows from the compactness and convexity of the space of all centrally additive B_2-states on P by using a standard argument found e.g. in [13]. ∎

To conclude this note, let us show by an example that our results are in a sense best possible. Let P be an OMP and B be a Boolean subalgebra of P. Let us call a mapping $s : P \to \langle 0, 1\rangle$ a *strong B-state* if

(1) $s(0) = 0$,
(2) $s(a') = 1 - s(a)$ for any $a \in P$,
(3) if $a \leq b$ then $s(a) \leq s(b)$, and
(4‴) if $a \leq b'$ and $b \in B$, then $s(a \vee b) = s(a) + s(b)$.

It turns out that there is no hope for a representation theorem via these states—there are finite OMP's which do not have an order determining set of two-valued strong B-states. We will show this using the Greechie paste technique (see [4]).

EXAMPLE 10. *Let us consider the OMP, P, given by the Greechie diagram indicated below. Let us consider elements a, b therein. Then $a \not\leq b'$. Let B be the maximal Boolean subalgebra of P containing the atom a. Then there is no two-valued strong B-state with $s(a) = 1$ and $s(b') = 0$.*



Proof. If $s(a) = 1$, then $s(c) = s(d) = 0$ (the elements c, a, d constitute all atoms of B). Suppose $s(b') = 0$. Then $s(b) = 1$. Since $e \leq b'$, we see that $s(e) = 0$. This implies that $s(f) = 1$, and therefore $s(g) = 0$. As $s(c) = s(g) = 0$, we infer that $s(h) = 1$. This yields $s(i) = 0$, and therefore $s(j) = 1$. As a consequence, $s(k) = 0$. Since $s(c) = s(k) = 0$, we have $s(l) = 1$. But $s(f) = s(l) = 1$, a contradiction. Thus, there is no two-valued strong B-state on P with $s(a) = 1$ and $s(b') = 0$. ■

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