COLLOQUIUM MATHEMATICUM

VOL. 89 2001 NO. 2

NON-ORBICULAR MODULES FOR GALOIS COVERINGS

BY

PIOTR DOWBOR (Toruń)

Abstract. Given a group G of k-linear automorphisms of a locally bounded k-category R, the problem of existence and construction of non-orbicular indecomposable R/G-modules is studied. For a suitable finite sequence B of G-atoms with a common stabilizer H, a representation embedding $\Phi^B : I_n\text{-spr}(H) \to \text{mod}(R/G)$, which yields large families of non-orbicular indecomposable R/G-modules, is constructed (Theorem 3.1). It is proved that if a G-atom B with infinite cyclic stabilizer admits a non-trivial left Kan extension $\widetilde{B}$ with the same stabilizer, then usually the subcategory of non-orbicular indecomposables in $\text{mod}_{\{\widetilde{B},B\}}(R/G)$ is wild (Theorem 4.1, also 4.5). The analogous problem for the case of different stabilizers is discussed in Theorem 5.5. It is also shown that if R is tame then $\widetilde{B} \simeq B$ for any infinite G-atom B with $\text{End}_R(B)/J(\text{End}_R(B)) \simeq k$ (Theorem 7.1). For this purpose the techniques of neighbourhoods (Theorem 7.2) and extension embeddings for matrix rings (Theorem 6.3) are developed.

Introduction. For more than twenty years now, the Galois coverings have remained one of the most efficient techniques in contemporary representation theory of algebras over a field and matrix problems. They were successfully used in solutions of various important classification and theoretical problems. The covering method often allowed a reduction of a given problem for modules over an algebra to an analogous one for its cover category, usually much simpler than the original one. Initially, the method was invented for studying representation-finite algebras [22, 15, 2, 17], later developed for the representation-infinite case ([11, 10, 12], also [3, 4, 6]) and effectively applied in [30, 31, 32, 16, 21, 19], in the meantime adopted for matrix problems [23, 24, 25, 14, 9].

The main interest in covering techniques was always concentrated on applications. The results answering theoretical questions, only indirectly important for applications, played a minor role. For a long time the central position in this area was occupied by the important, difficult and stimulating problem of determining if Galois coverings preserve the tame representation

2000 *Mathematics Subject Classification*: Primary 16G60.
Key words and phrases: Galois covering, locally finite-dimensional module, tame.
Supported by Polish KBN Grant 2 P03A 012 16.

type. An affirmative solution of this problem in full generality was announced by Drozd and Ovsienko more than ten years ago, but the preprint [13] containing a written version of the proof appeared only a few months ago (see also [3, 10, 12, 3, 4, 6] for partial results).

In the same time other, more detailed questions, closely related to the above one, were intensively studied. One of them is the so-called "stabilizer conjecture", which says that for a representation tame locally bounded category R over an algebraically closed field, the stabilizers of infinite G-atoms (see 1.3) with respect to a free action of a torsionfree group G on R are infinite cyclic groups (proved in [6, 8]).

Another group of interesting problems which have been studied recently concerns the notion of orbicular (resp. non-orbicular) module. A module X in $\mathrm{mod}(R/G)$ is called *orbicular* (resp. *non-orbicular*) if the "pull-up" $F_\bullet X$ of X with respect to the Galois covering $F : R \to R/G$ decomposes into a direct sum of indecomposable locally finite-dimensional modules which belong (resp. do not belong) to one G-orbit (see 1.3). One should recall that all indecomposable R/G-modules in the tame case (studied in terms of Galois coverings) are orbicular (with respect to G), and are formed in fact, according to a conjecture formulated long time ago, by use of one standard construction (see 1.3). In this context, posing the general question when all indecomposable R/G-modules are orbicular seems to be very natural. In particular, it is interesting to know if R/G admits indecomposable non-orbicular modules in the tame case (resp. in the case $G \simeq \mathbb{Z}$). Generally, it has been unknown how to construct non-orbicular indecomposables, and how the "bonds" which fix G-atoms into such modules could look like.

In this paper we study the problems described above. We present a construction of a representation embedding into the category $\mathrm{mod}(R/G)$ of finite-dimensional R/G-modules whose image contains a large, usually wild, subcategory consisting of non-orbicular indecomposable modules (see Theorem 3.1). This construction is based on the generalized tensor product functor, defined by a fixed finite sequence of non-isomorphic G-atoms with a common stabilizer H in G (see 2.4). In some situations, when H is an infinite cyclic group, we can describe the structure of this category, in fact of the image of the embedding, in terms of the generalized subspace problem for linearly ordered finite posets over the group algebra kH. The specialization of this result to the case of the canonical sequence of length 2 consisting of a G-atom B and its left Kan extension $\widetilde{B}$ (see Theorem 4.1) supplies, in the case $\widetilde{B} \not\simeq B$, a method of constructing "algebras" R/G, $G \simeq \mathbb{Z}$, which admit a large number of non-orbicular indecomposable modules (Corollary 4.4). It is proved that in this situation R/G is representation-wild, in fact the full subcategory formed by non-orbicular indecomposables in the category $\mathrm{mod}_2(R/G)$ (see [12]) is wild. We also discuss (on an example of the canonical sequence consisting of B and $\widetilde{B}$) how to construct non-orbicular indecomposable R/G-modules in case the members of the sequence have different stabilizers (see Theorem 5.5). Finally, we study the problem of how the properties of B and of the left Kan extension of B influence the representation type of R (see Theorems 7.1 and 7.6). We show that if the cover category R admits an infinite G-atom B such that $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$ and $B \not\simeq \widetilde{B}$ then R is representation-wild. To prove this result we apply the extension embeddings technique for matrix rings (see Theorem 6.3) and the neighbourhood approach to indecomposable locally finite-dimensional modules (see Theorem 7.2 and Proposition 7.5).

The paper is organized as follows. In Section 1 we recall basic definitions and fix notation used in the paper. There, a precise definition of a non-orbicular module is given. Section 2 is devoted to the construction of a generalized tensor product functor defined by a sequence of group representations, and R-modules with an R-action of a subgroup $G \subset \mathrm{Aut}_k(R)$, where R is a locally bounded category over a field k. In Section 3 the main result of the paper "on constructing indecomposable non-orbicular R/G-modules by use of a sequence of G-atoms with a common stabilizer" (Theorems 3.1) is formulated and proved. Section 4 is devoted to a specialization of Theorem 3.1 to the case of length 2 (resp. 3) sequences formed from a G-atom B by use of its Kan extensions (see Theorems 4.1 and 4.5, Corollary 4.4). The behaviour of the above construction in the case of different stabilizers, also in the context of the base field characteristic problem, is discussed in Section 5 (see Theorem 5.5). In Section 6 extension embeddings for matrix rings (a tool for the proof of Theorem 7.6) are studied and Theorem 6.3 is proved. Section 7 is devoted to the proofs of Theorems 7.1 and 7.6. For this purpose, we develop the technique of neighbourhoods, for the case where k is not algebraically closed; in particular, we prove Theorem 7.2 and Proposition 7.5.

Some of the results contained in this paper were presented in seminar talks at Toruń University, in May 1998.

1. Basic definitions and notation. Now we briefly describe the situation we are dealing with. Throughout the paper we use in principle the notation and definitions established in [4, 7]. For basic information concerning representation theory of algebras (resp. rings and modules, and notions of category theory) we refer to [26] (resp. [1], [18]).

1.1. Let k be a field (not necessarily algebraically closed) and R be a *locally bounded k-category*, i.e. all objects of R have local endomorphism rings, different objects are non-isomorphic, and the sums $\sum_{y\in R}\dim_k R(x,y)$ and $\sum_{y\in R}\dim_k R(y,x)$ are finite for each $x \in R$, where $R(x,y)$ is the k-linear space of morphisms from x to y in R. By an *R-module* we mean a contravariant k-linear functor from R to the category of all k-vector spaces. An R-module M is *locally finite-dimensional* (resp. *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x \in R$ (resp. the *dimension* $\dim_k M = \sum_{x\in R} \dim_k M(x)$ of M is finite). We denote by $\mathrm{MOD}\, R$ the category of all R-modules, by $\mathrm{Mod}\, R$ (resp. $\mathrm{mod}\, R$) the full subcategory of all locally finite-dimensional (resp. finite-dimensional) R-modules and by $\mathrm{Ind}\, R$ (resp. $\mathrm{ind}\, R$) the full subcategory of all indecomposable R-modules in $\mathrm{Mod}\, R$ (resp. $\mathrm{mod}\, R$). By the *support* of an object M in $\mathrm{MOD}\, R$ we mean the full subcategory $\mathrm{supp}\, M$ of R formed by the set $\{x \in R : M(x) \neq 0\}$. We denote by $\mathcal{J}_R$ the Jacobson radical of the category $\mathrm{Mod}\, R$.

For any k-algebra A we denote analogously by $\mathrm{MOD}\, A$ (resp. $\mathrm{mod}\, A$) the category of all (resp. all finite-dimensional) right A-modules and by $J(A)$ the Jacobson radical of A.

To any finite full subcategory C of R we can attach the finite-dimensional algebra $A(C) = \bigoplus_{x,y\in \mathrm{ob}\, C} R(x,y)$ endowed with the multiplication given by composition in R. It is well known that the mapping $M \mapsto \bigoplus_{x\in \mathrm{ob}\, C} M(x)$ yields an equivalence

$$\mathrm{mod}\, C \simeq \mathrm{mod}\, A(C).$$

1.2. Let G be a group of k-linear automorphisms of R acting freely on the objects of R. Then G acts on the category $\mathrm{MOD}\, R$ by translations ${}^g(-)$, which assign to each M in $\mathrm{MOD}\, R$ the R-module ${}^gM = M \circ g^{-1}$ and to each $f : M \to N$ in $\mathrm{MOD}\, R$ the R-homomorphism ${}^gf : {}^gM \to {}^gN$ given by the family $(f(g^{-1}(x)))_{x\in R}$ of k-linear maps.

Given M in $\mathrm{MOD}\, R$ the subgroup

$$G_M = \{g \in G : {}^gM \simeq M\}$$

of G is called the *stabilizer* of M.

Let R/G be the orbit category of the action of G on R. Then R/G is again a locally bounded k-category (see [15]). We can study the module category $\mathrm{mod}(R/G)$ in terms of the category $\mathrm{Mod}\, R$. The tool at our disposal is the pair of functors

$$\mathrm{MOD}\, R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \mathrm{MOD}(R/G)$$

where $F_\bullet : \mathrm{MOD}(R/G) \to \mathrm{MOD}\, R$ is the "pull-up" functor associated with the canonical Galois covering functor $F : R \to R/G$, assigning to each X in $\mathrm{MOD}(R/G)$ the R-module $X \circ F$, and the "push-down" functor $F_\lambda : \mathrm{MOD}\, R \to \mathrm{MOD}(R/G)$ is the left adjoint to $F_\bullet$.

The classical results from [15] state that if G acts freely on $(\mathrm{ind}\, R)/{\simeq}$ (i.e. $G_M = \{\mathrm{id}_R\}$ for every M in $\mathrm{ind}\, R$) then F_λ induces an embedding of the set $((\mathrm{ind}\, R)/{\simeq})/G$ of the G-orbits of isoclasses of objects in $\mathrm{ind}\, R$ into $(\mathrm{ind}(R/G))/{\simeq}$.

Let H be a subgroup of the stabilizer G_M of a given M in $\operatorname{MOD} R$. By an R-*action* of H on M we mean a family

$$\mu = (\mu_g : M \to {}^{g^{-1}}M)_{g \in H}$$

of R-homomorphisms such that $\mu_e = \mathrm{id}_M$, where $e = \mathrm{id}_R$ is the unit of H, and ${}^{g_1^{-1}}\mu_{g_2} \cdot \mu_{g_1} = \mu_{g_2 g_1}$ for all $g_1, g_2 \in H$ (see [15]). Observe that if H is a free group then M admits an R-action of H (see [3, Lemma 4.1]).

For any subgroup H of G we denote by $\operatorname{MOD}^H R$ (resp. $\operatorname{Mod}^H R$) the category consisting of the pairs (M, μ), where M is an R-module (resp. a locally finite-dimensional R-module) and μ an R-action of H on M. For any $M = (M, \mu)$ and $N = (N, \nu)$ in $\operatorname{MOD}^H R$ (resp. $\operatorname{Mod}^H R$) the space of morphisms from M to N in $\operatorname{MOD}^H R$ (resp. $\operatorname{Mod}^H R$) consists of all $f \in \operatorname{Hom}_R(M, N)$ such that ${}^{g^{-1}}f \cdot \mu_g = \nu_g \cdot f$ for every $g \in H$, and is denoted by $\operatorname{Hom}_R^H(M, N)$. By $\operatorname{Mod}_{\mathrm{f}}^G R$ we denote the full subcategory of the category $\operatorname{Mod}^H R$ formed by all (M, μ) such that $\operatorname{supp} M$ is contained in the union of a finite number of H-orbits of H in R (see [15, 12, 3]). Then the functor $F_\bullet$, associating with any X in $\operatorname{mod}(R/G)$ the R-module $F_\bullet X$ endowed with the trivial R-action of G, yields an equivalence

$$\operatorname{mod}(R/G) \simeq \operatorname{Mod}_{\mathrm{f}}^G R.$$

An important role in understanding the nature of objects from $\operatorname{Mod}_{\mathrm{f}}^G R$, and consequently from $\operatorname{mod}(R/G)$, is played by the G-atoms. Recall from [3] that an indecomposable R-module B in $\operatorname{Mod} R$ (with local endomorphism ring) is called a G-*atom* (over R) provided $\operatorname{supp} B$ is contained in the union of a finite number of G_B-orbits in R. The G-atom B is said to be *finite* (resp. *infinite*) if G_B (equivalently $\operatorname{supp} B$) is finite (resp. infinite).

Denote by $\mathcal{A}$ a fixed set of representatives of isoclasses of all G-atoms in $\operatorname{Mod} R$, by $\mathcal{A}_{\mathrm{o}}$ a fixed set of representatives of G-orbits of the induced action of G on $\mathcal{A}$ and for any $B \in \mathcal{A}_{\mathrm{o}}$ by S_B a fixed set of representatives of left cosets of G_B in G, containing the unit $e = \mathrm{id}_R$ of the group G. One can show that the category $\operatorname{mod}(R/G)$ is equivalent via $F_\bullet$ to the full subcategory of $\operatorname{Mod}_{\mathrm{f}}^G R$ formed by all possible pairs (M_n, μ), where $n = (n_B)_{B \in \mathcal{A}_{\mathrm{o}}}$ is a sequence of natural numbers such that almost all n_B are zeros, M_n the R-module given by the formula

$$M_n = \bigoplus_{B \in \mathcal{A}_{\mathrm{o}}} \Big(\bigoplus_{g \in S_B} {}^{g}(B^{n_B}) \Big)$$

and μ an arbitrary R-action of G on M_n. Therefore to any X in $\operatorname{mod}(R/G)$ one can attach the *direct summand support* $\operatorname{dss}(X)$ of X which is the finite set consisting of all $B \in \mathcal{A}_{\mathrm{o}}$ such that $n_B \neq 0$, and the *direct summand coordinate vector* $\operatorname{dsc}(X) = (\operatorname{dsc}(X)_B)_{B \in \mathcal{A}_{\mathrm{o}}}$ of X, given by the components $\operatorname{dsc}(X)_B = n_B$, $B \in \mathcal{A}_{\mathrm{o}}$, where $F_\bullet X \simeq M_n$.

For any $\mathcal{U} \subset \mathcal{A}_{\mathrm{o}}$ one can study the full subcategory $\mathrm{mod}_{\mathcal{U}}(R/G)$ of $\mathrm{mod}(R/G)$ consisting of all X in $\mathrm{mod}(R/G)$ such that $\mathrm{dss}(X) \subset \mathcal{U}$.

1.3. A module X in $\mathrm{mod}(R/G)$ is called *orbicular* (cf. [15]) provided $\mathrm{dss}(X) = \{B\}$ for some $B \in \mathcal{A}_{\mathrm{o}}$, i.e. in a decomposition of the R-module $F_{\bullet}X$ into a direct sum of idecomposables there occur only G-atoms contained, up to isomorphism, in one orbit of G in $\mathcal{A}$. The module X in $\mathrm{mod}(R/G)$ is called *non-orbicular* if X is not orbicular. The subcategory of all orbicular R/G-modules can be represented as a splitting union

$$\bigvee_{B\in\mathcal{A}_{\mathrm{o}}} \mathrm{mod}_{\{B\}}(R/G),$$

and the additive closure of the subcategory of all non-orbicular indecomposable modules as its complement

$$\mathrm{mod}(R/G) \setminus \bigvee_{B\in\mathcal{A}_{\mathrm{o}}} \mathrm{mod}_{\{B\}}(R/G),$$

in the sense explained below.

Let $\mathcal{C}$ be a Krull–Schmidt category and $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_i$, $i \in I$, full subcategories of $\mathcal{C}$ which are closed under direct sums, direct summands and isomorphisms. The notation $\mathcal{C}_0 = \mathcal{C}_1 \vee \mathcal{C}_2$ (resp. $\mathcal{C} = \bigvee_{i\in I} \mathcal{C}_i$) means that the set of indecomposable objects in $\mathcal{C}_0$ splits into the disjoint union of indecomposables in $\mathcal{C}_1$ and in $\mathcal{C}_2$ (resp. in $\mathcal{C}_i$, $i \in I$), and the notation $\mathcal{C}_2 = \mathcal{C}_0 \setminus \mathcal{C}_1$ that the set of indecomposables in $\mathcal{C}_2$ consists of all indecomposables in $\mathcal{C}_0$ which are not in $\mathcal{C}_1$. We denote by $[\mathcal{C}_0]$ the ideal of all morphisms in $\mathcal{C}$ which factor through an object from $\mathcal{C}_0$. For any ideal $\mathcal{I}$ in the category $\mathcal{C}$ and a subcategory $\mathcal{C}'$ of $\mathcal{C}$, the restriction of $\mathcal{I}$ to $\mathcal{C}'$ is denoted by $\mathcal{I}_{\mathcal{C}'}$.

The category of orbicular modules forms an essential part of the category $\mathrm{mod}(R/G)$. Recall that if R/G is representation-finite then all R/G-modules are orbicular, provided G acts freely on $(\mathrm{ind}\, R)/\simeq$. According to a general conjecture all R/G-modules in the tame case are orbicular (in particular those which belong to 1-parameter families). Roughly speaking all R/G-modules which have occurred up to now in the Galois covering context (in the representation-finite and tame cases) are orbicular. They have been described by use of the following construction.

Suppose that a G-atom B admits an R-action ν of G_B on itself (this is always the case if the group G_B is free). Then $F_\lambda B$ carries the structure of a kG_B-R/G-bimodule which is finitely generated free as a left kG_B-module, where kG_B is the group algebra of G_B over k (see [12, 3.6]). This bimodule induces a functor

$$\Phi^B = - \otimes_{kG_B} F_\lambda B : \mathrm{mod}\, kG_B \to \mathrm{mod}_B(R/G)$$

which is a representation embedding in the sense of [27] (see [4, Proposition 2.3]), provided the field $\mathrm{End}_R(B)/J(\mathrm{End}_R(B))$ is equal to k. Note that if G_B is trivial then $kG_B \simeq k$ and if G_B is an infinite cyclic group then kG_B is isomorphic to the algebra $k[T, T^{-1}]$ of Laurent polynomials. If G acts freely on $(\mathrm{ind}\, R)/\simeq$ then F_λ can be interpreted in terms of the representation embedding

$$\Phi^{\mathcal{A}_{\mathrm{o}}^{\mathrm{f}}} : \coprod_{B \in \mathcal{A}_{\mathrm{o}}^{\mathrm{f}}} \mathrm{mod}\, k \to \mathrm{mod}(R/G)$$

induced by the functors $\{\Phi^B\}_{B \in \mathcal{A}_{\mathrm{o}}^{\mathrm{f}}}$, where $\mathcal{A}_{\mathrm{o}}^{\mathrm{f}}$ consists of all finite G-atoms in $\mathcal{A}_{\mathrm{o}}$. It is well known that then the above embedding furnishes the classification of all indecomposables of the so-called first kind with respect to F (i.e. those from the image $\mathrm{Im}\, F_\lambda$). If all infinite G-atoms have cyclic stabilizers then the functors $\{\Phi^B\}_{B \in \mathcal{A}_{\mathrm{o}}^{\infty}}$, where $\mathcal{A}_{\mathrm{o}}^{\infty}$ consists of all infinite G-atoms in $\mathcal{A}_{\mathrm{o}}$, induce the representation embedding functor

$$\Phi^{\mathcal{A}_{\mathrm{o}}^{\infty}} : \coprod_{B \in \mathcal{A}_{\mathrm{o}}^{\infty}} \mathrm{mod}\, k[T, T^{-1}] \to \mathrm{mod}(R/G)$$

(see [4, 2.2]), which in nice situations (see [3, 4, 6, 12]) yields a description of all indecomposable R/G-modules of the second kind with respect to F (i.e. those “lying outside” $\mathrm{Im}\, F_\lambda$).

Recall that, if G acts freely on $(\mathrm{ind}\, R)/\simeq$, then we denote by $\mathrm{mod}_1(R/G)$ the additive closure of the class of all (indecomposable) R/G-modules of the form $F_\lambda M$ for some M in $\mathrm{ind}\, R$; $\mathrm{mod}_1(R/G)$ is called the subcategory of the *first kind modules with respect to* F. The additive closure of the class of remaining indecomposables (lying outside $\mathrm{mod}_1(R/G)$) is denoted by $\mathrm{mod}_2(R/G)$ and called the subcategory of the *second kind modules with respect to* F.

In this paper we present a construction of a functor (a generalization of Φ^B) whose image contains a large subcategory consisting of non-orbicular indecomposable R/G-modules. As one can expect it is mostly related to the case when R and R/G are wild.

1.4. The following notation is used in the paper. Given a full subcategory C of R and an R-module M we denote by $M_{|C}$ the C-module which is the restriction of M to C. For any R-homomorphism $f : M \to N$ we denote by $f_{|C} : M_{|C} \to N_{|C}$ the C-homomorphism which is the restriction of f to C.

We say that a full subcategory C of R is *non-trivial* (resp. *trivial*) provided the set $\mathrm{ob}\, C$ of all objects of C is non-empty (resp. empty).

Let C_1 and C_2 be full subcategories of a locally bounded k-category R. We denote by $C_1 \cup C_2$ (resp. $C_1 \cap C_2$ and $C_1 \setminus C_2$) the full subcategory of R formed by the union (resp. intersection and difference) of the sets $\mathrm{ob}\, C_1$ and $\mathrm{ob}\, C_2$. The notation $C_1 \subset C_2$ means that $\mathrm{ob}\, C_1$ is contained in

$\operatorname{ob} C_2$. The subcategories C_1 and C_2 are called *disjoint* (resp. *orthogonal*) if $\operatorname{ob} C_1 \cap \operatorname{ob} C_2 = \emptyset$ (resp. $R(x,y) = 0 = R(y,x)$ for all $x \in \operatorname{ob} C_1$, $y \in \operatorname{ob} C_2$). The union $C_1 \cup C_2$ is said to be a *disjoint union*, and denoted by $C_1 \vee C_2$, provided C_1 and C_2 are disjoint. If subcategories C_1 and C_2 are orthogonal then the union $C_1 \cup C_2$ $(= C_1 \vee C_2)$ is isomorphic to the coproduct of these subcategories and is denoted by $C_1 \sqcup C_2$.

For any full subcategory C of R, we denote by $\widehat{C}$ the full subcategory formed by all $x \in \operatorname{ob} R$ such that $R(x,y)$ or $R(y,x)$ is non-zero for some $y \in \operatorname{ob} S$. Note that $\widehat{C}$ is finite provided so is C (R is locally bounded!).

Let A be a k-algebra. For any $m, n \in \mathbb{N}$ we denote by $\mathrm{M}_{m\times n}(A)$ the set of all $m \times n$-matrices with coefficients in A, by $\mathrm{M}_n(A)$ the algebra of all square $n \times n$-matrices with coefficients in A and by $\mathrm{T}_n(A)$ the upper-triangular matrix subalgebra of $\mathrm{M}_n(A)$.

Let H be a group. Then for any subgroup H' of H the index of H' in H is denoted by $[H : H']$.

For any set X we denote by $|X|$ the cardinality of X.

1.5. We will frequently use the restriction and extension functors. For any full subcategories C and D of R such that $C \subset D$ we denote by $e_\lambda^{D,C} : \operatorname{MOD} C \to \operatorname{MOD} D$ the *left Kan extension functor* for the embedding $C \hookrightarrow D$ (see [18]), i.e. the left adjoint to the restriction functor $e_\bullet^{D,C} : \operatorname{MOD} D \to \operatorname{MOD} C$ ($e_\bullet^{D,C}(M) = M_{|C}$ and $e_\bullet^{D,C}(f) = f_{|C}$ for any R-module M and R-homomorphism $f : M \to N$). For any N in $\operatorname{MOD} C$ the D-module $e_\lambda^{D,C}(N)$ is defined by

$$e_\lambda^{D,C}(N)(x) = N \otimes_C D(x,-)_{|C}$$

for $x \in \operatorname{ob} D$ (see [20]), and consequently, $\operatorname{supp} e_\lambda^{D,C}(N) \subset \widehat{\operatorname{supp} N}$. Observe that $e_\lambda^{D,C}(\operatorname{mod} C) \subset \operatorname{mod} D$ and $e_\lambda^{D,C}(\operatorname{Mod} C) \subset \operatorname{Mod} D$) (clearly $e_\bullet^{D,C}(\operatorname{mod} D) \subset \operatorname{mod} C$ and $e_\bullet^{D,C}(\operatorname{Mod} D) \subset \operatorname{Mod} C$).

Denote by ϕ the natural family

$$\{\phi_{N,M} : \operatorname{Hom}_D(e_\lambda^{D,C}(N), M) \to \operatorname{Hom}_D(N, e_\bullet^{D,C}(M))\}_{N \in \operatorname{MOD} C,\ M \in \operatorname{MOD} D}$$

of standard isomorphisms, defining adjunction for the pair $(e_\lambda^{D,C}, e_\bullet^{D,C})$ of functors. Then the unit of the adjunction ϕ, i.e. the natural family

$$\alpha = \{\alpha(N) : N \to e_\bullet^{D,C} e_\lambda^{D,C}(N)\}_{N \in \operatorname{MOD} C}$$

of C-homomorphisms $\alpha(N) = \phi_{N, e_\lambda^{D,C}(N)}(\mathrm{id}_{e_\lambda^{D,C}(N)})$, yields a functor isomorphism

$$e_\bullet^{D,C} e_\lambda^{D,C} \simeq \mathrm{id}_{\operatorname{MOD} C}\,.$$

Consequently, the functor $e_\lambda^{D,C}$ is a right quasi-inverse for $e_\bullet^{D,C}$, moreover, it is full and faithful.

We will also frequently use the counit of the adjunction ϕ, i.e. the natural family

$$\beta = \{\beta(M) : e_\lambda^{D,C} e_\bullet^{D,C}(M) \to M\}_{M\in \mathrm{MOD}\, D}$$

of D-homomorphisms $\beta(M) = (\phi_{e_\bullet^{D,C}(M),M})^{-1}(\mathrm{id}_{e_\bullet^{D,C}(M)})$. Since α is an isomorphism of functors, the classical formulas $e_\bullet^{D,C}(\beta(M)) \circ \alpha(e_\bullet^{D,C}(M)) = \mathrm{id}_{e_\bullet^{D,C}(M)}$, M in $\mathrm{MOD}\, D$, and $\beta(e_\lambda^{D,C}(N)) \circ e_\lambda^{D,C}(\alpha(N)) = \mathrm{id}_{e_\lambda^{D,C}(N)}$, N in $\mathrm{MOD}\, C$, for the adjoint pair $(e_\lambda^{D,C}, e_\bullet^{D,C})$, imply that all $e_\bullet^{D,C}(\beta(M))$'s and $\beta(e_\lambda^{D,C}(N))$'s are isomorphisms. As a consequence, for any M, M' in $\mathrm{MOD}\, D$ the isomorphism $\phi_{e_\bullet^{D,C}(M),M'}$ has the factorization

$$\begin{aligned}\mathrm{Hom}_D(e_\lambda^{D,C} e_\bullet^{D,C}(M), M') \to \mathrm{Hom}_C(e_\bullet^{D,C} e_\lambda^{D,C} e_\bullet^{D,C}(M), e_\bullet^{D,C}(M'))\\ \to \mathrm{Hom}_C(e_\bullet^{D,C}(M), e_\bullet^{D,C}(M'))\end{aligned}$$

where the first map is given by the functor $e_\bullet^{D,C}$ and the second is induced by the isomorphism $e_\bullet^{D,C}(\beta(M))$.

If $D = R$ then for simplicity we denote the functors $e_\bullet^{D,C}$ and $e_\lambda^{D,C}$ by $e_\bullet^C$ and $e_\bullet^C$.

Throughout the paper we also use the *right Kan extension* $e_\varrho : \mathrm{MOD}\, C \to \mathrm{MOD}\, R$ for the embedding $C \hookrightarrow R$, i.e. the right adjoint functor to the restriction functor $e_\bullet^C : \mathrm{MOD}\, R \to \mathrm{MOD}\, C$. The functor e_ϱ is given by

$$e_\varrho(N) = \mathrm{Hom}_R(R(-,x)_{|S}, N)$$

for N in $\mathrm{MOD}\, C$, $x \in \mathrm{ob}\, R$, and has properties analogous to e_λ^C. The unit map

$$\beta' = \{\beta'(M) : M \to e_\bullet^C e_\varrho^C(M)\}_{M\in \mathrm{MOD}\, R}$$

given by $\beta'(M) = \phi'_{M,e_\bullet^C(M)}(\mathrm{id}_{e_\bullet^C(M)})$, where

$$\{\phi'_{M,N} : \mathrm{Hom}_C(e_\bullet^C(M), N) \to \mathrm{Hom}_R(M, e_\varrho^C(N))\}_{N\in \mathrm{MOD}\, C,\, M\in \mathrm{MOD}\, D}$$

is the standard adjunction for the pair $(e_\bullet^C, e_\varrho^C)$, yields a functor isomorphism

$$e_\bullet^C e_\lambda^C \simeq \mathrm{id}_{\mathrm{MOD}\, C}\,.$$

Consequently, for any M, M' in $\mathrm{MOD}\, D$ the isomorphism $(\phi'_{M,e_\bullet^C(M),M'})^{-1}$ has the factorization

$$\begin{aligned}\mathrm{Hom}_R(M, e_\varrho^C e_\bullet^C(M')) \to \mathrm{Hom}_C(e_\bullet^C(M), e_\bullet^C e_\varrho^C e_\bullet^C(M'))\\ \to \mathrm{Hom}_C(e_\bullet^C(M), e_\bullet^C(M'))\end{aligned}$$

where the first map is given by the functor $e_\bullet^C$ and the second is induced by the isomorphism $e_\bullet^C(\beta'(M'))$.

1.6. Recall that a k-algebra (resp. locally bounded k-category) Λ is called *representation-wild* (briefly *wild*) provided there exists a functor

$F : \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} \Lambda$, where $k\langle x,y\rangle$ is the free associative k-algebra in two non-commuting variables, satisfying the following two conditions:

(a) $F = - \otimes_{k\langle x,y\rangle} Q$, where Q is a $k\langle x,y\rangle$-Λ-bimodule which is a finitely generated free left $k\langle x,y\rangle$-module,

(b) F induces an injection on the sets of isoclasses.

In this paper, each Λ which is not wild will be called *tame* (k is not assumed to be algebraically closed!).

2. Generalized tensor product functors. We start by generalizing the notion of the tensor product of group representations. This construction gives a basis for a similar one for R-modules with an R-action of a group.

2.1. Let H be a group and kH be the group algebra of H. The category $\mathrm{MOD}(kH)^{\mathrm{op}}$ is equivalent to the category of all k-representations of H. Therefore each V in $\mathrm{MOD}(kH)^{\mathrm{op}}$ can be viewed as a pair (V,μ), where V is a k-vector space and $\mu : H \to \mathrm{Aut}_k(V)$ is a group homomorphism (equivalently, a k-linear action of H on V).

Suppose we are given a sequence

$$V: \quad V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{n-1} \subseteq V_n$$

of kH-submodules of the kH-module $V_n = (V_n,\mu)$ and a sequence

$$B: \quad B_1 \overset{\beta_2}{\leftarrow} B_2 \leftarrow \ldots \leftarrow B_{n-1} \overset{\beta_n}{\leftarrow} B_n$$

of kH-homomorphisms, where $B_i = (B_i,\nu_i)$ is in $\mathrm{MOD}(kH)^{\mathrm{op}}$ for every $i = 1,\ldots,n$. We shall construct a left kH-module $\underline{V} \otimes_k B = (\underline{V} \otimes_k B, \underline{\mu} \otimes_k \beta)$ which we call a *tensor product* of V and B.

Let $\underline{V} = (\underline{V}_i)_{i=1,\ldots,n}$ be a *sequence of complementary direct summands* for V, i.e. a sequence of subspaces $\underline{V}_i$ of V such that $\underline{V}_1 = V_1$ and $V_i = V_{i-1} \oplus \underline{V}_i$ for $i = 2,\ldots,n$. Then we have $V_i = \bigoplus_{l=1}^{i} \underline{V}_l$ for every $i = 1,\ldots,n$. Moreover, every automorphism $\mu(h) \in \mathrm{Aut}_k(\underline{V}_n)$, $h \in H$, has the matrix representation

$$\underline{\mu}(h) = [\mu(h)_{i,j}]_{1\le i,j\le n},$$

where each $\mu(h)_{i,j} : \underline{V}_j \to \underline{V}_i$ is the composition of $\mu(h)$ with the canonical jth embedding and ith projection. The matrix of $\mu(h)_{i,j}$'s is upper-triangular since $\mu(h)(V_j) \subseteq V_j$, hence $\mu(h)_{i,j} = 0$ for $i > j$. Note that we have

$$\text{(i)} \qquad \mu(hh')_{i,j} = \sum_{i\le l\le j} \mu(h)_{i,l} \cdot \mu(h')_{l,j}$$

for all $i \le j$, $h \in H$.

We denote by β a family of k-linear homomorphisms $\beta_{i,j}(h) = \nu_i(h)\cdot\beta_{i,j} : B_j \to B_i$, $1 \le i,j \le n$, $h \in H$, where the maps $\beta_{i,j} : B_j \to B_i$ are defined as follows:

(ii) $$\beta_{i,j} = \begin{cases} \beta_{i+1} \cdot \ldots \cdot \beta_j & \text{if } i < j, \\ \mathrm{id}_{B_i} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Note that

(iii) $$\beta_{i,l} \cdot \beta_{l,j} = \beta_{i,j},$$

(iv) $$\beta_{i,l} \cdot \beta_{l,j}(h) = \beta_{i,l}(h) \cdot \beta_{l,j} = \beta_{i,j}(h),$$

and

(v) $$\beta_{i,l}(h) \cdot \beta_{l,j}(h') = \beta_{i,j}(hh')$$

for all $i \le l \le j$; $h, h' \in H$.

We set

$$\underline{V} \otimes_k B = \bigoplus_{i=1}^{n} \underline{V}_i \otimes_k B_i.$$

For every $h \in H$ we denote by $(\underline{\mu} \otimes_k \beta)(h) : \underline{V} \otimes_k B \to \underline{V} \otimes_k B$ the k-linear homomorphism given by the matrix

$$(\underline{\mu} \otimes_k \beta)(h) = [\mu(h)_{i,j} \otimes_k \beta_{i,j}(h)]_{1 \le i,j \le n}$$

with components $\mu(h)_{i,j} \otimes_k \beta_{i,j}(h) : \underline{V}_j \otimes_k B_j \to \underline{V}_i \otimes_k B_i$, and we set

$$\underline{\mu} \otimes_k \beta = ((\underline{\mu} \otimes_k \beta)(h))_{h \in H}.$$

LEMMA. *$\underline{V} \otimes_k B = (\underline{V} \otimes_k B, \underline{\mu} \otimes_k \beta)$ is a kH-module.*

Proof. Note that $(\underline{\mu} \otimes_k \beta)(h)$ is a k-linear automorphism of the k-linear space $\underline{V} \otimes_k B$ since it is defined by an upper-triangular matrix with the isomorphisms $\mu(h)_{i,i} \otimes_k \nu_i(h)$, $i = 1, \ldots, n$, on the main diagonal. To show that $(\underline{\mu} \otimes_k \beta)(hh') = (\underline{\mu} \otimes_k \beta)(h) \cdot (\underline{\mu} \otimes_k \beta)(h')$ for all $h, h' \in H$, it suffices to check that the (i,j)th components of both maps are equal for all $1 \le i, j \le n$. The case $i > j$ is clear, the case $i \le j$ follows from the equalities

$$\begin{aligned} &\sum_{l=1}^{n} (\mu(h)_{i,l} \otimes_k \beta_{i,l}(h)) \cdot (\mu(h)_{l,j} \otimes_k \beta_{l,j}(h')) \\ &\qquad = \sum_{i \le l \le j} \mu(h)_{i,l}\, \mu(h)_{l,j} \otimes_k \beta_{i,l}(h)\, \beta_{l,j}(h') \\ &\qquad = \sum_{i \le l \le j} \mu(h)_{i,l}\, \mu(h)_{l,j} \otimes_k \beta_{i,j}(hh') = \mu(hh')_{i,j} \otimes_k \beta_{i,j}(hh') \end{aligned}$$

(see (i) and (v)). ∎

REMARK. (a) If $n \ge 2$, $B_1 = \ldots = B_n$ and $\beta_2 = \ldots = \beta_n = \mathrm{id}_{B_n}$ then $\underline{V} \otimes_k B \simeq V_n \otimes_k B_n$ in $\mathrm{MOD}(kH)^{\mathrm{op}}$ ($\underline{V} \otimes_k B = V_n \otimes_k B_n$ for $n = 1$).

(b) If $\underline{V}$, $\underline{V}'$ are two different sequences of complementary direct summands for V then $\underline{V} \otimes_k B \simeq \underline{V}' \otimes_k B$ in $\mathrm{MOD}(kH)^{\mathrm{op}}$.

2.2. Following [26] and [28], for any algebra A we denote by I_n-spr(A) the category whose objects are sequences of the form

$$V: \quad V_1 \subseteq V_2 \subseteq \ldots \subseteq V_{n-1} \subseteq V_n$$

where V_i, $i = 1, \ldots, n-1$, are A-submodules of a left finite-dimensional A-module V_n, and the set of morphisms from V to V' consists of all A-homomorphisms $f : V_n \to V_n'$ such that $f(V_i) \subseteq V_i'$ for every $i = 1, \ldots, n-1$. Note that I_n-spr(A) is equivalent to the full subcategory of $\mathrm{mod}\, \mathrm{T}_n(A^{\mathrm{op}})$ (see 1.4) formed by all modules whose structure maps are A-monomorphisms ($\mathrm{T}_n(A^{\mathrm{op}})$ can also be identified with the incidence algebra of the linear poset $I_n = \{1 < 2 < \ldots < n\}$ over A^{op}).

To any V in I_n-spr(A) we can assign the *coordinate vector*

$$\mathrm{cdn}(V) = (d_1, \ldots, d_n)$$

in $\mathbb{N}^n$, given by $d_i = \dim_k V_i/V_{i-1}$ ($V_0 = 0$). Then we denote by I_n-spr$'(A)$ the additive closure of the full subcategory formed by all indecomposable V in I_n-spr(A) such that $\mathrm{cdn}(V)$ has at least two non-zero coordinates.

We extend the construction of the generalized tensor product to a functor

$$- \otimes_k B : I_n\text{-spr}(kH) \to \mathrm{MOD}(kH)^{\mathrm{op}}$$

for B as in 2.1.

Let $f : V \to V'$ be a morphism in I_n-spr(kH). Suppose that $\underline{V} = (\underline{V}_i)_{i=1,\ldots,n}$ and $\underline{V}' = (\underline{V}_i')_{i=1,\ldots,n}$ are fixed sequences of complementary direct summands for V and V' respectively. Then the kH-homomorphism $f : \bigoplus_{i=1}^n \underline{V}_i \to \bigoplus_{i=1}^n \underline{V}_i'$ is given by the matrix representation

$$\underline{f} = [f_{i,j}]_{1 \le i,j \le n}$$

of f with respect to $\underline{V}$ and $\underline{V}'$, with components $f_{i,j} : \underline{V}_j \to \underline{V}_i'$ which are the compositions of f with the standard embeddings and projection. The matrix $\underline{f}$ is upper-triangular since $f(V_j) \subseteq V_j'$ ($V_j' = \bigoplus_{i=1}^j \underline{V}_i'$, consequently $f_{i,j} = 0$ for all $i > j$). Note that

$$\text{(i)} \qquad \sum_{i \le l \le j} \mu(h)_{i,l} \cdot f_{l,j} = \sum_{i \le l \le j} f_{i,l} \cdot \mu(h)_{l,j}$$

for all $1 \le i, j \le n$, $h \in H$.

Denote by $\underline{f} \otimes_k B : \underline{V} \otimes_k B \to \underline{V}' \otimes_k B$ the k-linear map given by the matrix

$$\underline{f} \otimes_k B = [f_{i,j} \otimes_k \beta_{i,j}]_{1 \le i,j \le n}$$

with k-linear components $f_{i,j} \otimes_k \beta_{i,j} : \underline{V}_j \otimes_k B_j \to \underline{V}_i' \otimes_k B_i$.

Lemma. *The map $\underline{f} \otimes_k B$ is a kH-homomorphism.*

Proof. It suffices to show that the (i,j)th components of the matrices $(\underline{\mu} \otimes_k \beta)(h) \cdot (\underline{f} \otimes_k B)$ and $(\underline{f} \otimes_k B) \cdot (\underline{\mu} \otimes_k \beta)(h)$, $h \in H$, are equal for all $1 \le i, j \le n$. In fact we can assume that $i \le j$ (all matrices are upper-triangular). Then by 2.1(iv) and 2.2(i) we have

$$\sum_{l=1}^{n} (\mu(h)_{i,l} \otimes_k \beta_{i,l}(h)) \cdot (f_{l,j} \otimes_k \beta_{l,j}) = \sum_{i \le l \le j} \mu(h)_{i,l}\, f_{l,j} \otimes_k \beta_{i,l}(h)\beta_{l,j}$$
$$= \sum_{i \le l \le j} \mu(h)_{i,l}\, f_{l,j} \otimes_k \beta_{i,j}(h) = \sum_{i \le l \le j} f_{i,l}\, \mu(h)_{l,j} \otimes_k \beta_{i,j}(h)$$
$$= \sum_{i \le l \le j} f_{i,l}\, \mu(h)_{l,j} \otimes_k \beta_{i,l}\, \beta_{l,j}(h) = \sum_{l=1}^{n} (f_{i,l} \otimes_k \beta_{i,l}) \cdot (\mu(h)_{l,j} \otimes_k \beta_{l,j}(h))$$

and the proof is complete. ∎

2.3. Now we define the tensor product functor $- \otimes_k B : I_n\text{-spr}(kH) \to \mathrm{MOD}(kH)^{\mathrm{op}}$. For every object V in I_n-spr(kH) we fix a sequence of complementary direct summands $\underline{V} = (\underline{V}_i)_{i=1,\dots,n}$. Then we set

$$V \otimes_k B = \underline{V} \otimes_k B$$

for any object V in I_n-spr(kH), and

$$f \otimes_k B = \underline{f} \otimes_k B$$

for any morphism $f : V \to V'$, where $\underline{f} = [f_{i,j}]_{1 \le i,j \le n}$ is the matrix representation of f with respect to $\underline{V}$ and $\underline{V}'$.

Proposition. *The mapping* $- \otimes_k B : I_n\text{-spr}(kH) \to \mathrm{MOD}(kH)^{\mathrm{op}}$ *is a* k*-linear functor.*

Proof. By Lemmas 2.1 and 2.2 the mapping $- \otimes_k B$ is well defined on objects and morphisms. The equality $\mathrm{id}_{V \otimes_k B} = \mathrm{id}_V \otimes_k B$ follows by an easy check on definitions. To show $(f' \otimes_k B) \cdot (f' \otimes_k B) = f'f \otimes_k B$ for morphisms $f : V \to V'$ and $f' : V' \to V''$ in I_n-spr(kH) note that the components of the matrix representations $\underline{f}$, $\underline{f}'$ and $\underline{f'f}$ (of f, f' and $f'f$ with respect to $\underline{V}$, $\underline{V}'$ and $\underline{V}''$ respectively) satisfy the equalities

$$\text{(i)} \qquad (f'f)_{i,j} = \sum_{i \le l \le j} f'_{i,l} \cdot f_{l,j}$$

for all $i \le j$. Now applying (i) and 2.1(iii) we check, as in the proofs of Lemmas 2.1 and 2.2, that the (i,j)th components of both maps from the required equality coincide for all $1 \le i, j \le n$. ∎

Remark. Different choices of sequences of complementary direct summands $\underline{V}$ for all V in I_n-spr(kH) lead to isomorphic functors.

2.4. From now on we assume that H is a subgroup of $\mathrm{Aut}_k(R)$. We generalize the above construction and define the tensor product functor

$$-\otimes_k B : I_n\text{-spr}(kH) \to \mathrm{Mod}^H R$$

for a sequence B in $\mathrm{Mod}^H R$.

This functor is related to the previous one by a "forgetful functor" from $\mathrm{Mod}^H R$ to $\mathrm{MOD}(kH)^{\mathrm{op}}$, which is also an efficient tool used in our further proofs.

We fix some notation. For an R-module M we set

$$M^{(k)} = \bigoplus_{x\in \mathrm{ob}\, R} M(x),$$

and for an R-homomorphism $f : M \to M'$ we denote by $f^{(k)}$ the k-linear map

$$\bigoplus_{x\in \mathrm{ob}\, R} f(x) : M^{(k)} \to M'^{(k)}.$$

Let $\mu = (\mu_h : M \to {}^{h^{-1}}M)_{h\in H}$ be a family of R-homomorphisms. Then we define a map $\mu^{(k)} : H \to \mathrm{End}_k(M^{(k)})$ assigning to $h \in H$ the matrix

$$\mu^{(k)}(h) = [\mu^{(k)}(h)_{x,y}]_{x,y\in \mathrm{ob}\, R}$$

with components $\mu^{(k)}(h)_{x,y} : M(y) \to M(x)$ given by

$$\mu^{(k)}(h)_{x,y} = \begin{cases} \mu_h(y) & \text{if } x = hy, \\ 0 & \text{if } x \neq hy. \end{cases}$$

Observe that for each $h \in H$ we have $\mu^{(k)}(h) = \xi_{h^{-1}}(M)\cdot \mu_h^{(k)}$, where $\xi_{h^{-1}}(M) : ({}^{h^{-1}}M)^{(k)} \xrightarrow{\sim} M^{(k)}$ is the canonical k-isomorphism.

LEMMA. (a) *Let $f : M \to M'$, $f' : M' \to M''$ and $f'' : M \to M''$ be R-homomorphisms. Then $f'' = f'f$ if and only if $f''^{(k)} = f'^{(k)} f^{(k)}$.*

(b) *Let μ be as above. Then μ is an R-action of H on M if and only if $\mu^{(k)}$ is a k-linear action of H on $M^{(k)}$.*

(c) *Let (M,μ), (M',μ') be in $\mathrm{MOD}^H R$ and $f : M \to M'$ be an R-homomorphism. Then $f : (M,\mu) \to (M',\mu')$ is a morphism in $\mathrm{MOD}^H R$ if and only if $f^{(k)} : M^{(k)} \to M'^{(k)}$ is a morphism in $\mathrm{MOD}(kH)^{\mathrm{op}}$.*

Proof. An easy check on definitions. ∎

It is clear (by the implications "⇒") that the mappings introduced above yield k-linear functors

$$(-)^{(k)} : \mathrm{MOD}\, R \to \mathrm{MOD}\, k \quad \text{and} \quad (-)^{(k)} : \mathrm{MOD}^H R \to \mathrm{MOD}(kH)^{\mathrm{op}}$$

(we use the same notation).

Remark. The kH-module $M^{(k)}$ is free for any $M = (M, \mu)$ in $\mathrm{MOD}^H R$ ($M^{(k)} \simeq (\bigoplus_{x \in R_o} M(x)) \otimes_k kH$, where R_o is a fixed set of representatives of H-orbits in $\mathrm{ob}\, R$). Moreover, the kH-module $M^{(k)}$ is finitely generated if and only if M belongs to $\mathrm{Mod}_f^H R$.

2.5. Suppose we are given a sequence

$$B: \quad B_1 \overset{\beta_2}{\leftarrow} B_2 \leftarrow \ldots \leftarrow B_{n-1} \overset{\beta_n}{\leftarrow} B_n$$

in $\mathrm{Mod}^H R$, i.e. all objects $B_i = (B_i, \nu_i)$ are in $\mathrm{Mod}^H R$ (B_i is an R-module and ν_i is an R-action of H on B_i) and all R-homomorphisms β_i are morphisms in $\mathrm{Mod}^H R$ (the β_i are compatible with the actions). We denote by β the family $(\beta_{i,j}(h) = (\nu_i)_h \cdot \beta_{i,j} : B_j \to {}^{h^{-1}}B_i)_{1 \le i,j \le n,\, h \in H}$ of R-homomorphisms, where the homomorphisms $\beta_{i,j} : B_j \to B_i$ are defined by 2.1(ii).

Recall that for any k-vector space W and an R-module M we denote by $W \otimes_k M$ the R-module which assigns to each $x \in \mathrm{ob}\, R$ the k-vector space $W \otimes_k M(x)$ and to each $r \in R(x,y)$ the k-linear homomorphism $\mathrm{id}_W \otimes_k M(r) : W \otimes_k M(y) \otimes W \otimes_k M(x)$.

Let $\underline{V} = (\underline{V}_i)_{i=1,\ldots,n}$ be a sequence of complementary direct summands for V in I_n-spr(kH). We set

$$\underline{V} \otimes_k B = \bigoplus_{i=1}^{n} \underline{V} \otimes_k B_i$$

and define an R-action $\underline{\mu} \otimes_k B$ of H on the R-module $\underline{V} \otimes_k B$ as follows.

Let $(\underline{\mu} \otimes_k B)_h : \underline{V} \otimes_k B \to {}^{h^{-1}}(\underline{V} \otimes_k B)$, $h \in H$, be the R-homomorphism given by the matrix

$$(\underline{\mu} \otimes_k B)_h = [\mu(h)_{i,j} \otimes_k \beta_{i,j}(h)]_{1 \le i,j \le n} : \bigoplus_{i=1}^{n} \underline{V}_j \otimes_k B_j \to \bigoplus_{i=1}^{n} {}^{h^{-1}}(\underline{V}_j \otimes_k B_j).$$

Lemma. *The family $\underline{\mu} \otimes_k B = ((\underline{\mu} \otimes_k B)_h)_{h \in H}$ is an R-action of H on $\underline{V} \otimes_k B$.*

Proof. We show that $((\underline{V} \otimes_k B)^{(k)}, (\underline{\mu} \otimes_k B)^{(k)})$ defines a left kH-module (cf. 2.4). Note that for any $h \in H$ the (x,y)th component $(\underline{\mu} \otimes_k B)^{(k)}(h)_{x,y} : (\underline{V} \otimes_k B)(y) \to (\underline{V} \otimes_k B)(x)$, $x, y \in \mathrm{ob}\, R$, of the k-linear endomorphism $(\underline{\mu} \otimes_k B)^{(k)}(h)$ of the k-vector space $(\underline{V} \otimes_k B)^{(k)} = \bigoplus_{x \in \mathrm{ob}\, R} (\bigoplus_{i=1}^{n} \underline{V}_i \otimes_k B_i(x))$ is given by

$$(\underline{\mu} \otimes_k \beta)^{(k)}(h)_{x,y} = \begin{cases} \mu(h)_{i,j} \otimes_k \beta_{i,j}(h)(y) & \text{if } x = hy, \\ 0 & \text{if } x \ne hy. \end{cases}$$

We denote by $B^{(k)}$ the image of B under the functor $(-)^{(k)} : \mathrm{MOD}^H R \to$

$\mathrm{MOD}(kH)^{\mathrm{op}}$, i.e. the sequence

$$B^{(k)}: \quad B_1^{(k)} \xleftarrow{\beta_2^{(k)}} B_2^{(k)} \leftarrow \ldots \leftarrow B_{n-1}^{(k)} \xleftarrow{\beta_n^{(k)}} B_n^{(k)}$$

where $B_i^{(k)} = (B_i^{(k)}, \nu_i^{(k)})$ and $\beta_i^{(k)} = \bigoplus_{x\in \mathrm{ob}\, R} \beta_i(x)$ for every i, and by $\beta^{(k)}$ the collection $(\beta_{i,j}^{(k)}(h))_{1\le i,j\le n,\, h\in H}$ of k-linear maps $\beta_{i,j}^{(k)}(h) = \nu_i^{(k)}(h)\cdot \beta_{i,j}^{(k)}$. Then $\underline{V}\otimes_k B^{(k)} = (\underline{V}\otimes_k B^{(k)}, \underline{\mu}\otimes_k \beta^{(k)})$ is a left kH-module (see Lemma 2.1). Denote by $\eta(V) = \eta(\underline{V}) : (\underline{V}\otimes_k B)^{(k)} \to \underline{V}\otimes_k B^{(k)}$ the canonical k-isomorphism

$$\bigoplus_{x\in \mathrm{ob}\, R}\Big(\bigoplus_{i=1}^{n} \underline{V}_i \otimes_k B_i(x)\Big) \simeq \bigoplus_{i=1}^{n} \underline{V}_i \otimes_k \Big(\bigoplus_{x\in \mathrm{ob}\, R} B_i(x)\Big).$$

Observe that $\eta(V)\cdot(\underline{\mu}\otimes_k B)^{(k)}(h) = (\underline{\mu}\otimes_k B^{(k)})(h)\cdot \eta(V)$ for all $h\in H$. Indeed, fix $1\le i,j\le n$, $i\le j$, and $h\in H$. Then $\beta_{i,j}^{(k)}(h)_{hy,y} = \beta_{i,j}(h)(y)$ for every $y\in \mathrm{ob}\, R$, where $\beta_{i,j}^{(k)}(h)_{hy,y}$ is the (hy,y)th component of $\beta_{i,j}^{(k)}(h)$ (these are the only non-zero components). Consequently, $(\underline{\mu}\otimes_k B)^{(k)}$ is a k-linear action of H on $(\underline{V}\otimes_k B)^{(k)}$ and, by Lemma 2.4(b), the proof is complete. ∎

Remark. If $B_1 = \ldots = B_n = X$ and $\beta_2 = \ldots = \beta_n = \mathrm{id}_X$, for X in $\mathrm{Mod}^H R$, then the canonical isomorphism $\bigoplus_{i=1}^{n} \underline{V}_i \simeq V$ induces an isomorphism $\upsilon_{V,X} : V\otimes_k B \to V_n \otimes_k X$ in $\mathrm{Mod}^H R$ (if $n=1$, then $\upsilon_{V,X}$ is the identity map $\underline{V}\otimes_k B \to V_n\otimes_k X$).

2.6. Let $\underline{V} = (\underline{V}_i)_{i=1,\ldots,n}$, $\underline{V}' = (\underline{V}_i')_{i=1,\ldots,n}$ be sequences of complementary direct summands for V, V' in I_n-spr(kH) respectively, and $f: V\to V'$ be a morphism in I_n-spr(kH) given by a matrix $\underline{f} = [f_{i,j}]_{1\le i,j\le n}$ (see 2.2). We denote by $\underline{f}\otimes_k B : \underline{V}\otimes_k B \to \underline{V}'\otimes_k B$ the R-homomorphism defined by

$$\underline{f}\otimes_k B = [f_{i,j}\otimes_k \beta_{i,j}]_{1\le i,j\le n}$$

with R-linear components $f_{i,j}\otimes_k \beta_{i,j} : \underline{V}_j \otimes_k B_j \to \underline{V}_i' \otimes_k B_i$.

Lemma. *The R-homomorphism $\underline{f}\otimes_k B$ belongs to* $\mathrm{Hom}_R^H(\underline{V}\otimes_k B, \underline{V}'\otimes_k B)$.

Proof. We prove that $(\underline{f}\otimes_k B)^{(k)} : (\underline{V}\otimes_k B)^{(k)} \to (\underline{V}'\otimes_k B)^{(k)}$ is a kH-homomorphism (cf. 2.4). Keeping the notation from the proof of Lemma 2.5 observe that $\eta(V) : (\underline{V}\otimes_k B)^{(k)} \to \underline{V}\otimes_k B^{(k)}$ is a kH-isomorphism (for any V). Next note that $(\underline{f}\otimes_k B^{(k)})\cdot \eta(V) = \eta(V')\cdot(\underline{f}\otimes_k B)^{(k)}$ where $\underline{f}\otimes_k B^{(k)} : \underline{V}\otimes_k B^{(k)} \to \underline{V}'\otimes_k B^{(k)}$. Since from Lemma 2.2, $\underline{f}\otimes_k B^{(k)}$ is a kH-homomorphism, the assertion follows by Lemma 2.4(c). ∎

2.7. To define the functor

$$-\otimes_k B : I_n\text{-spr}(kH) \to \mathrm{Mod}^H R$$

we proceed analogously to 2.3. For every object V in I_n-spr(kH) we fix a sequence of complementary direct summands $\underline{V} = (\underline{V}_i)_{i=1,\dots,n}$. Then we set

$$V \otimes_k B = \underline{V} \otimes_k B$$

for any object V in I_n-spr(kH), and

$$f \otimes_k B = \underline{f} \otimes_k B$$

for any morphism $f : V \to V'$, where $\underline{f} = [f_{i,j}]_{1\le i,j\le n}$ is the matrix representation of f with respect to $\underline{V}$ and $\underline{V}'$.

Proposition. *The mapping* $-\otimes_k B : I_n\text{-spr}(kH) \to \mathrm{Mod}^H R$ *is a* k*-linear functor.*

Proof. The mapping $-\otimes_k B$ is well defined on objects and morphisms by Lemmas 2.5 and 2.6. The equality $\mathrm{id}_{V\otimes_k B} = \mathrm{id}_V \otimes_k B$ is again easy to check. To show that

$$(f' \otimes_k B)\cdot(f\otimes_k B) = f'f \otimes_k B$$

for morphisms $f : V \to V'$ and $f' : V' \to V''$ in I_n-spr(kH), we consider the functor

$$-\otimes_k B^{(k)} : I_n\text{-spr}(kH) \to \mathrm{MOD}(kH)^{\mathrm{op}}$$

based on the same fixed selection of sequences of complementary direct summands (we keep the notation from the proof of Lemma 2.5). Since

$$(f' \otimes_k B^{(k)})\cdot(f\otimes_k B^{(k)}) = (f'f \otimes_k B^{(k)})$$

and $(f\otimes_k B^{(k)})\cdot\eta(V) = \eta(V')\cdot(f\otimes_k B)^{(k)}$, $(f'\otimes_k B^{(k)})\cdot\eta(V') = \eta(V'')\cdot(f'\otimes_k B)^{(k)}$, $(ff'\otimes_k B^{(k)})\cdot\eta(V) = \eta(V'')\cdot(f'f\otimes_k B)^{(k)}$ (see proof of Lemma 2.6) we have

$$(f'\otimes_k B)^{(k)}\cdot(f\otimes_k B)^{(k)} = (f'f\otimes_k B)^{(k)}$$

and by Lemma 2.4(a) the proof is complete. ∎

Remark. (a) The family $\eta = \{\eta(V)\}_{V\in \mathrm{ob}\, I_n\text{-spr}(kH)}$ yields an isomorphism of functors

$$(-\otimes_k B)^{(k)}, -\otimes_k B^{(k)} : I_n\text{-spr}(kH) \to \mathrm{MOD}(kH)^{\mathrm{op}}.$$

(b) Different choices of sequences of complementary direct summands $\underline{V}$ V in I_n-spr(kH) lead to isomorphic functors.

(c) If all B_i, $i = 1,\dots,n$, are in $\mathrm{Mod}_{\mathrm{f}}^H R$ then the functor $-\otimes_k B$ leads in fact to $\mathrm{Mod}_{\mathrm{f}}^H R$.

2.8. Suppose we are given another sequence

$$B' : \quad B'_1 \overset{\beta'_2}{\leftarrow} B'_2 \leftarrow \ldots \leftarrow B'_{n-1} \overset{\beta'_n}{\leftarrow} B'_n$$

in $\mathrm{Mod}^H R$ and a map $\phi : B \to B'$ of sequences, i.e. a sequence $\phi_i : B_i \to B'_i$, $i = 1, \ldots, n$, of morphisms in $\mathrm{Mod}^H R$ such that $\beta'_i \phi_i = \phi_{i-1}\beta_i$ for every $i > 2$. For any V in I_n-spr(kH) with a fixed sequence $\underline{V} = (\underline{V}_i)_{i=1,\ldots,n}$ of complementary direct summands, we define the R-homomorphism

$$V \otimes_k \phi : V \otimes_k B \to V \otimes_k B'$$

by setting

$$V \otimes_k \phi = \bigoplus_{i=1}^{n} \mathrm{id}_{\underline{V}_i} \otimes_k \phi_i.$$

Proposition. (a) *$V \otimes_k \phi$ belongs to $\mathrm{Hom}_R^H(V \otimes_k B, V \otimes_k B')$.*

(b) *The family $- \otimes_k \phi = \{V \otimes_k \phi\}_{V \in \mathrm{ob}\, I_n\text{-spr}(kH)}$ yields an isomorphism of functors*

$$- \otimes_k B, - \otimes_k B' : I_n\text{-spr}(KH) \to \mathrm{Mod}^H R.$$

Proof. Proceeding as before, one proves first the corresponding result when B, B' are sequences in $\mathrm{MOD}(KH)^{\mathrm{op}}$, and then, by applying Lemma 2.4, the assertions (a) and (b). ∎

Let X be in $\mathrm{Mod}^H R$ and $X^{[n]}$ be the sequence

$$X^{[n]} : \quad X \overset{\mathrm{id}_X}{\longleftarrow} X \overset{\mathrm{id}_X}{\longleftarrow} \ldots \overset{\mathrm{id}_X}{\longleftarrow} X \overset{\mathrm{id}_X}{\longleftarrow} X$$

of length n in $\mathrm{Mod}^H R$. Then for any morphism $f_1 : X \to B_n$ (resp. $f_2 : B_1 \to X$) in $\mathrm{Mod}^H R$ we denote by the $f_1^{[n]} : X^{[n]} \to B$ (resp. $f_2^{[n]} : B \to X^{[n]}$) the map of sequences given by the morphisms $\beta_{i,n} \cdot f_1 : X \to B_i$ (resp. $f_2 \cdot \beta_{1,i} : B_i \to X$), $i = 1, \ldots, n$. We denote by $V_n \otimes f_1^{[n]}$ the composite R-homomorphism

$$V_n \otimes_k X \xrightarrow{\upsilon^{-1}} V \otimes_k X^{[n]} \xrightarrow{V \otimes f_1^{[n]}} V \otimes_k B$$

and by $V_n \otimes f_2^{[n]}$ the composite R-homomorphism

$$V_n \otimes f_2^{[n]} : V \otimes_k B \xrightarrow{V \otimes f_2^{[n]}} V \otimes_k X^{[n]} \xrightarrow{\upsilon} V_n \otimes_k X$$

where $\upsilon = \upsilon_{V,X}$ (see Remark 2.5).

Corollary. *$V_n \otimes f_1^{[n]}$ and $V_n \otimes f_2^{[n]}$ are morphisms in $\mathrm{Mod}^H R$.*

2.9. For any $1 \le i \le j \le n$, we denote by $B^{[i,j]}$ the restriction of the sequence B to the interval $[i, j]$, i.e. the sequence

$$B^{[i,j]} : \quad B_i \overset{\beta_{i+1}}{\longleftarrow} \ldots \overset{\beta_j}{\longleftarrow} B_j$$

in $\mathrm{Mod}^H R$ of length $j - i + 1$.

Let V be an object in I_n-spr(kH). For any $i = 1, \dots, n$, we denote by $V_{(i)}$ the object $(V_1 \subseteq \dots \subseteq V_i)$ in I_i-spr(kH), and for any $i = 0, \dots, n-1$, by V/V_i the object $(V_{i+1}/V_i \subseteq \dots \subseteq V_n/V_i)$ in I_{n-i}-spr(kH), where $V_0 = 0$. If $\underline{V} = (\underline{V}_j)_{j=1,\dots,n}$ is a sequence of complementary direct summands for V then $(\underline{V}_j)_{j=1,\dots,i}$ is a sequence of complementary direct summands for $V_{(i)}$, and $(\underline{V}_{j,i})_{j=i+1,\dots,n}$, where $\underline{V}_{j,i} = (\underline{V}_j + V_i)/V_i$ $(= (\underline{V}_j \oplus V_i)/V_i \simeq \underline{V}_j)$, is a sequence of complementary direct summands for V/V_i.

For any $0 \le i < l \le j \le n$, let

$$v_{i,l,j} : \bigoplus_{t=i+1}^{l} \underline{V}_{t,i} \otimes_k B_t \to \bigoplus_{t=i+1}^{j} \underline{V}_{t,i} \otimes_k B_t$$

be the canonical embedding of R-modules, and, for any $0 \le i \le l < j \le n$,

$$r_{j,l,i} : \bigoplus_{t=i+1}^{l} \underline{V}_{t,i} \otimes_k B_t \oplus \bigoplus_{t=l+1}^{j} \underline{V}_{t,i} \otimes_k B_t \to \bigoplus_{t=l+1}^{j} \underline{V}_{t,l} \otimes_k B_t$$

the R-epimorphism given by the components $(0, \bigoplus_{t=l+1}^{j} \kappa_t \otimes B_t)$, where κ_t denotes the composition $\underline{V}_{t,i} \simeq \underline{V}_t \simeq \underline{V}_{t,l}$ of the canonical isomorphisms.

LEMMA. *For any $1 \le i \le l < j \le n$, the sequence*

$$0 \to V_{(l)}/V_{i-1} \otimes_k B^{[i,l]} \xrightarrow{v} V_{(j)}/V_{i-1} \otimes_k B^{[i,j]} \xrightarrow{r} V_{(j)}/V_l \otimes_k B^{[l+1,j]} \to 0$$

is an exact sequence in $\mathrm{Mod}^H R$, *where $v = v_{i-1,j,l}$ and $r = r_{j,l,i-1}$.*

Proof. The exactness (in $\mathrm{Mod}\, R$) and the fact that v, r are morphisms in $\mathrm{Mod}^H R$ follow immediately from definitions. ■

Let W be a sequence $W_1 \xrightarrow{p_1} W_2 \xrightarrow{p_2} \dots \xrightarrow{p_{n-1}} W_n$ of epimorphisms in $\mathrm{MOD}(kH)^{\mathrm{op}}$. With W we can associate the object $V(W) = (V_1 \subseteq \dots \subseteq V_n)$ in I_n-spr(kH) given by $V_i = \mathrm{Ker}(p_i \cdot \ldots \cdot p_1)$ for $i = 1, \dots, n$ (p_n is the map $W_n \to 0$). Then we define

$$W \otimes_k B = V(W) \otimes_k B.$$

In particular, for any morphism $f : B_1 \to X$ in $\mathrm{Mod}^H R$ we have a morphism $W_1 \otimes f^{[n]} : W \otimes_k B \to W_1 \otimes_k X$ in $\mathrm{Mod}^H R$, where $W_1 \otimes f^{[n]} = V_n \otimes f^{[n]}$ (see Corollary 2.8).

Conversely, with any $V = (V_1 \subseteq \dots \subseteq V_n)$ in I_n-spr(kH) we can associate the sequence $W(V)$ of the canonical projections

$$V_1 \xrightarrow{p_1} V_n/V_1 \xrightarrow{p_2} \dots \xrightarrow{p_{n-1}} V_n/V_{n-1}$$

in $\mathrm{MOD}(kH)^{\mathrm{op}}$, induced by the inclusions from V. Then $W(V) \otimes_k B = V \otimes_k B$, and consequently $W \otimes_k B = W(V(W)) \otimes_k B$ for every W.

For a given W as above and any $i = 1, \dots, n$, we denote by $W_{(i)}$ the sequence $W_i \xrightarrow{p_i} W_{i+1} \xrightarrow{p_{i+1}} \dots \xrightarrow{p_{n-1}} W_n$. Then applying the lemma to $V =$

$V(W)$, the canonical isomorphisms $W_i \simeq V_n/V_{i-1}$, $i = 1, \ldots, n$, yield the following result.

Corollary. *For any* $1 < i \le n$, *the sequence*

$$0 \to \operatorname{Ker} p_{i-1} \otimes_k B_{i-1} \xrightarrow{v} W_{(i-1)} \otimes_k B^{[i-1,n]} \xrightarrow{r} W_{(i)} \otimes_k B^{[i,n]} \to 0$$

is exact in $\operatorname{Mod}^H R$.

3. On some construction of non-orbicular modules. In this section we apply a generalized tensor product to construct a functor from I_n-spr(kH) to $\operatorname{mod}(R/G)$ whose image contains a large subcategory consisting of non-orbicular modules.

3.1. Let H be a subgroup of the group G, where $G \subseteq \operatorname{Aut}_k(R)$ is a group of k-linear automorphisms acting freely on R. Recall [3, 2.3] that we have at our disposal the induction functor

$$\theta = \theta_H^G : \operatorname{Mod}_{\mathrm{f}}^H R \to \operatorname{Mod}_{\mathrm{f}}^G R.$$

For any $M = (M, \mu)$ in $\operatorname{Mod}_{\mathrm{f}}^H R$, $\theta(M)$ is defined by setting $\theta(M) = (\bigoplus_{g_1 \in S_H} {}^{g_1}M, \mu^G)$. The R-isomorphisms $\mu_g^G : \bigoplus_{g_1 \in S_H} {}^{g_1}M \to \bigoplus_{g_2 \in S_H} {}^{g^{-1}g_2}M$, $g \in G$, are given by the families ${}^{g_1}\mu_h : {}^{g_1}M \to {}^{g^{-1}g_2}M$, $g_1 \in S_H$, where $g_2 \in S_H$ and $h \in H$ are determined by the equality $gg_1 = g_2 h$. Here S_H is a fixed set of representatives of left cosets in G/H containing the unit e.

Let B be a sequence

$$B: \quad B_1 \xleftarrow{\beta_2} B_2 \leftarrow \ldots \leftarrow B_{n-1} \xleftarrow{\beta_n} B_n$$

of morphisms in $\operatorname{Mod}_{\mathrm{f}}^H R$, where $B_i = (B_i, \nu_i)$ for all $i = 1, \ldots, n$. Then we denote by $\widetilde{\Phi}^B$ the composite functor

$$I_n\text{-spr}(kH) \xrightarrow{-\otimes_k B} \operatorname{Mod}_{\mathrm{f}}^H R \xrightarrow{\theta} \operatorname{Mod}_{\mathrm{f}}^G R$$

(see Remark 2.7(c)). We also set

$$\Phi^B = F_\bullet^{-1} \circ \widetilde{\Phi}^B : I_n\text{-spr}(KH) \to \operatorname{mod}(R/G)$$

where $F_\bullet^{-1}$ is a fixed quasi-inverse functor of $F_\bullet : \operatorname{mod}(R/G) \to \operatorname{Mod}_{\mathrm{f}}^G R$.

Set $\mathcal{B}_{\mathrm{o}} = \{B_1, \ldots, B_n\}$ and $\mathcal{B} = \{{}^g B_i\}_{i=1,\ldots,n;\, g \in S_H}$. Observe that if all B_i's are G-atoms (consequently H is a subgroup of G_{B_i} of finite index for every i) then $\operatorname{Im} \Phi^B \subset \operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)$ since $\operatorname{Im} \widetilde{\Phi}^B \subset \operatorname{Mod}_{\mathrm{f},\mathcal{B}_{\mathrm{o}}}^G R$. Here, $\operatorname{Mod}_{\mathrm{f},\mathcal{B}_{\mathrm{o}}}^G R$ denotes the subcategory of $\operatorname{Mod}_{\mathrm{f}}^G R$ corresponding via $F_\bullet$ to $\operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)$. Moreover, if $G_{B_i} = H$ for every i, then for any V in I_n-spr$(kH)^{\mathrm{op}}$ we have

$$\operatorname{dsc}(\Phi^B(V)) = \operatorname{cdn}(V)$$

under the identification via the canonical embedding $\mathbb{N}^n \hookrightarrow \mathbb{N}^{\mathcal{A}_{\mathrm{o}}}$ given by $i \mapsto B_i$.

We also denote by $\mathcal{B}_{\rm o}$ (resp. $\mathcal{B}$) the full subcategory of $\operatorname{Mod} R$ formed by $\mathcal{B}_{\rm o}$ (resp. $\mathcal{B}$). By $\widetilde{\mathcal{B}}$ we denote the full subcategory of $\operatorname{Mod} R$ formed by all R-modules M of the form

$$\text{(i)} \qquad M \simeq \bigoplus_{g\in S_H} \bigoplus_{i=1}^{n} {}^{g}B_i^{d_{i,g}},$$

$d_{i,g} \in \mathbb{N}$. Recall [12, 6] that in $\operatorname{Mod} R$ we have the uniqueness of decomposition into a direct sum of indecomposables.

Let $\mathcal{N}$ be an ideal in $\mathcal{B}$. Then we denote by $\mathcal{N}_{\rm o}$ the restriction of $\mathcal{N}$ to $\mathcal{B}_{\rm o}$. If $\mathcal{N}$ is summably closed (see definition below) then we denote by $\widetilde{\mathcal{N}}$ the ideal extension of $\mathcal{N}$ to $\widetilde{\mathcal{B}}$ given by the formula

$$\text{(ii)} \quad \widetilde{\mathcal{N}}\Big(\bigoplus_{g\in S_H} \bigoplus_{i=1}^{n} {}^{g}B_i^{d_{i,g}}, \bigoplus_{g'\in S_H} \bigoplus_{j=1}^{n} {}^{g'}B_j^{d'_{j,g'}} \Big) = \prod_{g,g'\in S_H} \prod_{i,j=1}^{n} \mathrm{M}_{d'_{j,g'}\times d_{i,g}}(\mathcal{N}({}^{g}B_i, {}^{g'}B_j))$$

(cf. [5]). Note that since $\mathcal{N}$ is summably closed, $\widetilde{\mathcal{N}}$ is a well defined ideal in $\widetilde{\mathcal{B}}$. In particular, the above formula uniquely (independently of the choice of the isomorphisms (i)) determines the value $\widetilde{\mathcal{N}}(M, M')$ for any M, M' in $\widetilde{\mathcal{B}}$.

Following [5], $\mathcal{N}$ is said to be *summably closed* provided each subspace $\mathcal{N}(B', B'') \subseteq \operatorname{Hom}_R(B', B'')$, $B', B'' \in \mathcal{B}$, is summably closed. This by definition means that for any summable family of R-homomorphisms $f_i \in \mathcal{N}(B', B'')$, $i \in I$, (i.e. for each $x \in \operatorname{ob} R$, $f_i(x) = 0$ for almost all i) the sum $f = \sum_{i\in I} f_i$ belongs to $\mathcal{N}(B', B'')$.

Let $G_{B_i} = H$ for every $i = 1, \dots, n$. We say that an ideal $\mathcal{N}$ in $\mathcal{B}$ is *determined* by the ideal $\mathcal{N}_{\rm o}$ in $\mathcal{B}_{\rm o}$ provided

$$\text{(iii)} \qquad \mathcal{N}({}^{g}B_i, {}^{g'}B_j) = \begin{cases} \operatorname{Hom}_R({}^{g}B_i, {}^{g'}B_j) & \text{if } g \neq g', \\ {}^{g}\mathcal{N}_{\rm o}(B_i, B_j) & \text{if } g = g', \end{cases}$$

where $i, j \in \{1, \dots, n\}$, $g, g' \in S_H$.

Remark. Any family $\mathcal{M}$ of subspaces $\mathcal{M}(B', B'') \subseteq \operatorname{Hom}_R(B, B'')$, $B', B'' \in \mathcal{B}_{\rm o}$, can be extended to the family $\mathcal{N}$ of subspaces $\mathcal{N}(B', B'') \subseteq \operatorname{Hom}_R(B, B'')$, $B', B'' \in \mathcal{B}$, by applying formula (iii). Then $\mathcal{N}$ is an ideal in $\mathcal{B}$ (and $\mathcal{N}_{\rm o} = \mathcal{M}$) if and only if $\mathcal{M}$ is an ideal in $\mathcal{B}_{\rm o}$ and for any $f \in \operatorname{Hom}_R(B_i, {}^{g}B_l)$, $f' \in \operatorname{Hom}_R({}^{g}B_l, B_j)$ the composition $f'f$ belongs to $\mathcal{M}(B_i, B_j)$ for all $B_i, B_j, B_l \in \mathcal{B}_{\rm o}$ such that $\mathcal{M}(B_i, B_j) \subsetneq \operatorname{Hom}_R(B_i, B_j)$, and $g \in S_H$, $g \neq e$. In this situation the ideal $\mathcal{N}$ is summably closed if and only if so is $\mathcal{N}_{\rm o} = \mathcal{M}$, and then $\widetilde{\mathcal{N}}$ is a well defined ideal in $\widetilde{\mathcal{B}}$ (also summably closed).

Recall (see [5]) that for any objects $M' = (M', \mu')$, $M'' = (M'', \mu'')$ in $\mathrm{Mod}^H R$ the space $\mathrm{Hom}_R(M', M'')$ carries the structure of a left kH-module which is given by $(h, f) \mapsto h * f = {}^h\mu''_h \cdot {}^h f \cdot \mu'_{h^{-1}}$ for $h \in H$ and $f \in \mathrm{Hom}_R(M', M'')$.

An ideal $\mathcal{M}$ in $\mathcal{B}_{\mathrm{o}}$ is called *H-invariant* provided $\mathcal{M}(B_i, B_j)$ is a kH-submodule of the kH-module $\mathrm{Hom}_R(B_i, B_j)$ for all $i, j = 1, \ldots, n$. Note that this definition does not depend on the choice of R-actions ν_i of H on B_i, $i = 1, \ldots, n$.

Following [5] we denote by $\mathcal{P}u$ the *pure-projective ideal* which by definition is the two-sided ideal in $\mathrm{MOD}\, R$ given by the subspaces $\mathcal{P}u(M, N) \subseteq \mathrm{Hom}_R(M, N)$, M, N in $\mathrm{MOD}\, R$, consisting of all R-homomorphisms $f : M \to N$ having a factorization through a direct sum of finite-dimensional R-modules. Note that the ideal $\mathcal{P}u_{\mathcal{B}_{\mathrm{o}}}$ is H-invariant, and by [5, Theorem A(ii)], $\mathcal{P}u_{\mathcal{B}_{\mathrm{o}}}$ is summably closed provided H is an infinite cyclic group. One can show (see Remark 3.5) that then $\mathcal{P}u_{\mathcal{B}}$ is also summably closed ($\mathcal{P}u_{\widetilde{\mathcal{B}}}$ is not necessarily so).

Now we are able to formulate our first main result of this paper.

THEOREM. *Let H be a subgroup of a group $G \subseteq \mathrm{Aut}_k(R)$ acting freely on R. Suppose we are given a sequence B in $\mathrm{Mod}_{\mathrm{f}}^H R$ as above such that all B_i's are G-atoms with $G_{B_i} = H$, $i = 1, \ldots, n$. Assume that $\beta_{i,j} \neq 0$ for all $1 \leq i \leq j \leq n$, and that $\mathcal{B}$ contains an ideal $\mathcal{N}$ determined by an H-invariant summably closed ideal $\mathcal{N}_{\mathrm{o}}$ in $\mathcal{B}_{\mathrm{o}}$ satisfying the condition*

$$(*) \qquad \mathrm{Hom}_R(B_j, B_i) = \mathcal{N}_{\mathrm{o}}(B_j, B_i) \oplus k\beta_{i,j}$$

for all $1 \leq i, j \leq n$ (see 2.1 for definition of $\beta_{i,j}$). *Then the functor*

$$\Phi^B : I_n\text{-spr}(kH) \to \mathrm{mod}(R/G)$$

is a representation embedding (*in the sense of* [27]). *Moreover*,

(a) *if $H \simeq \mathbb{Z}$ and $\mathcal{N} = \mathcal{P}u_{\mathcal{B}}$ then $\Phi^B : I_n\text{-spr}(kH) \to \mathrm{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)$ is dense and induces an equivalence*

$$I_n\text{-spr}(kG_B) \simeq \mathrm{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)/[\mathrm{mod}_{\mathcal{A}_{\mathrm{o}}^{\mathrm{f}}}(R/G)]_{\mathrm{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)},$$

(b) *if $G = H$ and $\mathcal{N}_{\mathrm{o}} = 0$, then Φ^B yields an equivalence $I_n\text{-spr}(kH) \simeq \mathrm{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)$*,

(c) *if $n \geq 2$ and H has a factor which is an infinite cyclic group* (*resp. a cyclic p-group of order greater than* 7, *if* $\mathrm{char}(k) = p > 0$) *then the full subcategory formed by all indecomposable non-orbicular modules in* $\mathrm{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)$ *is wild.*

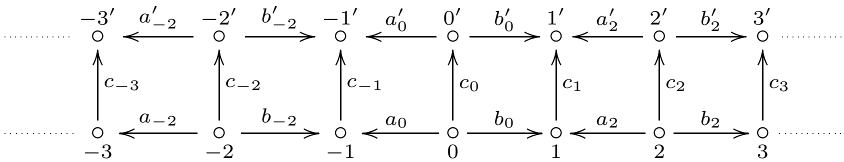
Note that the condition $(*)$ implies that all B_i's, $i = 1, \ldots, n$, are pairwise non-isomorphic ($\mathcal{N}_{\mathrm{o}}(B_i, B_i) \subsetneq \mathrm{End}_R(B_i)$ and $\mathcal{N}_{\mathrm{o}}(B_j, B_i) = \mathrm{Hom}_R(B_j, B_i)$ for $i > j$).

The proof of the theorem consists of several facts stated in 3.2–3.7. Its most important part is the construction of a functor $\widetilde{\Psi}^B$ (left quasi-inverse to $\widetilde{\Phi}^B$) which is done in a few steps. Therefore, we first formulate an immediate important consequence of Theorem 3.1 and illustrate it by an example.

Corollary. *Let R, G, and H be as in Theorem* 3.1. *Assume, in addition, that G acts freely on* $(\operatorname{ind} R)/\simeq$ *and H is an infinite group. Under the assumptions in* 3.1(c) *the category* $\operatorname{mod}_2(R/G)$ *is wild.*

Proof. Under the above assumption we have $\operatorname{mod}_{\mathcal{B}_\circ}(R/G) \subset \operatorname{mod}_2(R/G)$ (see [12, 2.3]). ∎

Example. Let R be the locally bounded k-category opposite to the category kQ/I, where Q is the quiver



and I is the ideal of the path category kQ generated by all elements of the form $c_{i-1}a_i - a'_i c_i$ and $c_{i+1}b_i - b'_i c_i$, $i \in 2\mathbb{Z}$. The category R is equipped with a natural free action of the infinite cyclic subgroup $G = \langle g \rangle$ of $\operatorname{Aut}_k(R)$, where g is defined by the equalities $g(i) = i+2$, $g(i') = (i+2)'$, for $i \in \mathbb{Z}$. Let B_1 be the "line" R-module given by $B_1(i) = k$, $B_1(i') = 0$, $B_1(a_{2i}) = B_1(b_{2i}) = \mathrm{id}_k$ for all $i \in \mathbb{Z}$, and $B_1(\gamma) = 0$ for all other arrows γ in Q. We also define the R-module B_2 by setting $B_2(i) = B_2(i') = k$ and $B_2(\gamma) = \mathrm{id}_k$ for all $i \in \mathbb{Z}$ and arrows γ in Q. Moreover, we consider the second "line" R-module B_3 given by $B_3(i') = k$, $B_3(i) = 0$, $B_3(a'_{2i}) = B_3(b'_{2i}) = \mathrm{id}_k$ for all $i \in \mathbb{Z}$, and $B_3(\gamma) = 0$ for all other arrows γ in Q. Clearly B_1, B_2, B_3 are G-atoms with the common stabilizer $H = G$ and they admit natural R-actions of H. Denote by $\beta_2 : B_2 \to B_1$ (resp. $\beta_3 : B_3 \to B_2$) the R-homomorphisms given by $\beta_2(i) = \mathrm{id}_k$ and $\beta_2(i') = 0$ (resp. $\beta_2(i') = \mathrm{id}_k$ and $\beta_2(i) = 0$) for $i \in \mathbb{Z}$. The maps β_1 and β_2 can be regarded as morphisms in $\operatorname{Mod}^H R$, but the sequence

$$B_1 \overset{\beta_2}{\leftarrow} B_2 \overset{\beta_3}{\leftarrow} B_3$$

does not satisfy the assumptions of the theorem ($\beta_2\beta_3 = 0$), in contrast to the sequence

$$B: \quad B_1 \overset{\beta_2}{\leftarrow} B_2$$

(take for $\mathcal{N}_\circ$ the zero ideal). Therefore the functor

$$\Phi^B : I_2\text{-spr}(kH) \to \operatorname{mod}(R/G)$$

is a representation embedding and $\operatorname{mod}_{\{B_1,B_2\}}(R/G)$ ($\subset \operatorname{mod}_2(R/G)$) contains a wild subcategory of non-orbicular modules (the same holds for the sequence $B : B_2 \overset{\beta_3}{\leftarrow} B_3$). Note that this example can be easily generalized (by adding "new layers" in the quiver Q) to obtain analogous embeddings for sequences B of arbitrary length $n \geq 2$.

3.2. Denote by $\boldsymbol{I} = \boldsymbol{I}_B$ the full subcategory of $\mathrm{Mod}_{\mathrm{f}}^G R$, contained in $\mathrm{Mod}_{\mathrm{f},\mathcal{B}_{\mathrm{o}}}^G R$, formed by all objects $\widetilde{\Phi}^B(V)$ with V in I_n-spr(KH). We construct a functor

$$\widetilde{\Psi}^B : \boldsymbol{I} \to I_n\text{-spr}(kH)$$

which is a left quasi-inverse of $\widetilde{\Phi}^B$.

For any $i = 1, \ldots, n$, we denote by

$$\mathcal{H}_i : \mathrm{Mod}_{\mathrm{f}}^G R \to \mathrm{MOD}(kH)^{\mathrm{op}}$$

the functor induced by the functor $\mathrm{Hom}_R(B_i, -) : \mathrm{Mod}\, R \to \mathrm{MOD}\, k$ which assigns to each $M = (M, \mu)$ in $\mathrm{Mod}_{\mathrm{f}}^G R$ the left kH-module $\mathrm{Hom}_R(B_i, M)$ with the kH-module structure given by the R-actions ν^i and $\mu_{|H}$ of H on B_i and M, respectively.

Lemma. *Let M be an object in $\boldsymbol{I}$. If an ideal $\mathcal{N}$ in $\mathcal{B}$ is determined by $\mathcal{N}_{\mathrm{o}}$ and $\mathcal{N}_{\mathrm{o}}$ is an H-invariant summably closed ideal in $\mathcal{B}_{\mathrm{o}}$ then the k-subspace $\widetilde{\mathcal{N}}(B_i, M) \subseteq \mathrm{Hom}_R(B_i, M)$ is a kH-submodule of $\mathcal{H}_i(M)$ for every $i = 1, \ldots, n$.*

Proof. Let V in I_n-spr(KH) be such that $\widetilde{\Phi}^B(V) = M$. Then $M = (\bigoplus_{g\in S_H} {}^g(V \otimes_k B), (\mu \otimes_k B)^G)$, where μ is a k-linear action defining the kH-module structure on V_n. Take any $h \in H$ and $f \in \widetilde{\mathcal{N}}(B_i, \bigoplus_{g\in S_H} {}^g(V \otimes_k B))$ with components $f_g \in \widetilde{\mathcal{N}}(B_i, {}^g(V \otimes_k B))$, $g \in S_H$. To show that $h * f : B_i \to \bigoplus_{g\in S_H} {}^g(V \otimes_k B)$ belongs to $\widetilde{\mathcal{N}}$ we have to verify that so do all components $(h * f)_g$, $g \in S_H$. In fact, we only need to show that $(h * f)_e \in \widetilde{\mathcal{N}}(B_i, {}^g(V \otimes_k B))$ since $\mathrm{Hom}_R(B, {}^gB_j) = \mathcal{N}(B, {}^gB_j)$ for all $j = 1, \ldots, n$ and $e \neq g \in S_H$. Note that M decomposes as $(V \otimes_k B) \oplus (\bigoplus_{e\neq g\in S_H} {}^g(V \otimes_k B))$ in $\mathrm{Mod}^H R$ (see definition of θ_H^G). Therefore $(h * f)_e = {}^h(\mu \otimes_k B)_h \cdot {}^hf_e \cdot (\nu_i)_{h^{-1}}$. The map f_e (resp. $(h * f)_e$) is given by components f_j (resp. $(h * f)_j$) in $\widetilde{\mathcal{N}}(B_i, \underline{V}_j \otimes_k B_j)$, $j = 1, \ldots, n$, where $\underline{V} = (\underline{V}_j)_{j=1,\ldots,n}$ is a fixed sequence of complementary direct summands for V. Then by definition we have

$$\text{(i)} \qquad (h * f)_j = \sum_{l=1}^{n} {}^h(\mu(h)_{j,l} \otimes_k \beta_{j,l}(h)) \cdot {}^hf_l \cdot (\nu_i)_{h^{-1}}$$

$$= \sum_{l=1}^{n} (\mu(h)_{j,l} \otimes_k {}^h\beta_{j,l}(h)) \cdot {}^hf_l \cdot (\nu_i)_{h^{-1}}$$

$$= \sum_{l=1}^{n} (\mu(h)_{j,l} \otimes_k (\beta_{j,l} \cdot {}^h(\nu_l)_h)) \cdot {}^hf_l \cdot (\nu_i)_{h^{-1}}.$$

We fix bases of the spaces $\underline{V}_l$, $l = 1, \dots, n$, and the induced isomorphisms $\underline{V}_l \otimes_k B_l \simeq B_l^{d_l}$, where $d_l = \dim_k \underline{V}_l$. Passing to components we conclude by (i) that each component of $(h * f)_j$ belongs to $\mathcal{N}_{\mathrm{o}}(B_i, B_j)$, since $f_l \in \mathcal{N}_{\mathrm{o}}(B_i, B_l)$ for all $l = 1, \dots, n$ ($\mathcal{N}_{\mathrm{o}}(B_i, B_l)$ is an H-invariant subspace of $\mathrm{Hom}_R(B_i, B_l)$). Consequently, $(h * f)_j \in \widetilde{\mathcal{N}}(B_i, \underline{V}_j \otimes_k B_j)$ for all $j = 1, \dots, n$, $(h * f)_e \in \widetilde{\mathcal{N}}(B_i, V \otimes_k B_j)$ and $h * f \in \widetilde{\mathcal{N}}(B_i, M)$. ∎

3.3. Suppose we are given an ideal $\mathcal{N}$ in $\mathcal{B}$ which satisfies the assumptions of Theorem 3.1. For any $i = 1, \dots, n$, we denote by $\overline{\mathcal{H}}_i$ the functor

$$\overline{\mathcal{H}}_i : \boldsymbol{I} \to \mathrm{MOD}(kH)^{\mathrm{op}}$$

which associates with M in $\boldsymbol{I}$ the kH-module $\mathcal{H}_i(M)/\widetilde{\mathcal{N}}(B_i, M)$ (see Lemma 3.2) and with any morphism $f : M \to M'$ in $\boldsymbol{I}$ the k-linear map $\overline{\mathcal{H}}_i(f) : \overline{\mathcal{H}}_i(M) \to \overline{\mathcal{H}}_i(M')$ induced by $\mathcal{H}_i(f) = \mathrm{Hom}_R(B_i, f)$. Note that $\overline{\mathcal{H}}_i(f)$ is well defined since $\mathcal{N}$ is an ideal, and that $\overline{\mathcal{H}}_i(f)$ is a kH-homomorphism, since so is $\mathcal{H}_i(f)$. Observe also that by analogous reasons the morphism $\beta_{i,j} : B_j \to B_i$ in $\mathrm{Mod}_{\mathrm{f}}^H R$ induces a kH-homomorphism $\iota_{j,i}(M) : \overline{\mathcal{H}}_i(M) \to \overline{\mathcal{H}}_j(M)$ for all $i \le j$. We set $\iota_{j,i} = \{\iota_{j,i}(M)\}_{M \in \mathrm{ob}\, \boldsymbol{I}}$. It is clear that each $\iota_{j,i}$ defines a natural transformation $\iota_{j,i} : \overline{\mathcal{H}}_i \to \overline{\mathcal{H}}_j$ of functors, and that by 2.1(iii) we have $\iota_{j,l} \cdot \iota_{l,i} = \iota_{j,i}$ for all $i \le l \le j$.

LEMMA. (a) $\mathrm{Im}(\overline{\mathcal{H}}_i) \subset \mathrm{mod}(kH)^{\mathrm{op}}$ *for every* $i = 1, \dots, n$.

(b) *Each* $\iota_{j,i}$ *is a natural embedding of functors*, *for* $i \le j$.

Proof. Fix $M = \widetilde{\Phi}^B(V)$ in $\mathrm{Im}\, \widetilde{\Phi}^B$, where V is in I_n-$\mathrm{spr}(kH)$. Then we have an R-isomorphism $M = \bigoplus_{g \in S_H} \bigoplus_{l=1}^n {}^g(\underline{V}_l \otimes_k B_l)$, where $(\underline{V}_l)_{\, l=1,\dots,n}$ is a fixed sequence of complementary direct summands for V and $d_l = \dim_k \underline{V}_l$. Then 3.1(ii), 3.1(∗) together with the isomorphisms

$$\text{(i)}_l \qquad \underline{V}_l \otimes_k B_l \simeq B_l^{d_l},$$

$l = 1, \dots, n$, given by fixing bases of the spaces $\underline{V}_l$, yield k-isomorphisms

$$\text{(ii)}_i \qquad \begin{aligned} \overline{\mathcal{H}}_i(M) &\simeq \prod_{g \in S_H} \prod_{l=1}^{n} \mathrm{Hom}_R(B_i, {}^gB_l)^{d_l} / \mathcal{N}(B_i, {}^gB_l)^{d_l} \\ &\simeq \bigoplus_{l=1}^{i} (k\beta_{l,i})^{d_l} \simeq \bigoplus_{l=1}^{i} \underline{V}_l \otimes_k k\beta_{l,i}, \end{aligned}$$

$i = 1, \dots, n$. Consequently, $\overline{\mathcal{H}}_i(M)$ is a finite-dimensional kH-module and (a) is proved.

To prove (b) note that the k-linear map $\iota_{j,i}(M)$ becomes, under the identifications (ii)_i and (ii)_j, the canonical embedding given by $\bigoplus_{l=1}^{i} \mathrm{id}_{\underline{V}_l} \otimes \cdot\, \beta_{i,j}$ for all $i \le j$. ∎

3.4. For every $i = 1, \ldots, n$, we denote by $\overline{\mathcal{H}}'_i$ the subfunctor $\iota_{n,i}(\overline{\mathcal{H}}_i)$ of $\overline{\mathcal{H}}_n$. We define the functor

$$\widetilde{\Psi}^B : \boldsymbol{I} \to I_n\text{-spr}(kH)$$

by setting

$$\widetilde{\Psi}^B(M) = \{\overline{\mathcal{H}}'_1(M) \subseteq \ldots \subseteq \overline{\mathcal{H}}'_n(M) = \overline{\mathcal{H}}_n(M)\}$$

for any object M in $\boldsymbol{I}$, and

$$\widetilde{\Psi}^B(f) = \overline{\mathcal{H}}_n(f)$$

for any morphism $f : M \to M'$ in $\boldsymbol{I}$. Note that $\widetilde{\Psi}^B(M)$ is an object of I_n-spr(kH), since $\iota_{j,i}$'s satisfy the commutativity condition, and $\overline{\mathcal{H}}_n(f)$ is a morphism in I_n-spr(kH), because $\iota_{j,i}$'s are natural transformations.

Remark. The functors $\overline{\mathcal{H}}_i$ and consequently $\widetilde{\Psi}^B$ can be extended, by the same formula, to the whole category $\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\circ} R$. In this way we obtain the functor $\Psi^B : \mathrm{mod}_{\mathcal{B}_\circ}(R/G) \to I_n\text{-spr}(kH)$, $\Psi^B = \widetilde{\Psi}^B \circ F_\bullet$, satisfying $\mathrm{cdn}(\Psi^B(X)) = \mathrm{dsc}(X)$ for X in $\mathrm{mod}_{\mathcal{B}_\circ}(R/G)$ (cf. 3.1).

To prove that Φ^B is a representation embedding it suffices to show the following.

Proposition. (a) *The functors* $\widetilde{\Psi}^B\widetilde{\Phi}^B$ *and* $\mathrm{id}_{I_n\text{-spr}(kH)}$ *are isomorphic.*

(b) $\mathrm{Ker}\,\widetilde{\Psi}^B$ *contains no non-zero idempotents.*

Proof. (a) Fix V in I_n-spr(kH) together with a sequence $\underline{V} = (\underline{V}_l)_{l=1,\ldots,n}$ of complementary direct summands for V ($d_l = \dim_k \underline{V}_l$). The identifications 3.3(ii)$_i$, $i = 1, \ldots, n$, yield $\overline{\mathcal{H}}'_i(\widetilde{\Phi}^B(V)) = \bigoplus_{l=1}^i \underline{V}_l \otimes_k k\beta_{l,n}$.

We show that the induced action of H on the k-vector space $\bigoplus_{l=1}^i \underline{V}_l \otimes_k k\beta_{l,i}$ is given by the family

$$\text{(i)}_i \qquad \{[\mu(h)_{m,l} \otimes_k \beta_{m,l}\, \cdot]_{m,l\in\{1,\ldots,i\}}\}_{h\in H}$$

of k-linear automorphisms (cf. 2.1).

For any $f \in \mathrm{Hom}_R(B_i, \bigoplus_{l=1}^n \underline{V}_l \otimes_k B_l)$ we denote by $\bar{f} = (\bar{f}_l)_{l=1,\ldots,i} \in \bigoplus_{l=1}^i (k\beta_{l,i})^{d_l}$ ($\bar{f}_l = a_l \cdot \beta_{l,i}$ for some $a_l \in k^{d_l}$) and $f' = (f'_l)_{l=1,\ldots,n} \in \bigoplus_{l=1}^n \mathcal{N}(B_i, B_l)^{d_l}$ the components of f under the isomorphism

$$\mathrm{Hom}_R\Big(B_i, \bigoplus_{l=1}^n \underline{V}_l \otimes_k B_l\Big) \simeq \bigoplus_{l=1}^i (k\beta_{l,i})^{d_l} \oplus \bigoplus_{l=1}^n \mathcal{N}(B_i, B_l)^{d_l},$$

induced by the identifications 3.3(i)$_l$ (cf. 3.1(∗)).

Recall that $\widetilde{\Psi}^B(V) = (\bigoplus_{l=1}^n \underline{V}_l \otimes_k B_l) \oplus (\bigoplus_{e\neq g\in S_H} \bigoplus_{l=1}^n {}^g(\underline{V}_l \otimes_k B_l))$ is a decomposition in $\mathrm{Mod}^H R$, therefore the kH-module $\mathrm{Hom}_R(B_i, \widetilde{\Psi}^B(V))$ decomposes as

$$\mathrm{Hom}_R\Big(B_i, \bigoplus_{l=1}^{n} \underline{V}_l \otimes_k B_l\Big) \oplus \mathrm{Hom}_R\Big(B_i, \bigoplus_{e\neq g\in S_H} \bigoplus_{l=1}^{n} {}^g(\underline{V}_l \otimes_k B_l)\Big),$$

and that

$$\mathrm{Hom}_R\Big(B_i, \bigoplus_{e\neq g\in S_H} \bigoplus_{l=1}^{n} {}^g(\underline{V}_l \otimes_k B_l)\Big) = \mathcal{N}\Big(B_i, \bigoplus_{e\neq g\in S_H} \bigoplus_{l=1}^{n} {}^g(\underline{V}_l \otimes_k B_l)\Big)$$

(cf. 3.1(ii) and 3.1(iii)).

Observe that to prove $(\mathrm{i})_i$ it suffices to show the formula

$$(\mathrm{ii})_{i,m} \qquad b_m = \sum_{l=1}^{i} \underline{\mu(h)}_{m,l} \cdot a_l$$

for all $h \in H$, $f \in \mathrm{Hom}_R(B_i, \underline{V}_l \otimes_k B_l)$ and $m = 1, \ldots, i$, where $\overline{(h * f)}_m = b_m \cdot \beta_{m,i}$, $b \in k^{d_m}$, and $\underline{\mu(h)}_{m,l} \in \mathrm{M}_{d_m\times d_l}(k)$ is the matrix of the k-linear map $\mu(h)_{m,l} : \underline{V}_l \to \underline{V}_m$ in the fixed bases.

Note that $\mu(h)_{m,l} \otimes_k \beta_{m,l}(h) : \underline{V}_l \otimes_k B_l \to {}^{h^{-1}}(\underline{V}_m \otimes_k B_m)$ corresponds via $3.3(\mathrm{i})_l$ and $3.3(\mathrm{i})_m$ to the map $\underline{\mu(h)}_{m,l} \cdot \beta_{m,l}(h) : B_l^{d_l} \to {}^{h^{-1}}B_m^{d_m}$. Therefore, by definition, the mth component $(h * f)_m \in \mathrm{Hom}_R(B_i, B_m)^{d_m}$ of $h * f$ is given by

$$(h * f)_m = \sum_{l=1}^{n} (\underline{\mu(h)}_{m,l} \cdot {}^h\beta_{m,l}(h)) \cdot {}^hf_l \cdot (\nu_i)_{h^{-1}}$$

$$= \sum_{l=1}^{n} (\underline{\mu(h)}_{m,l} \cdot {}^h\beta_{m,l}(h)) \cdot {}^hf'_l \cdot (\nu_i)_{h^{-1}} + \sum_{l=1}^{i} (\underline{\mu(h)}_{m,l} \cdot {}^h\beta_{m,l}(h)) \cdot (a_l \cdot {}^h\beta_{l,i}) \cdot (\nu_i)_{h^{-1}}$$

for every $m = 1, \ldots, n$. It is easily seen that the first summand of the above sum belongs to $\mathcal{N}(B_i, B_m)^{d_m}$. The second summand is equal to $\sum_{l=1}^{i} (\underline{\mu(h)}_{m,l} \cdot a_l) \cdot \beta_{m,i}$ $(= \overline{(h * f)_m})$, since

$$\begin{aligned} {}^h\beta_{m,l}(h) \cdot {}^h\beta_{l,i} \cdot (\nu_i)_{h^{-1}} &= {}^h((\nu_m)_h \cdot \beta_{m,i}) \cdot (\nu_i)_{h^{-1}} \\ &= {}^h({}^{h^{-1}}\beta_{m,i} \cdot (\nu_i)_h) \cdot (\nu_i)_{h^{-1}} = \beta_{m,i} \end{aligned}$$

(see 2.1). Consequently, $(\mathrm{ii})_{i,m}$ holds for every $m = 1, \ldots, i$, and the action we search for is just given by the family $(\mathrm{i})_i$.

To complete the proof of (a) observe that the composition of $3.3(\mathrm{ii})_n$ with the canonical isomorphism $\bigoplus_{l=1}^{i} \underline{V}_l \otimes_k k\beta_{l,n} \simeq \bigoplus_{l=1}^{i} \underline{V}_l = V$ yields a kH-isomorphism $\alpha(V) : \widetilde{\Psi}^B\widetilde{\Phi}^B(V) \to V$ (see the proof of Lemma 3.3(b)). It is easy to check that the family $(\alpha(V))_{V\in I_n\text{-spr}(KH)}$ is natural with respect to V and therefore defines the required isomorphism of functors.

(b) Observe first that since $\mathrm{End}_R(B_i)$ is local, we have $\mathcal{N}_{\mathrm{o}}(B_i, B_i) \subseteq J(\mathrm{End}_R(B_i))$ for every $i = 1, \ldots, n$. Then $\mathrm{Ker}\,\widetilde{\Psi}^B$ is nilpotent, by the lemma below, and (b) holds. ∎

Lemma. *Let $\mathcal{N}$ be an ideal in $\mathcal{B}$ determined by an ideal $\mathcal{N}_{\rm o}$ in $\mathcal{B}_{\rm o}$, where $\mathcal{B}_{\rm o}$ and $\mathcal{B}$ are as in Theorem* 3.1. *Then the following conditions are equivalent*:

(a) $\mathcal{N}_{\rm o}(B_i, B_i) \subseteq J(\mathrm{End}_R(B_i))$ *for every* $i = 1, \ldots, n$,
(b) $\mathcal{N}_{\rm o}$ *is nilpotent*,
(c) $\mathcal{N}$ *is nilpotent*,
(d) $\widetilde{\mathcal{N}}$ *is nilpotent.*

Moreover, *for a morphism* $f : M \to M'$ *in* $\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_{\rm o}} R$ *defined by the components* $f^{(g',g)}_{j,i} \in \mathrm{Hom}_R({}^g B_i^{d_i}, {}^{g'} B_j^{d'_j})$, *where* $M = (\bigoplus_{g\in S_H} {}^g(\bigoplus_{i=1}^n B_i^{d_i}), \mu)$ *and* $M' = (\bigoplus_{g\in S_H} {}^g(\bigoplus_{j=1}^n B_j^{d'_j}), \mu')$, *the following conditions are equivalent*:

(e) f *belongs to* $\widetilde{\mathcal{N}}$,
(f) $\widetilde{\Psi}^B(f) = 0$ (see Remark 3.4),
(g) $f^{(e,e)}_{j,i}$ *belongs to* $\mathrm{M}_{d'_j\times d_i}(\mathcal{N}(B_i, B_j))$ *for all* $i \geq j$.

In particular, $\mathrm{Ker}\,\widetilde{\Psi}^B \subset \widetilde{\mathcal{N}}$.

Sublemma. *Let H be a subgroup of $G \subset \mathrm{Aut}_k(R)$ acting freely on R, and L be a full subcategory of R. Suppose that H stabilizes L (i.e. $hL = L$ for all $h \in H$), and that $m = |\mathrm{ob}\, L/H|$ is a natural number. Then $\bigcap_{l=1}^{m+1} g_l L$ is a trivial subcategory for any pairwise different $g_1, \ldots, g_{m+1}$ in S_H.*

Proof. Let $\mathrm{ob}\, L = Hx_1 \cup \ldots \cup Hx_m$ be a splitting of $\mathrm{ob}\, L$ into a disjoint union of H-orbits. Suppose that $x \in \bigcap_{l=1}^{m+1} g_l L$, where $g_1, \ldots, g_{m+1}$ are as above. Then there exist $h_1, \ldots, h_{m+1} \in H$ and $i(1), \ldots, i(m+1) \in \{1, \ldots, m\}$ such that $x = g_1 h_1 x_{i(1)} = \ldots = g_{m+1} h_{m+1} x_{i(m+1)}$. Consequently, $i(l) = i(s)$ for some $1 \leq l < s \leq m+1$, and $g_l h_l = g_s h_s$. This contradicts $g_l H \neq g_s H$, therefore $\bigcap_{l=1}^{m+1} g_l L$ is trivial. ∎

Proof of Lemma. We start by observing that by [7, Theorem 2.9], each algebra $\mathrm{End}_R(B_i)$ is semiprimary (see [1]), so (a) is equivalent to $\mathcal{N}_{\rm o}(B_i, B_i)$ being a nilpotent ideal in $\mathrm{End}_R(B_i)$ for every $i = 1, \ldots, n$.

(a)⇒(b). The nilpotency degree of $\mathcal{N}_{\rm o}$ is bounded by nn', where n' is a common bound of the nilpotency degrees of the ideals $\mathcal{N}_{\rm o}(B_i, B_i) \subseteq \mathrm{End}_R(B_i)$, $i = 1, \ldots, n$. This follows from the fact that for any sequence $(i(j))_{j=0,1,\ldots,nn'}$ of elements of $\{1, \ldots, n\}$ there exists i such that $|\{j \in \{0, \ldots, nn'\} : i(j) = i\}| \geq n' + 1$.

(b)⇒(c). Denote by L the union $\bigcup_{i=1}^n \mathrm{supp}\, B_i$. Note that L satisfies the assumption of the Sublemma since all B_i's are G-atoms. We set $m = |\mathrm{ob}\, L/H|$ and denote by m' the nilpotency degree of $\mathcal{N}_{\rm o}$. We show that $f_{mm'} \cdot \ldots \cdot f_1 = 0$ for any collection $\{f_l \in \mathcal{N}({}^{g_{l-1}}B_{i(l-1)}, {}^{g_l}B_{i(l)})\}_{l=1,\ldots,mm'}$ of R-homomorphisms, where $B_{i(l)} \in \mathcal{B}_{\rm o}$ and $g_l \in S_H$ for $l = 0, 1, \ldots, mm'$.

Observe that if $|\{g_l\}_{l=0,1,\ldots,mm'}| > m$ then the claim follows immediately by the Sublemma. Consider the case $|\{g_l\}_{l=0,1,\ldots,mm'}| \le m$. Then there exists $g \in S_H$ such that $|\{l \in \{0,\ldots,mm'\} : g_l = g\}| \ge m'+1$. Consequently, the claim follows from the equality $\mathcal{N}_{\mathrm{o}}^{m'} = 0$ by definition of $\mathcal{N}$. Hence $\mathcal{N}$ is nilpotent.

The implication (c)⇒(d) follows easily from the definitions, (c)⇒(a) from the introductory remark.

To prove the second part of the lemma we fix f as above.

(e)⇒(f). Note that $\operatorname{Im}\operatorname{Hom}_R(B_n, f) \subseteq \widetilde{\mathcal{N}}(B_n, M')$ for $f \in \widetilde{\mathcal{N}}(M, M')$, and consequently $\widetilde{\Psi}^B(f) = 0$.

(f)⇒(g). We start by observing that 3.1(∗) induces the k-isomorphism

$$\operatorname{Hom}_R\Big(\bigoplus_{i=1}^{n} B_i^{d_i}, \bigoplus_{j=1}^{n} B_j^{d'_j}\Big) \simeq \prod_{1\le j\le i\le n} \mathrm{M}_{d'_j\times d_i}(k\beta_{j,i}) \oplus \prod_{1\le i,\, j\le n} \mathrm{M}_{d'_j\times d_i}(\mathcal{N}(B_i,B_j)).$$

Then the R-homomorphism $f^{(e,e)} : \bigoplus_{i=1}^{n} B_i^{d_i} \to \bigoplus_{j=1}^{n} B_j^{d'_j}$, defined by the components $f_{j,i}^{(e,e)}$, $1\le j,\ i\le n$, is given by the two collections $\{\overline{f_{j,i}^{(e,e)}}\}_{1\le j\le i\le n}$ and $\{(f_{j,i}^{(e,e)})'\}_{1\le j,\, i\le n}$, where $\overline{f_{j,i}^{(e,e)}} = a_{j,i}\cdot\beta_{j,i}$, $a_{j,i} \in \mathrm{M}_{d'_j\times d_i}(k)$ for all $1\le j\le i\le n$, and $(f_{j,i}^{(e,e)})' \in \mathrm{M}_{d'_j\times d_i}(\mathcal{N}(B_i,B_j))$ for all $1\le i,\ j\le n$. Then the morphism $\widetilde{\Psi}^B(f) : \widetilde{\Psi}^B(M) \to \widetilde{\Psi}^B(M')$ in I_n-spr(kH), under the identifications $\widetilde{\Psi}^B(M) \simeq \bigoplus_{i=1}^{n}(k\beta_{i,n})^{d_i}$ and $\widetilde{\Psi}^B(M') \simeq \bigoplus_{j=1}^{n}(k\beta_{j,n})^{d'_j}$ (see 3.1(∗) and 3.1(ii); cf. 3.3(ii)$_n$), is given by the k-linear block matrix map $a : \bigoplus_{i=1}^{n}(k\beta_{i,n})^{d_i} \to \bigoplus_{j=1}^{n}(k\beta_{j,n})^{d'_j}$, where $a = [a_{j,i}\cdot\beta_{j,i}]_{1\le i,\, j\le n}$ (we set $a_{j,i} = 0$ for $i<j$). Hence, if $\widetilde{\Psi}^B(f) = 0$, then $a_{j,i} = 0$ for all $1\le i,\ j\le n$, and consequently, $f_{j,i}^{(e,e)} \in \mathrm{M}_{d'_j\times d_i}(\mathcal{N}(B_i,B_j))$ for all $i\ge j$.

(g)⇒(e). Note that, by definition of $\mathcal{N}$ and $\widetilde{\mathcal{N}}$, we only have to show that $f_{j,i}^{(g,g)}$ belongs to $\widetilde{\mathcal{N}}$ for all $g \in S_H$ and $1\le i,j\le n$ (in fact $1\le j\le i\le n$; see 3.1(∗)). Since f is a morphism in $\mathrm{Mod}^G R$, we have ${}^{h^{-1}}f\cdot \mu_h = \mu'_h\cdot f$ for every $h\in G$. Then for any $g, g'_1 \in S_H$, looking at the (g'_1, g)-components of the above equality, we obtain the following equalities in $\operatorname{Hom}_R({}^{g}(\bigoplus_{i=1}^{n} B_i^{d_i}), {}^{h^{-1}g'_1}(\bigoplus_{j=1}^{n} B_j^{d'_j}))$:

$$\text{(iii)}_{(h,\,g'_1,\,g)} \qquad \sum_{g_1\in S_H} {}^{h^{-1}}f^{(g'_1,\,g_1)}\cdot \mu_h^{(g_1,\,g)} = \sum_{g'\in S_H} \nu_h^{(g'_1,\,g')}\cdot f^{(g',\,g)}$$

where $\mu_h^{(g_1,\,g)} : {}^{g}(\bigoplus_{i=1}^{n} B_i^{d_i}) \to {}^{h^{-1}g_1}(\bigoplus_{i=1}^{n} B_i^{d_i})$ (resp. ${\mu'_h}^{(g'_1,\,g')} : {}^{g'}(\bigoplus_{j=1}^{n} B_j^{d'_j})$

$\to {}^{h^{-1}g'_1}(\bigoplus_{j=1}^{n} B_j^{d'_j})$) is the (g_1, g)-component (resp. (g'_1, g')-component) of the R-isomorphism $\mu_h : \bigoplus_{g\in S_H} {}^{g}(\bigoplus_{i=1}^{n} B_i^{d_i}) \to {}^{h^{-1}}(\bigoplus_{g_1\in S_H} {}^{g_1}(\bigoplus_{i=1}^{n} B_i^{d_i}))$ (resp. $\mu'_h : \bigoplus_{g'\in S_H} {}^{g'}(\bigoplus_{j=1}^{n} B_j^{d'_j}) \to {}^{h^{-1}}(\bigoplus_{g'_1\in S_H} {}^{g'_1}(\bigoplus_{j=1}^{n} B_j^{d'_j}))$) defining the R-action μ (resp. μ') of H, and $f^{(g',g)} : {}^{g}(\bigoplus_{i=1}^{n} B_i^{d_i}) \to {}^{g'}(\bigoplus_{j=1}^{n} B_j^{d'_j})$ (resp. $f^{(g'_1,g_1)} : {}^{g_1}(\bigoplus_{i=1}^{n} B_i^{d_i}) \to {}^{g'_1}(\bigoplus_{j=1}^{n} B_j^{d'_j})$) is the R-homomorphism with components $f_{j,i}^{(g',g)}$ (resp. $f_{j,i}^{(g'_1,g_1)}$), $1 \le i, j \le n$. Assume now that $g'_1 = e$ and $h = g^{-1}$. Note that $\mu_h^{(g_1,g)}, {\mu'_h}^{(e,g')} \in \widetilde{\mathcal{N}}$ for $g_1 \ne e$ and $g' \ne g$; also ${}^{g}f^{(e,e)} \in \widetilde{\mathcal{N}}$ ($f^{(e,e)} \in \widetilde{\mathcal{N}}$!). Then $(\mathrm{iii})_{(g^{-1},e,g)}$ implies that ${\mu'_h}^{(e,g)} \cdot f^{(g,g)} \in \widetilde{\mathcal{N}}$. But by [7, Lemma 2.4], ${\mu'_h}^{(e,g)}$ is an R-isomorphism and therefore $f^{(g,g)} \in \widetilde{\mathcal{N}}$ for every $g \in S_H$; consequently, $f \in \widetilde{\mathcal{N}}$. ■

Corollary. *The functor Φ^B induces a representation embedding of the subcategory of all indecomposable objects in I_n-$\mathrm{spr}'(kH)$ into the full subcategory formed by all indecomposable non-orbicular modules in $\mathrm{mod}_{\mathcal{B}_o}(R/G)$.*

3.5. In this subsection we assume that $H \simeq \mathbb{Z}$ and $\mathcal{N} = \mathcal{P}u_{\mathcal{B}}$.

Remark. (a) For any $i = 1, \ldots, n$, and $M = (\bigoplus_{g\in S_H} {}^{g}(\bigoplus_{j=1}^{n} B_j^{d_j}), \mu)$ in $\mathrm{Mod}^{G}_{\mathrm{f},\mathcal{B}_o} R$, M and B_i satisfy the assumptions of [5, Theorem A(iii)] ($H \subseteq G_M = G$ and $\operatorname{supp} B_i \cap \operatorname{supp} M \subset \operatorname{supp} B_i$), therefore we have the equalities $\widetilde{\mathcal{N}}(B_i, M) = \mathcal{P}u(B_i, M)$ and $\widetilde{\mathcal{N}}(M, B_i) = \mathcal{P}u(M, B_i)$.

(b) Analogously, we obtain $\widetilde{\mathcal{N}}(V \otimes_k B, M) = \mathcal{P}u(V \otimes_k B, M)$ and $\widetilde{\mathcal{N}}(M, V \otimes_k B) = \mathcal{P}u(M, V \otimes_k B)$ for V in I_n-$\mathrm{spr}(kH)$. However generally, only the inclusion $\mathcal{P}u_{\widetilde{\mathcal{B}}} \subset \widetilde{\mathcal{N}}$ $(= \widetilde{\mathcal{P}u_{\mathcal{B}}})$ holds; it is not clear if $\mathcal{P}u_{\widetilde{\mathcal{B}}} = \widetilde{\mathcal{N}}$.

To prove the first statement of Theorem 3.1(a) it suffices to show the following (see first Remark 3.4).

Lemma. *$\widetilde{\Phi}^B \widetilde{\Psi}^B(M)$ is a direct summand of M for all M in $\mathrm{Mod}^{G}_{\mathrm{f},\mathcal{B}_o} R$.*

Sublemma. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{w_1} & C_2 & \xrightarrow{p_1} & C_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & D_1 & \xrightarrow{w_2} & D_2 & \xrightarrow{p_2} & D_3 & \longrightarrow & 0 \end{array}$$

be a commutative diagram (in an abelian category $\mathcal{C}$) whose rows are splittable exact sequences. Suppose that f_3 is a monomorphism and D_1 is an injective A-module. Then for any splitting $s_1 : C_3 \to C_2$ of p_1 there exists a splitting $s_2 : D_3 \to D_2$ of p_2 such that $f_2 s_1 = s_2 f_3$.

Proof. Let s_1 be as above. Fix $s'_2 : D_3 \to D_2$ such that $p_2 s'_2 = \mathrm{id}_{D_3}$. Since $p_2(f_2 s_1 - s'_2 f_3) = 0$, we have $f_2 s_1 - s'_2 f_3 = w_2 u$ for some $u : C_3 \to D_1$.

Then by our assumptions there exists $u' : D_3 \to D_1$ such that $u = u'f_3$. Now it is easily seen that $s_2 = s_2' + w_2 u'$ satisfies the assertion. ■

Proof of Lemma. To prove that, for a given M in $\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}} R$, $\widetilde{\Phi}^B\widetilde{\Psi}^B(M) = \theta(\widetilde{\Psi}^B(M) \otimes_k B)$ is a direct summand of M, we construct a splittable monomorphism $\varphi : \widetilde{\Psi}^B(M) \otimes_k B \to M$ in $\mathrm{Mod}^H R$. We may assume that $M = \bigoplus_{g\in S_H} {}^g(\bigoplus_{i=1}^n B_i^{d_i}) = \bigoplus_{i=1}^n B_i^{d_i} \oplus M'$, where $M' = \bigoplus_{e\neq g\in S_H} {}^g(\bigoplus_{i=1}^n B_i^{d_i})$.

For any $i = 1, \ldots, n-1$ consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathcal{N}}(B_i, M) & \xrightarrow{\varepsilon_i} & \mathrm{Hom}_R(B_i, M) & \xrightarrow{\pi_i} & \overline{\mathcal{H}}_i(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\mathcal{N}}(\beta_i, M)} & & \downarrow{\scriptstyle \mathrm{Hom}_R(\beta_i, M)} & & \downarrow{\scriptstyle \iota_{i+1,i}(M)} & & \\ 0 & \longrightarrow & \widetilde{\mathcal{N}}(B_{i+1}, M) & \xrightarrow{\varepsilon_i} & \mathrm{Hom}_R(B_{i+1}, M) & \xrightarrow{\pi_{i+1}} & \overline{\mathcal{H}}_{i+1}(M) & \longrightarrow & 0 \end{array}$$

in $\mathrm{MOD}(kH)^{\mathrm{op}}$. By [5, Theorem A(iv)] all kH-modules $\widetilde{\mathcal{N}}(B_i, M)$, $i = 1, \ldots, n$, are injective since $\widetilde{\mathcal{N}}(B_i, M) = \mathcal{P}u(B_i, M)$ (see Remark). Then by the Sublemma one can inductively construct a family $s_i : \overline{\mathcal{H}}_i(M) \to \mathrm{Hom}_R(B_i, M)$, $i = 1, \ldots, n$, such that $\pi_i s_i = \mathrm{id}_{\overline{\mathcal{H}}_i(M)}$ for every i, and $\mathrm{Hom}_R(\beta_{i+1}, M) \cdot s_i = s_{i+1} \cdot \iota_{i+1,i}(M)$ for $i < n$.

Let $\widetilde{s}_i : \overline{\mathcal{H}}_i(M) \otimes_k B_i \to M$ be the morphism in $\mathrm{Mod}^H R$ which is adjoint to s_i, $i = 1, \ldots, n$ (see [3, Lemma 2.4]). Then by the last equality we have

$$\text{(i)} \qquad \widetilde{s}_i \cdot (\overline{\mathcal{H}}_i(M) \otimes \beta_i) = \widetilde{s}_{i+1} \cdot (\iota_{i+1,i}(M) \otimes B_i)$$

for $i < n$.

For any $l = 1, \ldots, i$ and $t = 1, \ldots, d_l$, we denote by $\beta_{l,i,t}$ the composite map $B_i \xrightarrow{\beta_{l,i}} B_l \to \bigoplus_{j=1}^n B_j^{d_j}$, where the second map is the standard embedding into the tth component of $B_l^{d_l}$. Then the equality $\pi_i s_i = \mathrm{id}_{\overline{\mathcal{H}}_i(M)}$ implies that under the identifications $\overline{\mathcal{H}}_i(M) \simeq \bigoplus_{l=1}^i (k\beta_{l,i})^{d_l} = \bigoplus_{l=1}^i \bigoplus_{t=1}^{d_l} k\beta_{l,i,t}$ and $\mathrm{Hom}_R(B_i, M) \simeq \mathrm{Hom}_R(B_i, \bigoplus_{j=1}^n B_j^{d_j}) \oplus \mathrm{Hom}_R(B_i, M')$ of k-linear spaces, we have

$$s_i(\beta_{l,i,t}) = (\beta_{l,i,t} + \varphi_{l,i,t},\, \varphi'_{l,i,t})$$

for all $i = 1, \ldots, n$, $l = 1, \ldots, i$, $t = 1, \ldots, d_l$, where $\varphi_{l,i,t} \in \widetilde{\mathcal{N}}(B_i, \bigoplus_{j=1}^n B_j^{d_j})$ and $\varphi'_{l,i,t} \in \mathrm{Hom}_R(B_i, M')$. Note that, via the R-isomorphism $\overline{\mathcal{H}}_i(M) \otimes_k B_i \simeq \bigoplus_{l=1}^i B_i^{d_l}$, the R-homomorphism $\widetilde{s}_i$ regarded as a map $\bigoplus_{l=1}^i B_i^{d_l} \to \bigoplus_{j=1}^n B_j^{d_j} \oplus M'$ has components $(\beta_{l,i,t} + \varphi_{l,i,t},\, \varphi'_{l,i,t})_{l\in\{1,\ldots,i\},\, t\in\{1,\ldots,d_l\}}$.

Set for simplicity $V = \widetilde{\Psi}^B(M)$. From now on we will identify the k-spaces $V_i = \overline{\mathcal{H}}'_i(M)$ and $\overline{\mathcal{H}}_i(M)$ (via $\iota_{n,i}(M)$), $i = 1, \ldots, n$.

To define φ we construct inductively a family $\{\varphi_i : V_{(i)} \otimes_k B^{[1,i]} \to M\}_{i=1,\dots,n}$ of kH-homomorphisms such that

(ii) $$\varphi_{i|V_{(i-1)}\otimes B^{[1,i-1]}} = \varphi_{i-1}$$

for $i > 1$, and

(iii) $$\widetilde{s}_i = \varphi_i \cdot (V_i \otimes \mathrm{id}_{B_i}^{[i]})$$

for every i (see 2.8 for definition of $V_i \otimes \mathrm{id}_{B_i}^{[i]}$).

We set $\varphi_i = \widetilde{s}_1$. To construct φ_{i+1} from φ_i, for $1 < i < n$, we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_i \otimes_k B_{i+1} & \xrightarrow{w_i} & V_{i+1} \otimes_k B_{i+1} & \longrightarrow & (V_{i+1}/V_i) \otimes_k B_{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V_i \otimes \beta_{i+1}^{[i]}} & & \downarrow{\scriptstyle V_{i+1} \otimes \mathrm{id}_{B_i}^{[i]}} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & V_{(i)} \otimes_k B^{[1,i]} & \xrightarrow{v_i} & V_{(i+1)} \otimes_k B^{[1,i+1]} & \xrightarrow{r_i} & (V_{i+1}/V_i) \otimes_k B_{i+1} & \longrightarrow & 0
\end{array}$$

in $\mathrm{Mod}^H R$ with exact rows where $w_i = \iota_{i+1,i}(M) \otimes B_{i+1}$ (see Lemma 2.9 for definition of the lower row). Observe that $\widetilde{s}_{i+1} w_i = \varphi_i \cdot (V_i \otimes \beta_{i+1}^{[i]})$ since $\widetilde{s}_{i+1} \cdot (\iota_{i+1,i}(M) \otimes B_{i+1}) = \varphi_i \cdot (V_i \otimes \mathrm{id}_{B_i}^{[i]}) \cdot (V_i \otimes \beta_{i+1})$ by (i) and (iii). Consequently, there exists a unique morphism $f : V_{(i+1)} \otimes_k B^{[1,i+1]} \to M$ satisfying $f v_i = \varphi_i$, $f \cdot (V_{i+1} \otimes \mathrm{id}_{B_{i+1}}^{[i+1]}) = \widetilde{s}_{i+1}$, and we set $\varphi_{i+1} = f$.

Now we define $\varphi : \widetilde{\Psi}^B(M) \otimes_k B \to M$ by setting $\varphi = \varphi_n$.

To give a direct description of φ recall that, under the identification $V_n = \overline{\mathcal{H}}_n(M) \simeq \bigoplus_{l=1}^n (k\beta_{l,n})^{d_l}$, each V_i corresponds to $\bigoplus_{l=1}^i (k\beta_{l,n})^{d_l}$ (see 3.3), and we can assume that the sequence $(\underline{V}_i)_{i=1,\dots,n}$ of complementary direct summands for V is given by $\underline{V}_i = (k\beta_{i,n})^{d_i}$, $i = 1,\dots,n$. Consequently, we have R-isomorphisms $V_{(i)} \otimes_k B^{[1,i]} \simeq \bigoplus_{l=1}^i B_l^{d_l}$, and in particular $\widetilde{\Psi}^B(M) \otimes_k B \simeq \bigoplus_{l=1}^n B_l^{d_l}$. Now it is easily seen that by (ii) and (iii), φ regarded as an R-homomorphism $\bigoplus_{l=1}^n B_l^{d_l} \to \bigoplus_{j=1}^n B_j^{d_j} \oplus M'$ is given by the components $(\beta_{l,l,t} + \varphi_{l,l,t},\ \varphi'_{l,l,t})_{l\in\{1,\dots,n\},\, t\in\{1,\dots,d_l\}}$ ($\beta_{l,l} = \mathrm{id}_{B_l}$!).

To show that φ is a splittable monomorphism in $\mathrm{Mod}^H R$ we construct a morphism $\psi : M \to \widetilde{\Phi}^B \widetilde{\Psi}^B(M)$ in $\mathrm{Mod}^H R$ such that $\psi\varphi$ is an invertible R-homomorphism. In the construction we apply, in contrast to the previous case, the functors

$$\overline{\mathcal{H}}^i : \mathrm{Mod}_{\mathrm{f},\mathcal{B}_\mathrm{o}}^G R \to \mathrm{MOD}(kH)^{\mathrm{op}},$$

$i = 1,\dots,n$, which are defined by setting

$$\overline{\mathcal{H}}^i(N) = \mathrm{Hom}_R(N, B_i)/\widetilde{\mathcal{N}}(N, B_i)$$

for N in $\mathrm{Mod}_{\mathrm{f},\mathcal{B}_\mathrm{o}}^G R$. By similar arguments to those for $\overline{\mathcal{H}}_i$, we have the canonical k-isomorphism

(iv) $$\overline{\mathcal{H}}^i(M) \simeq \bigoplus_{l=i}^{n} (k\beta_{i,l})^{d_l}.$$

We also have at our disposal the natural compatible monomorphisms

$$\iota^{i,j} : \overline{\mathcal{H}}^j \to \overline{\mathcal{H}}^i$$

of functors induced by $\beta_{i,j}$, $i \leq j$, which evaluated at M correspond to the canonical k-linear monomorphisms

(v) $$\bigoplus_{l=j}^{n} (k\beta_{j,l})^{d_l} \to \bigoplus_{l=i}^{n} (k\beta_{i,l})^{d_l}$$

given by $\beta_{i,j}\cdot$.

Now applying analogous arguments as before one can inductively construct kH-homomorphisms $s^i : \overline{\mathcal{H}}^i(M) \to \mathrm{Hom}_R(M, B_i)$, $i = n, \ldots, 1$, such that s^i splits the canonical projection $\pi^i : \mathrm{Hom}_R(M, B_i) \to \overline{\mathcal{H}}^i(M)$ for every i, and $\mathrm{Hom}_R(M, \beta_i) \cdot s^i = s^{i-1} \cdot \iota^{i-1,i}(M)$ for $i > 1$.

For any $i = 1, \ldots, n$, consider the composite map

$$u_i : M \otimes_R B_i^* \overset{\varepsilon}{\hookrightarrow} \mathrm{Hom}_R(M, B_i)^* \xrightarrow{(s^i)^*} \overline{\mathcal{H}}^i(M)^*$$

of left kH-modules, where ε is the embedding from [3, Corollary 2.4] (see [7, 5.1] for the definitions). Denote by $\widetilde{u}_i$ the composition

$$M \to \mathrm{Hom}_k(B_i^*, \overline{\mathcal{H}}^i(M)^*) \to \overline{\mathcal{H}}^i(M)^* \otimes_k B_i$$

where the first map is adjoint to u_i and the second is given by the functor isomorphism from [3, Lemma 2.2]. It is easily seen that by the commutativity condition for the s^i's we have

(vi) $$(\overline{\mathcal{H}}^i(M)^* \otimes \beta_i) \cdot \widetilde{u}_i = ((\iota^{i,i-1}(M))^* \otimes B_{i-1}) \cdot \widetilde{u}_{i-1}$$

for every $i > 1$. For any $l = i, \ldots, n$ and $t = 1, \ldots, d_l$, we denote by $\beta_{i,l}^t$ the composite map $\bigoplus_{j=1}^n B_j^{d_j} \to B_l \overset{\beta_{i,l}}{\to} B_i$, where the first map is the standard projection onto the tth component of $B_l^{d_l}$. Then the equality $\pi^i w^i = \mathrm{id}_{\overline{\mathcal{H}}^i(M)}$ implies that, under the identifications $\overline{\mathcal{H}}^i(M) \simeq \bigoplus_{l=i}^n (k\beta_{i,l})^{d_l} \simeq \bigoplus_{l=i}^n \bigoplus_{t=i}^{d_l} k\beta_{i,l}^t$ and $\mathrm{Hom}_R(M, B_i) \simeq \mathrm{Hom}_R(\bigoplus_{j=1}^n B_j^{d_j}, B_i) \oplus \mathrm{Hom}_R(M', B_i)$ of k-linear spaces, we have

$$s^i(\beta_{i,l}^t) = (\beta_{i,l}^t + \psi_{i,l}^t,\, \psi'^{\,t}_{i,l})$$

for all $i = 1, \ldots, n$, $l = i, \ldots, n$, $t = 1, \ldots, d_l$, where $\psi_{i,l}^t \in \widetilde{\mathcal{N}}(\bigoplus_{j=1}^n B_j^{d_j}, B_i)$ and $\psi'^{\,t}_{i,l} \in \mathrm{Hom}_R(M', B_i)$. It is easily seen that under the R-isomorphism $\overline{\mathcal{H}}^i(M)^* \otimes_k B_i \simeq \bigoplus_{l=i}^n B_i^{d_l}$ induced by the k-linear isomorphism $\overline{\mathcal{H}}^i(M)^* \simeq \bigoplus_{l=i}^n (k\beta_{i,l}^*)^{d_l}$ (dual to (iv)), the R-homomorphism $\widetilde{u}_i$ regarded as a map

$\bigoplus_{j=1}^{n} B_i^{d_j} \oplus M' \to \bigoplus_{l=i}^{n} B_i^{d_l}$ is given by the components $(\beta_{i,l}^t + \psi_{i,l}^t,\ \psi'^{\,t}_{i,l})$, $l \in \{i, \ldots, n\}, t \in \{1, \ldots, d_l\}$.

Denote by $W = (W_1 \xrightarrow{p_1} \ldots \xrightarrow{p_{n-1}} W_n)$ the sequence of epimorphisms in $\operatorname{mod}(kH)^{\mathrm{op}}$ given by $W_i = \overline{\mathcal{H}}^i(M)^*$ and $p_i = (\iota^{i+1,i}(M))^*$. To define ψ we proceed analogously as in the case of φ, and construct inductively kH-homomorphisms $\psi_i : M \to W_{(i)} \otimes_k B^{[i,n]}$, $i = n, \ldots, 1$, such that

$$r_i \psi_i = \psi_{i+1} \tag{vii}$$

for $i < n$, and

$$(W_i \otimes \mathrm{id}_{B_i}^{[n-i+1]}) \cdot \psi_i = \widetilde{u}_i \tag{viii}$$

for every i (see 2.9 for definitions of $W_i \otimes \mathrm{id}_{B_i}^{[n-i+1]} : W_{(i)} \otimes_k B^{[i,n]} \to W_i \otimes_k B_i$ and $r_i : W_{(i)} \otimes_k B^{[i,n]} \to W_{(i+1)} \otimes_k B^{[i+1,n]}$). We set $\psi_n = \widetilde{u}_n$. To construct ψ_{i-1} from ψ_i, for $1 < i < n$, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} p_{i-1} \otimes_k B_{i-1} & \longrightarrow & W_{(i-1)} \otimes_k B^{[i-1,n]} & \xrightarrow{r_{i-1}} & W_{(i)} \otimes_k B^{[i,n]} & \longrightarrow & 0 \\
 & & \Big\downarrow{=} & & \Big\downarrow{W_{i-1} \otimes \mathrm{id}_{B_{i-1}}^{[n-i+2]}} & & \Big\downarrow{W_i \otimes \beta_i^{[n-i+1]}} & & \\
0 & \longrightarrow & \operatorname{Ker} p_{i-1} \otimes_k B_{i-1} & \longrightarrow & W_{i-1} \otimes_k B_{i-1} & \xrightarrow{p_{i-1} \otimes B_{i-1}} & W_i \otimes_k B_{i-1} & \longrightarrow & 0
\end{array}$$

in $\operatorname{Mod}^H R$ with exact rows (see Corollary 2.9).

Note that $(p_{i-1} \otimes B_{i-1}) \cdot \widetilde{u}_i = (W_i \otimes \beta_i^{[n-i+1]}) \cdot \psi_i$, as $(p_{i-1} \otimes B_{i-1}) \cdot \widetilde{u}_i = (W_i \otimes \beta_i) \cdot (W_i \otimes \mathrm{id}_{B_i}^{[n-i+1]}) \cdot \psi_i$ by (vi) and (viii). Consequently, there exists a unique map $f' : M \to W_{(i-1)} \otimes_k B^{[i-1,n]}$ satisfying $r_{i-1} f' = \psi_i$, $(W_{i-1} \otimes \mathrm{id}_{B_{i-1}}^{[n-i+2]}) \cdot f' = \widetilde{u}_{i-1}$, and we set $\psi_{i-1} = f'$.

Now we define $\psi : M \to \widetilde{\Psi}^B(M)$ by setting $\psi = \psi_1$.

To give a direct description of ψ note that, under the k-linear isomorphisms $\overline{\mathcal{H}}^i(M)^* \simeq \bigoplus_{l=i}^{n} (k\beta_{i,l}^*)^{d_l}$ (dual to (iv)), p_i corresponds to the standard k-linear epimorphism $\bigoplus_{l=i}^{n} (k\beta_{i,l}^*)^{d_l} \to \bigoplus_{l=i+1}^{n} (k\beta_{i+1,l}^*)^{d_l}$ (dual to (v)) with kernel $(k\beta_{i,l}^*)^{d_i}$. In this way we obtain the induced R-isomorphisms $W_{(i)} \otimes_k B^{[i,n]} \simeq \bigoplus_{l=i}^{n} B_l^{d_l}$, and in particular $\widetilde{\Psi}^B(M) \otimes_k B \simeq \bigoplus_{l=1}^{n} B_l^{d_l}$. It is easily seen that by (vii) and (viii), ψ regarded as an R-homomorphism $\bigoplus_{j=1}^{n} B_j^{d_j} \oplus M' \to \bigoplus_{l=1}^{n} B_l^{d_l}$ is given by the components $(\beta_{l,l}^t + \psi_{l,l}^t,\ \psi'^{\,t}_{l,l})$, $l = 1, \ldots, n$, $t = 1, \ldots, d_l$ ($\beta_{l,l} = \mathrm{id}_{B_l}!$).

In conclusion, $\varphi\psi$ is an isomorphism in $\operatorname{Mod}^H R$, since by [7, Lemma 2.4] it is an invertible R-homomorphism ($\widetilde{\mathcal{N}} \subset \mathcal{J}_R$). In this way we constructed a splittable monomorphism $\varphi : \widetilde{\Psi}^B(M) \otimes_k B \to M$ in $\operatorname{Mod}^H R$ and now the assertion follows immediately from [7, Lemma 6.2]. ■

The result below completes the proof of Theorem 3.1(a).

Proposition. (a) $\operatorname{Ker}\Psi^B = [\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}_{\mathrm{o}}}(R/G)]_{\operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)}$.

(b) *The functors Φ^R and Ψ^B induce an equivalence*

$$I_n\text{-spr}(kH) \simeq \operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)/\,[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}_{\mathrm{o}}}(R/G)]_{\operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)}.$$

Proof. We start by observing that, by Proposition 3.4(a) and Lemma 3.5, (a) immediately implies (b). Moreover, by Lemma 3.4, we have the inclusion $[\operatorname{mod}_{\mathcal{A}^{\mathrm{f}}_{\mathrm{o}}}(R/G)]_{\operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)} \subset \operatorname{Ker}\Psi^B$ $(\mathcal{P}u_{\widetilde{\mathcal{B}}} \subset \widetilde{\mathcal{N}}!)$. To prove the inverse inclusion we show first that any morphism $\varphi : \theta^G_H(V \otimes_k B) \to kH \otimes_k B_m$ in $\operatorname{MOD}^H R$, $m \in \{1,\dots,n\}$, V in I_n-spr(kH), factors through $\theta^H_e(Z)$, for some Z in $\operatorname{mod} R$ (here e denotes the trivial subgroup of G). Observe that for this purpose, it suffices to show that the map $\psi : V \otimes_k B \to \theta^G_H(kH \otimes_k B_m)$ which corresponds to φ under the natural isomorphisms

$$\begin{aligned}\operatorname{Hom}^H_R(\theta^G_H(V \otimes_k B), kH \otimes_k B_m) &\simeq \operatorname{Hom}^G_R(\theta^G_H(V \otimes_k B), \theta^G_H(kH \otimes_k B_m)) \\ &\simeq \operatorname{Hom}^H_R(V \otimes_k B, \theta^G_H(kH \otimes_k B_m)),\end{aligned}$$

(see [3, Lemma 2.3]; $(\operatorname{supp} kH \otimes_k B_m)/H$ is finite!) factors through $\theta^H_e(Z)$ for some Z in $\operatorname{mod} R$.

Fix a map ψ as above. Note that $kH \otimes_k B_m \simeq \theta^H_e(B_m)$ in $\operatorname{MOD}^H R$; an isomorphism is given by $\bigoplus_{h\in H}(\nu_m)_{h^{-1}} : \bigoplus_{h\in H} B_m \to \bigoplus_{h\in H} {}^hB_m$, under the identification $kH \otimes_k B_m \simeq \bigoplus_{h\in H} h \otimes B_m \simeq \bigoplus_{h\in H} B_m$. Consequently, $\theta^G_H(kH \otimes_k B_m) \simeq \theta^G_e(B_m)$, since $\theta^G_e = \theta^G_H \circ \theta^H_e$. The module $\theta^G_e(B_m) = \bigoplus_{g\in G} {}^gB_m$, as an object in $\operatorname{MOD}^H R$, decomposes into a direct sum $\bigoplus_{g'\in U_H} \theta^G_H({}^{g'}B_m) = \bigoplus_{g'\in U_H}(\bigoplus_{h\in H} {}^{hg'}B_m)$, where U_H is a fixed set of representatives of right cosets H/G containing e. Then the map $\psi : V \otimes_k B \to \theta^G_e(B_m)$, under the k-isomorphism

$$\operatorname{Hom}^H_R(V \otimes_k B, \theta^G_e(B_m)) \simeq \operatorname{Hom}^H_R\Big(V \otimes_k B, \bigoplus_{g'\in U_H} \theta^G_H({}^{g'}B_m)\Big),$$

is given by the components $\psi_{g'} = (\psi_{h,g'})_{h\in H}$, $g' \in U_H$. Since $(V \otimes_k B)^{(k)}$ is a finitely generated kH-module (see Remarks 2.7(a) and 2.4), there exist $g_1,\dots,g_{t_0} \in U_H$ such that $\psi_{g'} = 0$ for all $g' \in U_H \setminus \{g_1,\dots,g_{t_0}\}$. Note that ψ_{g_t} factors through $\bigoplus_{h\in H} \operatorname{Im}\psi_{h,g_t} = \theta^H_e(\operatorname{Im}\psi_{e,g_t})$ $(\operatorname{Im}\psi_{g_t} \subseteq \bigoplus_{h\in H} \operatorname{Im}\psi_{h,g_t}$ and $\operatorname{Im}\psi_{h,g_t} = {}^h(\operatorname{Im}\psi_{e,g_t}))$. Hence, ψ factors through $\bigoplus_{t=1}^{t_0} \theta^H_e(\operatorname{Im}\psi_{e,g_t})$.

To complete the proof of our claim, it suffices to show that $\dim_k(\operatorname{Im}\psi_{e,g'})$ is finite for every $g' \in U_H$. Set $L = \operatorname{supp} B_1 \cup \ldots \cup \operatorname{supp} B_n$ (clearly, $\operatorname{supp}(V \otimes_k B_m) \subset L$ and L/H is finite). Note that if $G_{{}^{g'}B_m} \cap H = e$ then, by [3, Lemma 3.6], $L \cap \operatorname{supp} {}^{g'}B_m$ is finite, and consequently $\dim_k(\operatorname{Im}\psi_{e,g'})$ is finite. Consider the case $H' = G_{{}^{g'}B_m} \cap H \neq e$. Then L is contained in the union of a finite number of H'-orbits, since $[H : H']$ is finite ($H \simeq \mathbb{Z}$!). Suppose that $\dim_k(\operatorname{Im}\psi_{e,g'})$ is infinite, equivalently, $\operatorname{supp}(\operatorname{Im}\psi_{e,g'})$ is infinite. Then there exist $x \in \operatorname{ob} L$ and pairwise different elements $h_s \in H'$, $s \in \mathbb{N}$, such that $\psi_{e,g'}(h_s x) \neq 0$ for all s. This implies that $\psi_{h_s^{-1},g'}(x) \neq 0$ for all $s \in \mathbb{N}$, a contradiction $((\operatorname{Im}\psi_{g'})(x) \subseteq (\bigoplus_{h\in H} {}^{hg'}B_m)(x) = \bigoplus_{h\in H} {}^{g'}B_m(h^{-1}x)!)$. Consequently, all modules $\operatorname{Im}\psi_{e,g_t}$, $t = 1,\dots,t_0$, are finite-dimensional, and ψ factors through $\theta_e^H(Z)$, where $Z = \bigoplus_{t=1}^{t_0} \operatorname{Im}\psi_{e,g_t}$; the claim is proved.

Next we prove that any morphism $\varphi : M \to N$ in $\operatorname{Mod}^H R$, between M in $\operatorname{Mod}^G_{\mathrm{f},\mathcal{B}_\circ} R$ and $N = V \otimes_k B$, V in I_n-spr(KH), factors through $\bigoplus_{i=1}^n P_i \otimes_k B_i$, where all P_i's are finitely generated free kH-modules, provided the R-homomorphism φ belongs to $\widetilde{\mathcal{N}}$.

Consider first the case $N = W \otimes_k B_m$, $m \in \{1,\dots,n\}$, where W is in $\operatorname{mod}(kH)^{\mathrm{op}}$. Recall that B_m^* stands for the object in $\operatorname{Mod}^H R^{\mathrm{op}}$ which consists of the k-dual to B_m, the R^{op}-module B_m^* ($B_m^*(x) = \operatorname{Hom}_k(B_m(x), k)$ for every $x \in \operatorname{ob} R$), and the standard R^{op}-action of H on B_m^* (see [7, 5.1], also [3, 2.1], where the notation $B_m^{\circledast}$ is used). Then the image φ' of the map φ via the natural isomorphisms

$$\operatorname{Hom}_R^H(M, W\otimes_k B_m) \simeq \operatorname{Hom}_R^H(M, \operatorname{Hom}_k(B_m^*, W)) \simeq \operatorname{Hom}_{kH}(M\otimes_R B_m^*, W)$$

(see [3, 2.2 and 2.4]) admits a factorization $\varphi' = (\mathrm{id}_N)' \cdot (\varphi \otimes_R B_m^*)$, where $(\mathrm{id}_N)'$ corresponds to id_N via $\operatorname{Hom}_R^H(N, W\otimes_k B_m) \simeq \operatorname{Hom}_{kH}(N\otimes_R B_m^*, W)$. We prove that $\varphi\otimes_R B_m^*$ factors through a free finitely generated kH-module. Since $M \otimes_R B_m^*$ is a finitely generated kH-module ($(\operatorname{supp} B_m)/H$ is finite), $M \otimes_R B_m^*$ decomposes into a direct sum $M \otimes_k B_m^* = P \oplus F$ of kH-submodules, where P is free finitely generated and F is finite-dimensional ($kH \simeq k[T, T^{-1}]$ is a principal ideal domain). Consequently, $\varphi \otimes_R B_m^*$ can be regarded as a matrix map $[s_1, s_2] : P \oplus F \to N \otimes_k B_m^*$. We show that $s_2 = 0$. For this purpose consider the dual map $(\varphi\otimes_R B_m^*)^* : (N\otimes_R B_m^*)^* \to (M \otimes_R B_m^*)^*$, which can now be viewed in the form $\left[\begin{smallmatrix} s_1^* \\ s_2^* \end{smallmatrix}\right] : (N \otimes_k B_m^*)^* \to P^* \oplus F^*$. Observe that, under the natural kH-isomorphisms $\eta_N : \operatorname{Hom}_R(N, B_m) \to (N \otimes_R B_m^*)^*$ and $\eta_M : \operatorname{Hom}_R(M, B_m) \to (M \otimes_R B_m^*)^*$ (see [3, 2.4]), the map $(\varphi \otimes_R B_m^*)^*$ corresponds to $\operatorname{Hom}_R(\varphi, B_m)$. Since by Remark 3.5, $\widetilde{\mathcal{N}}(M, B_m) = \mathcal{P}u(M, B_m)$, the kH-submodule $U = \widetilde{\mathcal{N}}(M, B_m)$ of $\operatorname{Hom}_R(M, B_m)$ is injective (see [5, Theorem A(iv)]), and $\operatorname{Hom}_R(M, B_m)$ has a decomposition $\operatorname{Hom}_R(M, B_m) = U \oplus U_0$, where U_0 is a finite-dimensional kH-module ($U_0 \simeq \bigoplus_{i=m}^n (k\beta_{m,i})^{d_i}$ as k-vector spaces, where $d_i = \operatorname{dsc}(F_\bullet^{-1}(M))_{B_i}$, $i = 1,\dots,n$). Then $\operatorname{Hom}_R(\varphi, B_m)$ is given by the matrix map $\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] : \operatorname{Hom}_R(N, B_m) \to U\oplus U_0$ (φ belongs to $\widetilde{\mathcal{N}}$ and $\operatorname{Im}\operatorname{Hom}_R(\varphi, B_m) \subseteq \widetilde{\mathcal{N}}(M, B_m)$). Moreover, the isomorphism η_M is given by the matrix map $\left[\begin{smallmatrix} w_{11} & w_{12} \\ 0 & w_{22} \end{smallmatrix}\right] : U \oplus U_0 \to P^* \oplus F^*$ ($\operatorname{Hom}_{kH}(U, F^*) = 0$, because there is no non-trivial divisible finite-dimensional kH-module). Consequently,

$$\begin{bmatrix} s_1^* \\ s_2^* \end{bmatrix} \cdot \eta_N = \begin{bmatrix} w_{11} & w_{12} \\ 0 & w_{22} \end{bmatrix} \cdot \begin{bmatrix} u \\ 0 \end{bmatrix},$$

and $s_2 = 0$.

Now we consider the general case. For any non-zero morphism $\varphi : M \to N$, where $N = V \otimes_k B$ for V in I_n-spr(kH), we denote by $m = m(\varphi)$ the smallest $i \in \{1, \dots, n\}$ such that $\operatorname{Im}\varphi \subseteq V_{(i)} \otimes_k B^{[1,i]}$ (we set $m(\varphi) = 0$ if $\varphi = 0$). We show by induction on m that φ factors through $\bigoplus_{i=1}^{m} P_i \otimes_k B_i$, where all P_i's are finitely generated free kH-modules, provided φ belongs to $\widetilde{\mathcal{N}}$.

By the previous considerations (the case $N = W \otimes_k B_m$) we can assume that $m \geq 2$. Moreover, by the same reason, the map $r\varphi$ has a factorization

$$M \xrightarrow{\psi} P_m \otimes_k B_m \xrightarrow{\psi'} \overline{V}_m \otimes_k B_m$$

where

$$(*) \qquad 0 \to V_{(m-1)} \otimes_k B^{[1,m-1]} \xrightarrow{v} V_{(m)} \otimes_k B^{[1,m]} \xrightarrow{r} \overline{V}_m \otimes_k B_m \to 0$$

is an exact sequence in $\operatorname{Mod}^H R$ defined in 2.9 (here $\overline{V}_m = V_{(m)}/V_{(m-1)}$) and P_m is a finitely generated free kH-module. Observe that the map

$$\operatorname{Hom}_{kH}(P_m, \operatorname{Hom}_R(B_m, r)) : \operatorname{Hom}_{kH}(P_m, \operatorname{Hom}_R(B_m, V_{(m)} \otimes_k B^{[1,m]})) \\ \to \operatorname{Hom}_{kH}(P_m, \operatorname{Hom}_R(B_m, \overline{V}_m \otimes_k B_m)),$$

which corresponds under the standard adjunction isomorphisms to

$$\operatorname{Hom}_R(P_m \otimes_k B_m, r) : \\ \operatorname{Hom}_R^H(P_m \otimes_k B_m, V_{(m)} \otimes_k B^{[1,m]}) \to \operatorname{Hom}_R^H(P_m \otimes_k B_m, \overline{V}_m \otimes_k B_m),$$

is surjective (P_m is kH-projective, and $\operatorname{Hom}_R(B_m, r)$ is a kH-epimorphism since $(*)$ is R-splittable). Therefore, there exists $\psi'' : P_m \otimes_k B_m \to V_{(m)} \otimes_k B^{[1,m]}$ such that $r\psi'' = \psi'$, and consequently $\varphi' : M \to V_{(m-1)} \otimes_k B^{[1,m-1]}$ such that $v\varphi' = \varphi - \psi''\psi$, because $r(\varphi - \psi''\psi) = 0$. Note that $m(\varphi') \leq m-1$, and that by Remark 3.5, φ' belongs to $\widetilde{\mathcal{N}}$, since $\psi''\psi \in \mathcal{P}u$ by the first part of the proof; therefore all components of $\varphi - \psi''\psi$ belong to $\mathcal{N}$. Hence, by the inductive assumption, φ' factors through $\bigoplus_{i=1}^{m-1} P_i \otimes_k B_i$, where all P_i are finitely generated free kH-modules, and $\varphi = v\varphi' + \psi''\psi$ factors through $\bigoplus_{i=1}^{m} P_i \otimes_k B_i$.

Now we can prove the inclusion $\operatorname{Ker}\Psi^B \subset [\operatorname{mod}_{\mathcal{A}_{\mathrm{o}}^{\mathrm{f}}}(R/G)]_{\operatorname{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)}$. Let $f : M \to N$ be a morphism in $\operatorname{Mod}^G_{\mathrm{f},\mathcal{B}_{\mathrm{o}}} R$ such that $\widetilde{\Psi}^B(f) = 0$. Then, by Lemma 3.5, $M \simeq \theta_H^G(V \otimes_k B)$ and $N \simeq \theta_H^G(V' \otimes_k B)$ for some V, V' in I_n-spr(kH). By Lemma 3.4, all components of f belong to $\mathcal{N}$; therefore, the morphism $\varphi \in \operatorname{Hom}_R^H(\theta_H^G(V \otimes_k B), V' \otimes_k B)$ which corresponds to f via the isomorphism from [3, 2.3] belongs to $\widetilde{\mathcal{N}}$. Consequently, by the second part of the proof, φ factors through $\bigoplus_{i=1}^{m} P_i \otimes_k B_i$, where P_i's are as above, and by the first, through $\theta_e^H(Z)$, for some Z in $\operatorname{mod} R$. Hence, f factors through $\theta_e^G(Z) = \theta_H^G(\theta_e^H(Z))$ (apply [3, 2.3]), and the proof is complete. ∎

3.6. The next result proves Theorem 3.1(b).

LEMMA. *The functors* $\widetilde{\Psi}^B\widetilde{\Phi}^B$, $\mathrm{id}_{\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}}R} : \mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}}R \to \mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}}R$ *are isomorphic provided* $G = H$ *and* $\mathcal{N}_\mathrm{o} = 0$ (see Remark 3.4).

Proof. By Proposition 3.4(a), it suffices to show that the functors $\widetilde{\Psi}^B(-) \otimes_k B_{|\boldsymbol{M}}$ and $\mathrm{id}_{|\boldsymbol{M}}$ are isomorphic, where $\boldsymbol{M}$ is the full (dense) subcategory of $\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}}R$ formed by all $M = (M, \mu)$ such that $M = \bigoplus_{i=1}^n B_i^{d_i}$, $d_i \in \mathbb{N}$, $i = 1, \ldots, n$.

Fix any M in $\boldsymbol{M}$. Then for any $h \in H$ the composite R-homomorphism

$$\bigoplus_{i=1}^n B_i^{d_i} \xrightarrow{\mu_h} {}^{h^{-1}}\Big(\bigoplus_{i=1}^n B_i^{d_i}\Big) \simeq \bigoplus_{i=1}^n {}^{h^{-1}}B_i^{d_i} \xrightarrow{v} \bigoplus_{i=1}^n B_i^{d_i},$$

$v = \bigoplus_{i=1}^n ((\nu_i)_h^{-1})^{d_i}$, is given by the components $A_{i,j}(h) \cdot \beta_{i,j} : \bigoplus_{j=1}^n B_j^{d_j} \to \bigoplus_{i=1}^n B_i^{d_i}$, where $A_{i,j}(h) \in \mathrm{M}_{d_j \times d_i}(k)$, $i, j = 1, \ldots, n$, are uniquely determined by the equalities $\mathrm{Hom}_R(B_j, B_i) = k\beta_{i,j}$ for $i \le j$, and $A_{i,j}(h) = 0$ for $j < i$. Consequently, we have $(\mu_h)_{i,j} = ((\nu_i)_h)^{d_i}\,(A_{i,j}(h) \cdot \beta_{i,j}) = A_{i,j}(h) \cdot \beta_{i,j}(h)$, where $(\mu_h)_{i,j} : B_j^{d_j} \to {}^{h^{-1}}B_i^{d_i}$ is the (i,j)th component of μ_h, $i, j = 1, \ldots, n$. Applying the k-isomorphisms $\overline{\mathcal{H}}_i(M) \simeq \bigoplus_{l=1}^i (k\beta_{l,i})^{d_l}$, $\overline{\mathcal{H}}'_i(M) \simeq \bigoplus_{l=1}^i (k\beta_{l,n})^{d_l}$ and passing to components, we obtain $\widetilde{\Psi}^B(M) \simeq V$, where V in $I_n\text{-spr}(KH)$ is the object given by the spaces $V_i = \bigoplus_{l=1}^i k^{d_i}$, $i = 1, \ldots, n$, and the linear maps $\nu(h) : V_n \to V_n$, $h \in H$, with components $A_{i,j}(h)\cdot : k^{d_j} \to k^{d_i}$. It is easily seen that if we set $\underline{V}_i = k^{d_i}$, $i = 1, \ldots, n$, then the standard R-isomorphism

$$\bigoplus_{i=1}^n \underline{V}_i \otimes_k B_i \simeq \bigoplus_{i=1}^n B_i^{d_i}$$

is an isomorphism in $\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}}R$.

We denote by $\xi(M)$ the composite isomorphism

$$\widetilde{\Psi}^B(M) \otimes_k B \simeq \underline{V} \otimes_k B \simeq M$$

in $\mathrm{Mod}^G_{\mathrm{f},\mathcal{B}_\mathrm{o}}R$ and show that $\xi = \{\xi(M)\}_{M \in \mathrm{ob}\,\boldsymbol{M}}$ yields the required isomorphism of functors.

Fix any morphism $f : M \to M'$ in $\boldsymbol{M}$, where $M = \bigoplus_{i=1}^n B_i^{d_i}$ and $M' = \bigoplus_{i=1}^n B_i^{d'_i}$. Then the R-homomorphism f is given by the R-components $F_{i,j} \cdot \beta_{i,j} : B_i^{d_i} \to B_i^{d'_i}$, $i, j = 1, \ldots, n$, where $F_{i,j} \in \mathrm{M}_{d'_i \times d_j}(k)$ are uniquely determined by the equalities $\mathrm{Hom}_R(B_j, B_i) = k\beta_{i,j}$ for $i \le j$, and $F_{i,j} = 0$ for $i < j$. Consequently, the kH-homomorphism $\widetilde{\Psi}^B(f)$, regarded as a map $V \to V'$ under the isomorphisms $\widetilde{\Psi}^B(M) \simeq V$, $\widetilde{\Psi}^B(M') \simeq V'$ as above, is given by the components $F_{i,j}\cdot : k^{d_j} \to k^{d'_i}$, $i, j = 1, \ldots, n$. Now the equality $f \cdot \xi(M) = \xi(M') \cdot (\widetilde{\Psi}^B(f) \otimes_k B)$ follows by an easy check on definitions. ∎

3.7. To prove 3.1(c), recall that any surjective k-algebra homomorphism $A \to A_0$ induces a full and faithful embedding of categories

$$\mathrm{mod}(A_0)^{\mathrm{op}} \hookrightarrow \mathrm{mod}(A)^{\mathrm{op}},$$

and consequently

$$I_n\text{-spr}(A_0) \hookrightarrow I_n\text{-spr}(A).$$

Therefore, a surjective homomorphism $H \to H_0$ of groups induces a full and faithful embedding

$$I_n\text{-spr}(kH_0) \hookrightarrow I_n\text{-spr}(kH)$$

which preserves the coordinate vectors.

It is also well known that, for a k-algebra A and $m \le n$, any $s = (s_i)_{i=1,\dots,m} \in \mathbb{N}^m$ such that $1 \le s_1 < \ldots < s_m \le n$ yields the full embedding

$$\varepsilon_s^n : I_m\text{-spr}(A) \hookrightarrow I_n\text{-spr}(A)$$

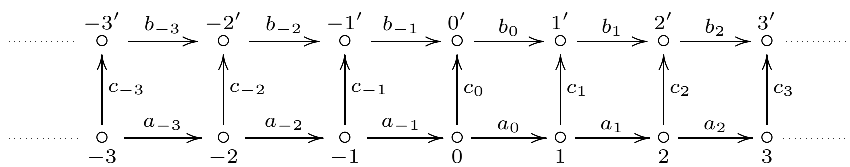
given by $(V_1 \subseteq \ldots \subseteq V_m) \mapsto (V_1' \subseteq \ldots \subseteq V_n')$, where $V_j' = 0$ for $j < s_1$, $V_j' = V_i$ for $s_i \le j < s_{i+1}$, $i = 1, \dots, n-1$, and $V_j' = V_m$ for $j \ge s_m$. Note that ε_s^n preserves the coordinate vectors, i.e. $\mathrm{cdn}(\varepsilon_s^n(V))_j = \mathrm{cdn}(V)_i$ if $j = s_i$ for some i and $\mathrm{cdn}(\varepsilon_s^n(V))_j = 0$ otherwise, for V in I_m-spr(A).

In consequence, the result below completes the proof of Theorem 3.1.

LEMMA. *Let H be an infinite cyclic group* (*resp. a cyclic p-group of order $|H| \ge 8$ if* $\mathrm{char}(k) = p > 0$). *Then the category I_2-spr$'(kH)$ is wild.*

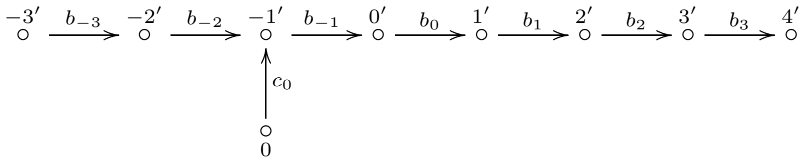
Proof. It is enough to show that the category I_2-spr$'(A)$ is wild, where $A = k[T]/(T^8)$ ($k[T]$ is the polynomial algebra in one variable T). The algebra A can be regarded as a factor algebra of $kH \simeq k[T, T^{-1}]$ (resp. of $kH \simeq k[T]/(T^{p^m} - 1)$ for $m \in \mathbb{N}$ large enough if $\mathrm{char}(k) = p > 0$) and then the category I_2-spr$'(kH)$ is also wild.

To prove our claim we apply the arguments suggested by D. Simson and consider the universal covering $F' : R' \to \overline{R}' = R'/G'$ of the algebra $\mathrm{T}_2(A^{\mathrm{op}})$ ($A(\overline{R}') \simeq \mathrm{T}_2(A^{\mathrm{op}})$, $G' \simeq \mathbb{Z}$). The cover category R' can be regarded as the locally bounded k-category opposite to kQ/I, where Q is the quiver



and I is the ideal in the path category kQ generated by all elements of the form $c_{i+1}a_i - b_i c_i$, $a_{i+7} \cdot \ldots \cdot a_i$ and $b_{i+7} \cdot \ldots \cdot b_i$, $i \in \mathbb{Z}$. Denote by $\mathcal{C}$ the full subcategory of $\mathrm{mod}\, R'$ formed by all representations V such that $V(c_0)$ is injective and $V(0), V(-1') \ne 0$, satisfying the following conditions: $V(i) = 0$ for $i \ge 5$ and $i \le -4$, $V(i') = 0$ for $i \ge 5$ and $i \le -1$, $V(i) = V(i')$ and

$V(c_i) = \mathrm{id}_{V_i}$ for $1 \le i \le 4$, $V(a_i) = V(b_i)$ for $1 \le i \le 3$ and $V(a_0) = V(b_0)\,V(c_0)$. It is easily seen that $\mathcal{C}$ is equivalent to the wild subcategory $\mathcal{D}$ of $\mathrm{mod}(kQ')^{\mathrm{op}}$ formed by all representations W of Q' such that $W(c_0)$ is injective and $W(0), W(-1') \neq 0$, where Q' is the quiver



Observe that $F'_\lambda(\mathcal{C}) \subset I_2\text{-spr}'(A)$, where $F'_\lambda : \mathrm{mod}\, R' \to \mathrm{mod}\, \mathrm{T}_2(A^{\mathrm{op}})$ is the "push-down" functor associated with F'. Moreover, F'_λ preserves the indecomposability (G' is torsionfree) and $(F'_\lambda)_{|\mathcal{C}}$ sends non-isomorphic indecomposables into non-isomorphic ones, since ${}^gV \not\simeq V'$ for all V, V' in $\mathcal{C}$ and $e \neq g \in G'$ (see [15]). Consequently, the category $I_2\text{-spr}'(kH)$ is wild (see [11]). ∎

Corollary. *If H is as is* 3.1(c) *then, for any $1 \le i < j \le n$, the full subcategory of all indecomposable non-orbicular modules in the category* $\mathrm{mod}_{\{B_i,B_j\}}(R/G)$ *is wild.*

Remark. (a) One can show that if H is as above then for any sequence $1 \le i_1 < \ldots < i_m \le n$, $2 \le m \le n$, the full subcategory formed by all indecomposable non-orbicular modules X in $\mathrm{mod}_{\{B_{i_1},\ldots,B_{i_m}\}}(R/G)$ such that $\mathrm{dss}(X) = \{B_{i_1}, \ldots, B_{i_m}\}$ is wild.

(b) The minimal value of $n \in \mathbb{N}$ such that $I_2\text{-spr}(k[T]/(T^n))$ is wild is not known to the author (clearly $n \ge 5$, by [29]).

4. Non-orbicular modules in $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$ and $\mathrm{mod}_{\{\widetilde{\widetilde{B}},B,\widetilde{B}\}}(R/G)$.

We apply Theorem 3.1 to the sequence of length 2 (resp. 3) induced by a G-atom B, which consists of B and its Kan extensions.

4.1. Let B be a G-atom. For simplicity we set $S = \mathrm{supp}\, B$ and denote by $\widetilde{B}$ the R-module $e^S_\lambda(B_{|S})$, where $e^S_\lambda : \mathrm{MOD}\, S \to \mathrm{MOD}\, R$ is the left adjoint to the restriction functor $e^S_\bullet : \mathrm{MOD}\, R \to \mathrm{MOD}\, S$. The module $\widetilde{B}$ belongs to $\mathrm{Ind}\, R$, $\mathrm{End}_R(\widetilde{B}) \simeq \mathrm{End}_S(B_{|S}) \simeq \mathrm{End}_R(B)$ (e^S_λ is a full and faithful embedding of $\mathrm{Mod}\, S$ into $\mathrm{Mod}\, R$), and the support $\widetilde{S} = \mathrm{supp}\, \widetilde{B}$ is contained in $\widehat{S}$ (see 1.5). Observe that $G_{\widetilde{B}}$ contains G_B; consequently, $\widetilde{B}$ is a G-atom, since $\widehat{S}$ is the union of a finite number of G_B-orbits in R (R is locally bounded and S/G_B is finite).

Note that iterating this construction we always get $e^{\widetilde{S}}_\lambda(\widetilde{B}_{|\widetilde{S}}) \simeq \widetilde{B}$, where $e^{\widetilde{S}}_\lambda : \mathrm{MOD}\, \widetilde{S} \to \mathrm{MOD}\, R$ is the left adjoint to the restriction functor $e^{\widetilde{S}}_\bullet$:

$\mathrm{MOD}\, R \to \mathrm{MOD}\, \widetilde{S}$ (for any $x \in \mathrm{ob}\, S$, we have $e_\lambda^S(S(-,x)) \simeq R(-,x) \simeq e_\lambda^{\widetilde{S}}(\widetilde{S}(-,x))$ and $e_\bullet^{\widetilde{S}}(R(-,x))$ is equal to the projective module $\widetilde{S}(-,x)$).

Suppose that B admits an R-action ν of G_B. Then ν induces an R-action $\widetilde{\nu} = (\widetilde{\nu}_h)_{h\in G_B}$ on $\widetilde{B}$, where each $\widetilde{\nu}_h$ is a family

$$\{\widetilde{\nu}_h(x) : B \otimes_S R(x,-) \to B \otimes_S R(hx,-)\}_{x\in \mathrm{ob}\, R}$$

of k-linear maps given by $\widetilde{\nu}_h(x)(b\otimes\alpha) = \nu_h(b)\otimes h\alpha$ for $y \in \mathrm{ob}\, S$, $b \in B(y)$, $\alpha \in R(x,y)$. Note that the counit map $\beta(B) : \widetilde{B} \to B$ (see 1.5) is a morphism from $\widetilde{B} = (\widetilde{B},\widetilde{\nu})$ to $B = (B,\nu)$ in $\mathrm{Mod}_{\mathrm{f}}^{G_B} R$.

Fix ν as above and denote by B the sequence

$$B : \quad B_1 \overset{\beta_2}{\leftarrow} B_2$$

of length 2, where $B_1 = B$, $B_2 = \widetilde{B}$ and $\beta_2 = \beta(B)$. Then according to the notation introduced in 3.1 we have $\mathcal{B}_{\mathrm{o}} = \{B, \widetilde{B}\}$ and $\mathcal{B} = \{{}^gB, {}^g\widetilde{B}\}_{g\in S_B}$, where $S_B = S_{G_B}$.

Now we are able to formulate our second main result of the paper.

Theorem. *Let $G \subseteq \mathrm{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on R. Suppose that B is a G-atom which admits an R-action ν of G_B, and B satisfies the following conditions*:

(a) $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$,

(b) $\widetilde{B} \not\simeq B$ (*equivalently*, $S \subsetneq \widetilde{S}$),

(c) $B_{|S}$ *is not a direct summand of any* ${}^g\widetilde{B}_{|S}$, *for* $g \in S_B \setminus \{e\}$.

Then the functor $\Phi^B : I_2\text{-}\mathrm{spr}(kG_B) \to \mathrm{mod}(R/G)$ is a representation embedding. In particular 3.1(c) *holds. If*, *in addition*, *G is torsionfree then the non-orbicular indecomposable modules in $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$ form a wild subcategory of $\mathrm{mod}_2(R/G)$.*

Remark. The condition (c) immediately implies

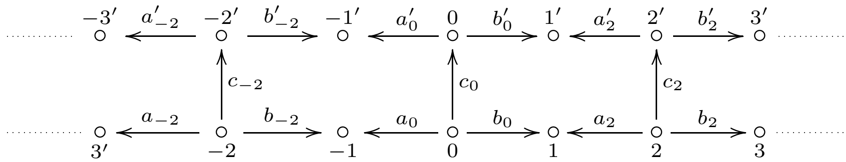
(d) $G_{\widetilde{B}} = G_B$,

since otherwise $B_{|S}$ is a direct summand of ${}^g\widetilde{B}_{|S}$ ($\simeq \widetilde{B}_{|S}$) for any $g \in (G_{\widetilde{B}} \setminus G_B) \cap S_B$ ($\neq \emptyset$). (For better understanding of (c) we also refer to Corollary 4.2.)

The proof (see 4.3) needs some preparation. We first illustrate the above result, and also the meaning of the conditions (c) and (d), by presenting several examples.

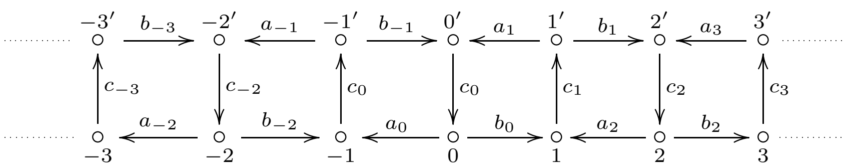
Example (i). Let R be the locally bounded k-category from Example 3.1. Keeping the notation from 3.1, we set $B = B_1$. It is easily seen that this example fits exactly into the context of Theorem 4.1. Note that all assumptions are trivially satisfied ($\widetilde{B} \simeq B_2$ and $\beta(B) = \beta_2$, under this identification).

Exaample (ii). Let R be the opposite (locally bounded) k-category to the path category kQ of the following quiver Q:



The category R is equipped with a natural free action of the infinite cyclic subgroup $G = \langle g \rangle$ of $\mathrm{Aut}_k(R)$, where g is defined by $g(i) = i+2$, $g(i') = (i+2)'$ for $i \in \mathbb{Z}$. Let B be the indecomposable R-module given by $B(i) = B((4i)') = k$ for all $i \in \mathbb{Z}$, $B(i') = 0$ for all $i \notin 4\mathbb{Z}$, and $B(a_{2i}) = B(b_{2i}) = B(c_{4i}) = \mathrm{id}_k$, $B(a'_{2i}) = B(b'_{2i}) = B(c_{4i+2}) = 0$ for all $i \in \mathbb{Z}$. The module B is a G-atom with stabilizer $G_B = \langle g^2 \rangle$. Then $\widetilde{B}$ can be viewed as an R-module given by setting $\widetilde{B}_{|\mathrm{supp}\, B} \simeq B_{|\mathrm{supp}\, B}$, $\widetilde{B}((4i+2)') = k$, $\widetilde{B}((2i+1)') = k^2$, and $\widetilde{B}(c_{4i+2}) = \mathrm{id}_k$, $\widetilde{B}(a'_{2i}) = w_1$, $\widetilde{B}(b'_{2i}) = w_2$ for all $i \in \mathbb{Z}$, where $w_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\cdot$ (resp. $w_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\cdot$) are the canonical embeddings. Consequently, $G_{\widetilde{B}} = G$ ($\supsetneq G_B$) and ${}^g\widetilde{B}_{|\mathrm{supp}\, B} \simeq B_{|\mathrm{supp}\, B}$ ($S_B = \{e, g\}$).

Example (iii). Let R be the locally bounded k-category opposite to the category kQ/I, where Q is the quiver



and I the ideal of the path category kQ generated by the elements $b_{i-1}c_{i-1}a_i - a_{i+1}c_{i+1}b_i$, $c_i b_{i-1} c_{i-1} a_i$, $i \in \mathbb{Z}$. The category R is equipped with a free action of the infinite cyclic subgroup $G = \langle g \rangle$ of $\mathrm{Aut}_k(R)$, where g is defined by $g(i) = (i+1)'$, $g(i') = i+1$ for $i \in \mathbb{Z}$. Let B be the "line" R-module given by $B(i) = k$, $B(i') = 0$, $B(a_{2i}) = B(b_{2i}) = \mathrm{id}_k$ for all $i \in \mathbb{Z}$, and $B(\gamma) = 0$ for all other arrows γ in Q. Then $\widetilde{B}$ can be viewed as an R-module given by $\widetilde{B}(i) = \widetilde{B}(i') = k$, $\widetilde{B}(a_i) = \widetilde{B}(b_i) = \widetilde{B}(c_{2i+1}) = \mathrm{id}_k$ and $\widetilde{B}(c_{2i}) = 0$ for all $i \in \mathbb{Z}$. Both modules B and $\widetilde{B}$ are G-atoms with stabilizers $G_B = \langle g^2 \rangle = G_{\widetilde{B}}$, but ${}^g\widetilde{B}_{|\mathrm{supp}\, B} \simeq B_{|\mathrm{supp}\, B}$ ($S_B = \{e, g\}$).

4.2. Lemma. *Let G be as above, H be a subgroup of G, and L a nontrivial full subcategory of R. Suppose that H stabilizes L and that L is contained in the union of a finite number of H-orbits in R. Then $gL \subset L$ if and only if $gL = L$, for any $g \in G$.*

Proof. Fix $x_1, \ldots, x_n \in \operatorname{ob} L$ such that $L = Hx_1 \cup \ldots \cup Hx_n$, an object x in L, and an element $g \in G$ such that $gL \subset L$. Then for every $l \in \mathbb{N}$ we have a descending sequence of inclusions

$$L \supset gL \supset g^2L \supset \ldots \supset g^lL$$

of subcategories of R. Note that, for every $m \in \mathbb{N}$, $g^m x = h_m x_{i(m)}$ for some $h_m \in H$ and $1 \le i(m) \le n$. Then $i(p) = i(m)$ for some $m > p$ and $h_p^{-1} g^p x = h_m^{-1} g^m x$. Since $G_x = \{e\}$, we have $g^{m-p} \in H$ and $g^l L = L$, where $l = m - p > 0$. Consequently, $gL = L$ and the proof is complete. ■

COROLLARY. *Let B be a G-atom, $\widetilde{B} = e_\lambda^S(B_{|S})$ and $g \in G$. If ${}^g\widetilde{B} \simeq B$ or $B_{|S}$ isomorphic to a direct summand of ${}^gB_{|S}$ then $g \in G_B$.*

Proof. In the case ${}^g\widetilde{B} \simeq B$, we have $gS \subset g\widetilde{S} \subset S$. Then by the lemma $gS = g\widetilde{S} = S$. This implies $\widetilde{B} \simeq B$ since $\widetilde{S} = S$, and so $g \in G_B$.

In the second case we have $gS \supset S$ and then by the lemma $gS = S$. This implies ${}^gB \simeq B$ and consequently $g \in G_B$. ■

4.3. *Proof of Theorem 4.1.* We construct an ideal $\mathcal{N}$ in $\mathcal{B}$ which satisfies the assumptions of Theorem 3.1. For simplicity we set $E = \operatorname{End}_S(B_{|S})$ ($\simeq \operatorname{End}_R(B)$) and $J = J(E)$. We denote by I the inverse image of J under the canonical isomorphism

$$\operatorname{Hom}_R(\widetilde{B}, B) \simeq E$$

which can also be viewed as the composition

$$\operatorname{Hom}_R(\widetilde{B}, B) \to \operatorname{Hom}_S(\widetilde{B}_{|S}, B_{|S}) \to E,$$

where the first map is given by the restriction functor $e_\bullet^S$ and the second is induced by an isomorphism $\beta(B)_{|S} : \widetilde{B}_{|S} \to B_{|S}$ (see 1.5).

We first define a family $\mathcal{N}_\circ = \{\mathcal{N}_\circ(B', B'') \subseteq \operatorname{Hom}_R(B', B'')\}_{B', B'' \in \mathcal{B}_\circ}$ of k-subspaces by setting

$$\mathcal{N}_\circ(B', B'') = \begin{cases} \operatorname{Hom}_R(B', B'') & \text{if } B' = B,\ B'' = \widetilde{B}, \\ I & \text{if } B' = \widetilde{B},\ B'' = B, \\ J(\operatorname{End}_R(B')) & \text{if } B' = B''. \end{cases}$$

We denote by $\mathcal{N}$ the family $\{\mathcal{N}(B', B'') \subseteq \operatorname{Hom}_R(B', B'')\}_{B', B'' \in \mathcal{B}}$ of k-subspaces given by 3.1(iii). To prove that $\mathcal{N}$ is an ideal in $\mathcal{B}$ we show first that $\mathcal{N}_\circ$ is an ideal in $\mathcal{B}_\circ$, equivalently, that

$$N = \begin{pmatrix} \mathcal{N}_\circ(B, B) & \mathcal{N}_\circ(\widetilde{B}, B) \\ \mathcal{N}_\circ(B, \widetilde{B}) & \mathcal{N}_\circ(\widetilde{B}, \widetilde{B}) \end{pmatrix}$$

is an ideal in the endomorphism algebra

$$\mathbb{E} = \operatorname{End}_R(B \oplus \widetilde{B}) = \begin{pmatrix} \operatorname{Hom}_R(B, B) & \operatorname{Hom}_R(\widetilde{B}, B) \\ \operatorname{Hom}_R(B, \widetilde{B}) & \operatorname{Hom}_R(\widetilde{B}, \widetilde{B}) \end{pmatrix}.$$

Consider the algebra homomorphism

$$r : \mathbb{E} \to \mathrm{M}_2(E)$$

which is the composition of the restriction map $\mathrm{End}_R(B \oplus \widetilde{B}) \to \mathrm{End}_S(B_{|S} \oplus \widetilde{B}_{|S})$ given by $e_\bullet^S$ and the isomorphism $\mathrm{End}_S(B_{|S} \oplus \widetilde{B}_{|S}) \to \mathrm{M}_2(E)$ induced by $\beta(B)_{|S}$. Observe that the first map, and then also r, is an embedding since $e_\bullet^S$ induces the isomorphism $\mathrm{End}_R(\widetilde{B}) \simeq \mathrm{End}_S(\widetilde{B}_{|S})$ and $S = \operatorname{supp} B$. Moreover, we have

$$r(\mathbb{E}) = \mathbb{E}' = \begin{pmatrix} E & E \\ U & E \end{pmatrix} \quad \text{and} \quad r(N) = N' = \begin{pmatrix} J & J \\ U & J \end{pmatrix}$$

where U is the image of $\mathrm{Hom}_R(B, \widetilde{B})$ under the $(2,1)$th component of r. The space U forms a two-sided ideal in E, since multiplication in $\mathbb{E}$ is well defined. Note that U is contained in J, since otherwise there exists $f \in \mathrm{Hom}_R(B, \widetilde{B})$ such that $\beta(B)_{|S} f_{|S} \in \mathrm{End}_S(B_{|S})$ is an isomorphism, and consequently $\beta(B) f \in \mathrm{End}_R(B)$ is an isomorphism and $\widetilde{B} \simeq B$, a contradiction.

Now it is easy to check that N' is an ideal in $\mathbb{E}'$. Consequently, the same holds for N in $\mathbb{E}$ and $\mathcal{N}_{\mathrm{o}}$ in $\mathcal{B}_{\mathrm{o}}$.

Next we show that the ideal $\mathcal{N}_{\mathrm{o}}$ is H-invariant, where $H = G_B = G_{\widetilde{B}}$ (see Remark 4.1). Note that since $\mathcal{J}_R$ is a G-invariant ideal in $\mathrm{Mod}\, R$ and $\mathcal{N}_{\mathrm{o}}(B, \widetilde{B}) = \mathrm{Hom}_R(B, \widetilde{B})$ we only need to check that $\mathcal{N}_{\mathrm{o}}(\widetilde{B}, B) = I$ is an H-invariant subspace of $\mathrm{Hom}_R(\widetilde{B}, B)$. In order to show that $h * f \in I$ for all $f \in I$ and $h \in H$, it suffices to show that $(h * f)_{|S} = ({}^h\nu_{h\,|S})({}^h f_{|S})(\widetilde{\nu}_{h\,|S})^{-1}$ is a non-isomorphism. Observe that $({}^h f)_{|\,(hS)}$ is a non-isomorphism since by definition of I so is $f_{|S}$. Consequently, $({}^h f)_{|S}$ is a non-isomorphism since $hS = S$, and therefore so is $(h * f)_{|S}$ (ν_h, $\widetilde{\nu}_h$ are isomorphisms).

Recall that $\mathcal{J}_R$ and Hom_R are summably closed ideals in $\mathrm{Mod}\, R$ (see [5, 7]). Therefore to show that the ideal $\mathcal{N}_{\mathrm{o}}$ is summably closed we have to check only that $\mathcal{N}_{\mathrm{o}}(\widetilde{B}, B) = I$ is a summably closed subspace of $\mathrm{Hom}_R(\widetilde{B}, B)$. Fix a summable family $f_i \in \mathrm{Hom}_R(\widetilde{B}, B)$, $i \in T$, such that $f_i \in I$ for every i. Then $\{f_{i\,|S}\}_{i \in T}$ is a summable family in $\mathrm{Hom}_S(\widetilde{B}_{|S}, B_{|S})$ and $\sum_{i \in T} f_{i\,|S} = f_{|S}$, where $f = \sum_{i \in T} f_i$. Since all $f_{i\,|S}$ are in $J \circ \beta(B)_{|S}$ and J is a summably closed subspace of E, $f_{|S}$ also belongs to $J \circ \beta(B)_{|S}$. Consequently, $f \in I$ and the claim is proved.

Finally, we show that $\mathcal{N}$ is an ideal in $\mathcal{B}$. Since $\mathcal{N}_{\mathrm{o}}$ is an ideal in $\mathcal{B}_{\mathrm{o}}$ we have to check that for any $f \in \mathrm{Hom}_R(B_i, {}^g B_l)$ and $f' \in \mathrm{Hom}_R({}^g B_l, B_j)$, the composition $f'f$ belongs to $\mathcal{N}$, for all $e \neq g \in S_H$ and $B_i, B_l, B_j \in \mathcal{B}_{\mathrm{o}}$ as in Remark 3.1. We first consider the case $B_i = B_j$. Suppose that $f'f \notin \mathcal{N}(B_i, B_j)$. Then ${}^g B_l \simeq B_i$ (B_l is indecomposable). Since $g \neq e$, we have $B_l \neq B_i$ and then either ${}^g\widetilde{B} \simeq B$ or ${}^{g^{-1}}\widetilde{B} \simeq B$, hence, by Corollary 4.2, g is in H, a contradiction. Consequently, $f'f \in \mathcal{N}$. It remains to consider the case $B_i = \widetilde{B}$, $B_j = B$, since $\mathcal{N}_{\circ}(B, \widetilde{B}) = \mathrm{Hom}_R(B, \widetilde{B})$. Suppose again that $f'f \notin \mathcal{N}(B_i, B_j) = I$. This means that the composition

$$\text{(i)} \qquad B_{|S} \xrightarrow{\beta(B)^{-1}_{|S}} \widetilde{B}_{|S} \xrightarrow{f_{|S}} {}^{g}B_{l\,|S} \xrightarrow{f'_{\;|S}} B_{|S}$$

does not belong to J and $B_{|S}$ is isomorphic to a direct summand of ${}^{g}B_{l\,|S}$. Then Corollary 4.2 (the case $B_l = B$) and the assumption (c) (the case $B_l = \widetilde{B}$) imply $g = e$, a contradiction. In consequence, $f'f \in \mathcal{N}$, and $\mathcal{N}$ is an ideal in $\mathcal{B}$.

Note that by construction the ideal $\mathcal{N}$ satisfies the remaining assumptions of Theorem 3.1, in particular $(*)$, and the proof is complete. ∎

REMARK. If $G = G_B$, the situation discussed in Theorem 4.1 is fully controlled by the subalgebra $\mathbb{E}' \subseteq \mathrm{M}_2(E)$ and the ideal N' (see 4.3).

4.4. COROLLARY. *Let $G \subseteq \mathrm{Aut}_k(R)$ be an infinite cyclic group acting freely on R. Suppose that there exists a G-atom such that $G_B = G$, $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$ and $\widetilde{B} \not\simeq \widetilde{B}$. Then $\mathrm{mod}_2(R/G)$ contains a wild subcategory consisting of non-orbicular indecomposable modules which is contained in $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$. Moreover, if $J(\mathrm{End}_R(B)) = \mathcal{P}u(B,B)$ and $\mathrm{Hom}_R(B,\widetilde{B}) = \mathcal{P}u(B,\widetilde{B})$, then the faithful embedding $\Phi^B : I_2\text{-spr}(kG) \to \mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$ is dense and induces an equivalence*

$$I_2\text{-spr}(kG) \simeq \mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)/\,[\mathrm{mod}_1(R/G)]_{\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)}.$$

Proof. The first assertion is an immediate consequence of Theorem 4.1. The second can be derived from Theorems 3.1(b) and 4.1, once we show that $I = \mathcal{P}u(\widetilde{B}, B)$ and $J(\mathrm{End}_R(\widetilde{B})) = \mathcal{P}u(\widetilde{B}, \widetilde{B})$. But these equalities follow easily from the definition of I and the isomorphism $\mathrm{End}_R(\widetilde{B}) \simeq \mathrm{Hom}_S(B_S, \widetilde{B}_S) \simeq \mathrm{End}_S(B_S)$, by the lemma below. ∎

We denote by $\mathcal{P}u'$ the pure-projective ideal in the category $\mathrm{MOD}\,S$.

LEMMA. (a) *For any M in $\mathrm{MOD}\,R$ and N in $\mathrm{MOD}\,S$, the canonical adjunction isomorphism $\mathrm{Hom}_R(e^S_\lambda(N), M) \simeq \mathrm{Hom}_S(N, e^S_\bullet(M))$ induces an isomorphism $\mathcal{P}u(e^S_\lambda(N), M) \simeq \mathcal{P}u'(N, e^S_\bullet(M))$.*

(b) *For any M in $\mathrm{Mod}\,R$ and N in $\mathrm{Mod}\,S$, the canonical adjunction isomorphism $\mathrm{Hom}_R(M, e^S_\varrho(N)) \simeq \mathrm{Hom}_S(e^S_\bullet(M), N)$ induces an isomorphism $\mathcal{P}u(M, e^S_\varrho(N)) \simeq \mathcal{P}u'(e^S_\bullet(M), N)$* (see 1.5 for definition of $e^S_\bullet$).

(c) *For any M, M' in $\mathrm{MOD}\,R$ such that $\mathrm{supp}\,M, \mathrm{supp}\,M' \subset S$, the restriction functor $e^S_\bullet$ induces an isomorphism $\mathcal{P}u(M, M') \simeq \mathcal{P}u'(e^S_\bullet(M), e^S_\bullet(M'))$.*

Proof. The statements (a) and (b) follow easily from the basic properties of the functors $e_\bullet$ and e_ϱ (to prove (b) apply the fact that each morphism in $\mathcal{P}u'_{\mathrm{Mod}\,S}$ factorizes through a locally finite-dimensional module which decomposes into a direct sum of finite-dimensional modules).

(c) It is clear that the restriction map $\mathcal{P}u(M, M') \to \mathcal{P}u'(M_{|S}, M'_{|S})$ is well defined and injective. To show that it is surjective we fix an S-homomorphism $f \in \mathcal{P}u(M_{|S}, M'_{|S})$. It admits a factorization $M_{|S} \xrightarrow{u} Z \xrightarrow{v} M'_{|S}$, where $Z = \bigoplus_{t\in T} Z_t$, $u = (u_t)_{t\in T}$, $v = (v_t)_{t\in T}$ and all Z_t's are in $\mathrm{mod}\, S$. Therefore f factors through the S-module $Z' = \bigoplus_{t\in T} Z'_t$, where $Z'_t = \mathrm{Im}\, u_t$ for every $t \in T$ ($f = v'u'$, $u' = (u'_t)_{t\in T}$, $v' = (v'_t)_{t\in T}$). Since $\mathrm{supp}\, M' \subset S$, each Z'_t as an S-factor of M can be extended by zeros to a module Z''_t in $\mathrm{mod}\, R$. Then all S-homomorphisms u'_t, v'_t, $t \in T$, and $f, u'v'$ can be regarded as R-homomorphisms and f factors through $Z'' = \bigoplus_{t\in T} Z''_t$. ■

We prove that, under the above assumptions (generally if G acts freely on $\mathrm{ind}(R/G)/\simeq$, G_B is infinite and $B \not\simeq \widetilde{B}$), also the category $\mathrm{mod}_1(R/G)$ is wild since so is $\mathrm{mod}\, R$ (see Theorems 7.1 and 7.6).

4.5. For a given G-atom B we can also consider the functor Φ^B relating to the dual construction, namely the sequence

$$\mathbb{B}: \quad \widetilde{\widetilde{B}} \overset{\beta'(B)}{\longleftarrow} B$$

where $\widetilde{\widetilde{B}} = e^S_\varrho(B)$ (see 1.5 for definition of $e^S_\varrho : \mathrm{Mod}\, S \to \mathrm{Mod}\, R$ and $\beta'(B)$). Observe that $\widetilde{\widetilde{B}}$, analogously to $\widetilde{B}$, is a G-atom and $\beta'(B)$ can be regarded as a morphism in $\mathrm{Mod}^{G_B} R$ provided B is equipped with a fixed R-action ν of G_B (if it admits any) and $\widetilde{\widetilde{B}}$ with the R-action $\widetilde{\widetilde{\nu}}$ of G_B which is induced by ν.

It is rather easily seen that for the sequence $\mathbb{B}$ as above we can prove results analogous to Theorem 4.1 and Corollary 4.4.

One can also study properties of the functor Φ^B for the "full" sequence induced by the G-atom B, i.e. the sequence

$$\mathbb{B}: \quad \widetilde{\widetilde{B}} \overset{\beta'(B)}{\longleftarrow} B \overset{\beta(B)}{\longleftarrow} \widetilde{B}$$

of length 3 in $\mathrm{MOD}^{G_B} R$. It is clear that now $\mathrm{Im}\, \Phi^B \subset \mathrm{mod}_{\widetilde{\widetilde{B}}, B, \widetilde{B}}(R/G)$ ($\mathcal{B}_{\mathrm{o}} = \{\widetilde{\widetilde{B}}, B, \widetilde{B}\}$).

The following result extends Theorem 4.1 in a natural way.

Theorem. *Let $G \subseteq \mathrm{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on R. Suppose that B is a G-atom which admits an R-action ν of G_B and B satisfies the following conditions*:

(a) $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$,

(b) $\widetilde{B} \not\simeq B \not\simeq \widetilde{\widetilde{B}}$,

(c) $G_{\widetilde{B}} = G_B = G_{\widetilde{\widetilde{B}}} = G$.

Then the functor $\Phi^B : I_3\text{-spr}(kG) \to \text{mod}(R/G)$ *is a representation embedding. In particular* 3.1(c) *holds and, if additionally* G *is torsionfree, then the non-orbicular indecomposable modules in* $\text{mod}_{\{\widetilde{\widetilde{B}},B,\widetilde{B}\}}(R/G)$ *form a wild subcategory of* $\text{mod}_2(R/G)$. *Moreover, if* G *is an infinite cyclic group and* $J(\text{End}_R(B) = \mathcal{P}u(B,B)$, $\text{Hom}_R(B,\widetilde{B}) = \mathcal{P}u(B,\widetilde{B})$, $\text{Hom}_R(\widetilde{\widetilde{B}},B) = \mathcal{P}u(\widetilde{\widetilde{B}},B)$, $\text{Hom}_R(\widetilde{\widetilde{B}},\widetilde{B}) = \mathcal{P}u(\widetilde{\widetilde{B}},\widetilde{B})$, *then the functor* $\Phi^B : I_3\text{-spr}(kG) \to \text{mod}_{\{\widetilde{\widetilde{B}},B,\widetilde{B}\}}(R/G)$ *is dense and induces an equivalence*

$$I_3\text{-spr}(kG) \simeq \text{mod}_{\{\widetilde{\widetilde{B}},B,\widetilde{B}\}}(R/G)/\left[\text{mod}_1(R/G)\right]_{\text{mod}_{\{\widetilde{\widetilde{B}},B,\widetilde{B}\}}(R/G)}.$$

Proof. Keeping all notation from 4.3 we construct, as in the proof of Theorem 4.1, the ideal $\mathcal{N}$ in $\mathcal{B}$ satisfying the assumptions of Theorem 3.1.

We denote by I' the inverse image of J under the standard adjunction isomorphism

$$\text{Hom}_R(B,\widetilde{\widetilde{B}}) \simeq E$$

(see 1.5 for the factorization), and by I'' the inverse image of J under the composite map

$$\text{Hom}_R(\widetilde{B},\widetilde{\widetilde{B}}) \to \text{Hom}_S(\widetilde{B}_{|S},\widetilde{\widetilde{B}}_{|S}) \to E,$$

where the first map is given by the restriction functor $e^S_\bullet$ and the second is induced by the isomorphisms $\beta(B)_{|S}$ and $\beta'(B)_{|S}$ (see 1.5). Then we let $\mathcal{N}_\circ = \{\mathcal{N}_\circ(B',B'') \subseteq \text{Hom}_R(B',B'')\}_{B',B''\in\mathcal{B}_\circ}$ be the family of k-subspaces given by

$$\mathcal{N}_\circ(B',B'') = \begin{cases} J(\text{End}_R(B')) & \text{if } B' = B'', \\ I & \text{if } B' = \widetilde{B},\ B'' = B, \\ I' & \text{if } B' = B,\ B'' = \widetilde{\widetilde{B}}, \\ I'' & \text{if } B' = \widetilde{B},\ B'' = \widetilde{\widetilde{B}}, \\ \text{Hom}_R(B',B'') & \text{otherwise.} \end{cases}$$

To show that $\mathcal{N}_\circ$ is an ideal we consider the subspace N of the endomorphism algebra

$$\mathbb{E} = \text{End}_R(\widetilde{\widetilde{B}} \oplus B \oplus \widetilde{B}) = \begin{pmatrix} \text{Hom}_R(\widetilde{\widetilde{B}},\widetilde{\widetilde{B}}) & \text{Hom}_R(B,\widetilde{\widetilde{B}}) & \text{Hom}_R(\widetilde{B},\widetilde{\widetilde{B}}) \\ \text{Hom}_R(\widetilde{\widetilde{B}},B) & \text{Hom}_R(B,B) & \text{Hom}_R(\widetilde{B},B) \\ \text{Hom}_R(\widetilde{\widetilde{B}},\widetilde{B}) & \text{Hom}_R(B,\widetilde{B}) & \text{Hom}_R(\widetilde{B},\widetilde{B}) \end{pmatrix}$$

defined by $\mathcal{N}_\circ$, and the algebra homomorphism

$$r : \mathbb{E} \to \text{M}_3(E)$$

which is the composition of the restriction map $\text{End}_R(\widetilde{\widetilde{B}} \oplus B \oplus \widetilde{B}) \to \text{End}_S(\widetilde{\widetilde{B}}_{|S} \oplus B_{|S} \oplus \widetilde{B}_{|S})$ defined by $e^S_\bullet$ and the isomorphism $\text{End}_S(\widetilde{\widetilde{B}}_{|S} \oplus B_{|S} \oplus \widetilde{B}_{|S}) \to \text{M}_3(E)$ induced by $\beta(B)_{|S}$ and $\beta'(B)_{|S}$.

Observe that all components $r_{i,j}$ $(i,j=1,2,3)$ of r but $r_{3,1}$ are injective (the map $r_{1,3}$ has a factorization $\mathrm{Hom}_R(\widetilde{B},\widetilde{\widetilde{B}})\simeq\mathrm{Hom}_S(B_{|S},\widetilde{\widetilde{B}}_{|S})\simeq E$, for the remaining ones apply arguments from 4.3). Then

$$r(\mathbb{E})=\begin{pmatrix}E&E&E\\U&E&E\\U''&U'&E\end{pmatrix},\quad r(N)=\begin{pmatrix}J&J&J\\U&J&J\\U''&U'&J\end{pmatrix},$$

where $U=\mathrm{Im}\,r_{2,1}$, $U'=\mathrm{Im}\,r_{3,2}$, $U''=\mathrm{Im}\,r_{3,1}$. The spaces U, U' form two-sided ideals in $\mathbb{E}$ which are contained in J (see 4.3).

Finally observe that $\mathrm{Hom}_R(\widetilde{\widetilde{B}},\widetilde{B})=\mathcal{J}_R(\widetilde{\widetilde{B}},\widetilde{B})$ since by (b), $\widetilde{B}$ and $\widetilde{\widetilde{B}}$ are not isomorphic.

By the above remarks it is easily seen that $\mathcal{N}_{\mathrm{o}}$ forms an ideal in $\mathcal{B}_{\mathrm{o}}$. As in 4.3, the ideal $\mathcal{N}=\mathcal{N}_{\mathrm{o}}$ satisfies the remaining assumptions of Theorem 3.1.

To complete the proof one shows that $\mathcal{N}_{\mathrm{o}}=\mathcal{P}u_{\mathcal{B}_{\mathrm{o}}}$ (this follows by Lemma 4.4 and definitions of I, I' and I''). ∎

5. The case of different stabilizers. In this section we briefly discuss the problem of how to construct indecomposable non-orbicular modules in $\mathrm{mod}_{\mathcal{B}_{\mathrm{o}}}(R/G)$, by use of generalized tensor product, in the case when the stabilizers G_{B_i} of G-atoms B_i, $i=1,\ldots,n$, are not all equal to H (see 3.1). We study more carefully the very special situation when $\mathcal{B}_{\mathrm{o}}=\{B,\widetilde{B}\}$ for a G-atom B (as in the previous section), but in contrast (to 3.1 and 4.1) we now assume $G_B\subsetneq G_{\widetilde{B}}$ (see Example 4.1(ii)).

5.1. Keeping the notation from 4.1 and assumptions (a) and (b) from Theorem 4.1 (we drop assumption (c)), we assume that there exists an R-action ν of $H=G_B$ on B such that the R-action $\widetilde{\nu}=\widetilde{\nu}_H$ of H on $\widetilde{B}$ can be extended to an R-action $\widetilde{\nu}_{G_B}$ of $G_{\widetilde{B}}$ on $\widetilde{B}$ (i.e. $(\widetilde{\nu}_{G_{\widetilde{\nu}}})_{|H}=\widetilde{\nu}_H$).

We fix ν and $\widetilde{\nu}_{G_B}$ as above and assume for simplicity that $G_{\widetilde{B}}=G$. Then the morphism $\beta=\beta(B):(\widetilde{B},\widetilde{\nu}_H)\to(B,\nu)$ in $\mathrm{Mod}_{\mathrm{f}}^H R$ induces the morphism

$$\beta^G:(\widetilde{B},\widetilde{\nu}_G)\to(B^G,\nu^G)\ (=\theta^G_{G_B}(B,\nu))$$

in $\mathrm{Mod}_{\mathrm{f}}^G R$ given by components $\beta_g={}^g\beta\cdot\nu_{g^{-1}}:\widetilde{B}\to{}^gB$, where $B^G=\bigoplus_{g\in S_H}{}^gB$ (see [3, Lemma 2.3]).

From now on we use the notation $\widetilde{\nu}$ also for $\widetilde{\nu}_{G_{\widetilde{B}}}$.

Denote by B the sequence

$$B:\quad B_1\xleftarrow{\beta_2}B_2$$

of length 2, where $B_1=B^G$, $B_2=\widetilde{B}$ and $\beta_2=\beta^G$. According to 3.1, the sequence B induces the functors $\widetilde{\Phi}^B:I_2\text{-spr}(kG)\to\mathrm{Mod}_{\mathrm{f}}^G R$, $\widetilde{\Phi}^B=-\otimes_k B$, and $\Phi^B:I_2\text{-spr}(kG)\to\mathrm{mod}(R/G)$, $\Phi^B=F_\bullet^{-1}\circ\widetilde{\Phi}^B$. It is easily seen that similarly to 4.1 we have $\mathrm{Im}\,\Phi^B\subset\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$ and $\mathrm{dsc}(\Phi^B(V))=$

$\operatorname{cdn}(V)$ for V in $I_2\text{-spr}(kG)$ (cf. 3.1). Moreover, $\Phi^B(V)$ is in $\operatorname{mod}_B(R/G)$ if and only if V is in $I_2\text{-spr}_1(kG)$, where $I_2\text{-spr}_1(kG)$ is the full subcategory of $I_2\text{-spr}(kG)$ formed by all objects $V = (V_1 \subseteq V_2)$ such that $V_1 = V_2$ ($I_2\text{-spr}_1(kG) = \operatorname{Im} \varepsilon^2_{(1)}$, cf. 3.7). Nevertheless, we cannot expect such nice behaviour of the functor Φ^B as in Theorem 4.1 and Corollary 4.4 (see Theorem 5.5). To study it we will proceed analogously and define a functor $\Psi^B : \operatorname{mod}_{\{B,\widetilde{B}\}}(R/G) \to I_2\text{-spr}(kG)$ (see 5.2).

Denote by $\mathcal{B}$ the full subcategory of $\operatorname{Mod} R$ formed by $\{B\} \cup \{{}^gB\}_{g\in S_H}$, and by $\mathcal{N}$ the family $\mathcal{N}(B', B'') \subseteq \operatorname{Hom}_R(B, B'')$, $B', B'' \in \mathcal{B}$, of k-subspaces defined by

$$\text{(i)} \qquad \mathcal{N}(B', B'') = \begin{cases} \mathcal{J}_R(B', B'') & \text{if } B' = B'', \\ {}^gI \cdot \widetilde{\nu}_{g^{-1}} & \text{if } B' = \widetilde{B},\ B'' = {}^gB, \\ \operatorname{Hom}_R(B', B'') & \text{otherwise,} \end{cases}$$

where I is as in 4.3. Note that the definition of $\mathcal{N}$ does not depend on the choice of the isomorphisms $\widetilde{\nu}_{g^{-1}}$, $g \in S_H$ ($\varphi I = I$ for any $\varphi \in \operatorname{Aut}_R(B)$), and that the restriction of $\mathcal{N}$ to the full subcategory $\mathcal{B}_\text{o}$ of $\mathcal{B}$ formed by the set $\{B, \widetilde{B}\}$ is equal to the ideal $\mathcal{N}_\text{o}$ from 4.3. We also have the formulas

$$\text{(ii)} \qquad \operatorname{Hom}_R(B', B'') = k\beta_{B'',B'} \oplus \mathcal{N}(B', B'')$$

where

$$\text{(iii)} \qquad \beta_{B'',B'} = \begin{cases} \operatorname{id}_{B'} & \text{if } B' = B'', \\ \beta_g & \text{if } B' = \widetilde{B},\ B'' = {}^gB, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA. *$\mathcal{N}$ is an ideal in $\mathcal{B}$.*

Proof. Since $\mathcal{N}_\text{o}$ is an ideal in $\mathcal{B}_\text{o}$, the restriction of $\mathcal{N}$ to the full subcategory formed by $\{{}^gB, \widetilde{B}\}$ is an ideal for every $g \in S_H$. Therefore to show that $\mathcal{N}$ is an ideal in $\mathcal{B}$, it suffices to know that $f'f$ belongs to $\operatorname{Hom}_R(\widetilde{B}, B)$ for all $f \in \operatorname{Hom}_R(\widetilde{B}, {}^gB)$, $f' \in \operatorname{Hom}_R({}^gB, B)$, $e \neq g \in S_H$. But this has already been proved (see 4.3(i)). ∎

5.2. Let $\widetilde{\mathcal{B}}$ denote the additive closure of $\mathcal{B}$ (i.e. the full subcategory of $\operatorname{Mod} R$ formed by all R-modules M of the form $M \simeq \bigoplus_{g\in S_H} {}^gB^{d_g} \oplus \widetilde{B}^d$, where $d_g, d \in \mathbb{N}$), and $\widetilde{\mathcal{N}}$ the ideal in $\widetilde{\mathcal{B}}$ which is the unique extension of $\mathcal{N}$ to $\widetilde{\mathcal{B}}$ ($|S_H| = [G_{\widetilde{B}} : G_B]$ is finite!).

LEMMA. *The k-subspace $\widetilde{\mathcal{N}}(M, M') \subseteq \operatorname{Hom}_R(M, M')$ is a kG-submodule of $\operatorname{Hom}_R(M, M')$ for any M, M' in $\operatorname{Mod}^G_\text{f} R$.*

Proof. First we show that $\varphi_2^{-1} \cdot {}^gf \cdot \varphi_1 \in \widetilde{\mathcal{N}}({}^{g_1}B_1, {}^{g_2}B_2)$ for any $g \in G$, B_1, B_2 in $\mathcal{B}$, $f \in \widetilde{\mathcal{N}}(B_1, B_2)$ and R-isomorphisms $\varphi_i : {}^{g_i}B_i \to {}^gB_i$, $i = 1, 2$, where g_i represents gg' in S_H in case $B_i = {}^{g'}B$, $g' \in S_H$, or $g_i = e$ in case $B_i = \widetilde{B}$. Since $\mathcal{J}_R$ is an ideal in $\operatorname{Mod} R$ and $\mathcal{N}$ is an ideal in $\mathcal{B}$, it suffices to check the case $B_1 = \widetilde{B}$, $B_2 = {}^{g'}B$, and $\varphi_1 = \widetilde{\nu}_{g^{-1}}$, $\varphi_2 = ({}^{gg'}\nu_{h_2})^{-1}$, where $g_2h_2 = gg'$, $g_2 \in S_H$, $h \in H$. Fix $f = {}^{g'}f_0 \cdot \widetilde{\nu}_{g'_{-1}} \in \mathcal{N}(\widetilde{B}, {}^{g'}B)$, where $f_0 \in \mathcal{N}(\widetilde{B}, B)$. Then

$$ {}^{gg'}\nu_{h_2} \cdot {}^{g}({}^{g'}f_0 \cdot \widetilde{\nu}_{g'_{-1}}) \cdot \widetilde{\nu}_{g^{-1}} = {}^{g_2h_2}\nu_{h_2} \cdot {}^{gg'}f_0 \cdot \widetilde{\nu}_{(gg')^{-1}} = {}^{g_2}({}^{h_2}\nu_{h_2} \cdot {}^{g_2h_2}f_0 \cdot \widetilde{\nu}_{h_2^{-1}}) \cdot \widetilde{\nu}_{g_2^{-1}}. $$

Note that ${}^{h_2}\nu_{h_2} \cdot {}^{g_2h_2}f_0 \cdot \widetilde{\nu}_{h_2^{-1}} \in \mathcal{N}(\widetilde{B}, B)$ since $\mathcal{N}_{\mathrm{o}}$ is a kH-invariant ideal in $\mathcal{B}_{\mathrm{o}}$ (see 4.3). Consequently, ${}^{g_2}({}^{h_2}\nu_{h_2} \cdot {}^{g_2h_2}f_0 \cdot \widetilde{\nu}_{h_2^{-1}}) \cdot \widetilde{\nu}_{g_2^{-1}} \in \mathcal{N}(\widetilde{B}, {}^{g_2}B)$.

Now we prove the main assertion. Fix $g \in G$ and a morphism $f : (M, \mu) \to (M', \mu')$ in $\mathrm{Mod}_{\mathrm{f}}^{G} R$ which belongs to $\widetilde{\mathcal{N}}(M, M')$. To check that $g * f = {}^{g}\mu'_{g} \cdot {}^{g}f \cdot \mu_{g^{-1}}$ belongs to $\widetilde{\mathcal{N}}(M, M')$, we show that each component $(g * f)' : B_1 \to B_2$, B_1, B_2 in $\mathcal{B}$, of the R-homomorphism $g * f$, under the fixed R-module identifications $M \simeq \bigoplus_{g \in S_H} {}^{g}B^{n} \oplus \widetilde{B}^{\widetilde{n}}$, $M' \simeq \bigoplus_{g \in S_H} {}^{g}B^{n'} \oplus \widetilde{B}^{\widetilde{n}'}$, $n, n', \widetilde{n}, \widetilde{n}' \in \mathbb{N}$, belongs to $\mathcal{N}(B_1, B_2)$. Observe that, for each component $f' : B_1 \to B_2$, B_1, B_2 in $\mathcal{B}$, of f, the R-homomorphism ${}^{g}(f') : {}^{g}B_1 \to {}^{g}B_2$ (under the above identifications) can be represented in the form $\varphi_2 \cdot \varphi_2^{-1} \cdot {}^{g}f \cdot \varphi_1 \cdot \varphi_2^{-1}$, where φ_1, φ_2 are as in the first part of the proof. Now passing to components of the R-isomorphisms $\mu_{g^{-1}}$, ${}^{g}\mu'_{g}$, and applying the fact that all $\varphi_2^{-1} \cdot {}^{g}(f') \cdot \varphi_1$'s belong to the ideal N, we immediately obtain our claim. ■

To define Ψ^B we denote by

$$ \overline{\mathcal{H}}_1, \overline{\mathcal{H}}_2 : \mathrm{Mod}^{G}_{\mathrm{f}, \{B, \widetilde{B}\}} R \to \mathrm{MOD}(kG)^{\mathrm{op}} $$

the functors $\overline{\mathcal{H}}_1 = \mathrm{Hom}_R(B^G, -)/\widetilde{\mathcal{N}}(B^G, -)$, $\overline{\mathcal{H}}_2 = \mathrm{Hom}_R(\widetilde{B}, -)/\widetilde{\mathcal{N}}(\widetilde{B}, -)$ and by

$$ \iota : \overline{\mathcal{H}}_1 \to \overline{\mathcal{H}}_2 $$

the natural transformation of functors induced by the morphism $\beta^G : \widetilde{B} \to B^G$ in $\mathrm{Mod}_{\mathrm{f}}^{G} R$ ($\overline{\mathcal{H}}_1, \overline{\mathcal{H}}_2$ and ι are well defined by Lemmas 5.1 and 5.2). Note that $\overline{\mathcal{H}}_i(\mathrm{Mod}^{G}_{\mathrm{f}, \{B, \widetilde{B}\}} R) \subset \mathrm{mod}(kG)^{\mathrm{op}}$ for $i = 1, 2$ (see 5.1(ii)).

Now we define the functor

$$ \widetilde{\Psi}^B : \mathrm{Mod}^{G}_{\mathrm{f}, \{B, \widetilde{B}\}} R \to I_2\text{-spr}(kG). $$

We set

$$ \widetilde{\Psi}^B(M) = (\mathrm{Im}\, \iota(M) \subseteq \overline{\mathcal{H}}_2(M)) $$

for M in $\mathrm{Mod}^{G}_{\mathrm{f}, \{B, \widetilde{B}\}}$.

Let $f : M \to M'$ be a morphism in $\mathrm{Mod}^{G}_{\mathrm{f}, \{B, \widetilde{B}\}}$. Since ι is a natural transformation, we have $\overline{\mathcal{H}}_2(f)(\mathrm{Im}\, \iota(M)) \subseteq \mathrm{Im}\, \iota(M')$. We set

$$\widetilde{\Psi}^B(f) = \overline{\mathcal{H}}_2(f).$$

It is easily seen that $\widetilde{\Psi}^B$ is a k-linear functor.

We denote by $\Psi^B : \mathrm{mod}_{\{B,\widetilde{B}\}}(R/G) \to I_2\text{-spr}(kG)$ the functor

$$\Psi^B = \widetilde{\Psi}^B \circ F_\bullet.$$

REMARK. (a) Let M be in $\mathrm{Mod}^G_{\mathrm{f},\,\{B,\widetilde{B}\}}$. Then $\overline{\mathcal{H}}_1(M) \simeq \bigoplus_{g\in S_H}(k\,\mathrm{id}_{{}^gB})^n$, $\overline{\mathcal{H}}_2(M) \simeq \bigoplus_{g\in S_H}(k\beta_g)^n \oplus (k\,\mathrm{id}_{\widetilde{B}})^{\widetilde{n}}$, where $M \simeq \bigoplus_{g\in S_H} {}^gB^n \oplus \widetilde{B}^{\widetilde{n}}$, $n, \widetilde{n} \in \mathbb{N}$ (we have $\mathrm{Hom}_R(B^G, M) \simeq \bigoplus_{g\in S_H}(k\,\mathrm{id}_{{}^gB})^n \oplus \widetilde{\mathcal{N}}(B^G, M)$ and $\mathrm{Hom}_R(\widetilde{B}, M)$ $\simeq \bigoplus_{g\in S_H}(k\beta_g)^n \oplus\ (k\,\mathrm{id}_{\widetilde{B}})^{\widetilde{n}} \oplus \widetilde{\mathcal{N}}(\widetilde{B}, M)$, by 5.1(ii)).

(b) The map $\iota(M)$ is a kG-monomorphism for any M in $\mathrm{Mod}^G_{\mathrm{f},\,\{B,\widetilde{B}\}}R$ (by the identifications from (a), $\iota(M)$ maps $a\cdot\mathrm{id}_{{}^gB}$ onto $a\cdot\beta_g$ for any $g \in S_H$ and $a \in k^n$).

(c) For any X in $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$, we have

$$\mathrm{cdn}(\Psi^B(X))_2 = \dim_k \overline{\mathcal{H}}_2(F_\bullet(X)) - \dim_k \mathrm{Im}\,\iota(F_\bullet(X)) = \mathrm{dsc}(X)_{\widetilde{B}},$$
$$\mathrm{cdn}(\Psi^B(X))_1 = \dim_k \mathrm{Im}\,\iota(F_\bullet(X)) = [G : H]\cdot \mathrm{dsc}(X)_B.$$

In particular, $\Psi^B(X)$ is in $I_2\text{-spr}_1(kG)$ if and only if X is in $\mathrm{mod}_{\{B\}}(R/G)$, and $\Psi^B(X)$ is in $I_2\text{-spr}'(kG)$ if and only if X is non-orbicular.

5.3. Now we compute the composition $\Psi^B \circ \Phi^B$.

For any $W = (W,\mu)$ in $\mathrm{mod}(kG)^{\mathrm{op}}$ we denote by $W^{G/H}$ the kG-module defined by the k-space $k(G/H) \otimes_k W = \bigoplus_{\gamma\in G/H} \gamma \otimes W$ together with the linear action $\mu^{G/H} = (\mu^{G/H}(g))_{g\in G}$ of G given by $(g, \gamma\otimes w) \mapsto g\gamma \otimes gw$, $g \in G$, $\gamma \in G/H$, $w \in W$ (G acts on the set G/H of left cosets by left shifts). Note that we have a kG-isomorphism $KG \otimes_{kH} W \simeq W^{G/H}$ given by $g\otimes w \mapsto gH \otimes gw$, $g \in G$, $w \in W$, which is natural with respect to W.

We denote by $\nabla = \nabla_W : W^{G/H} \to W$ and $\Delta = \Delta_W : W \to W^{G/H}$ ($[G : H] = [G_{\widetilde{B}} : G_B]$ is finite) the standard (natural with respect to W) kG-homomorphisms given by $\gamma\otimes w \mapsto w$ and $w \mapsto \sum_{\gamma\in G/H} \gamma\otimes w$. Note that $\nabla\Delta = [G : H]\cdot \mathrm{id}_W$.

We define the functor

$$\Gamma : I_2\text{-spr}(kG) \to I_2\text{-spr}(kG)$$

setting

$$\Gamma(V) = (\mathrm{Im}\, i\ \subseteq\ V_1^{G/H} \sqcup_{V_1} V_2)$$

for $V = (V_1 \subseteq V_2)$ in $I_2\text{-spr}(kG)$, where $i = i(V) : V_1^{G/H} \to V_1^{G/H} \sqcup_{V_1} V_2$ is the first canonical embedding into an amalgamated sum (defined by the maps $V_1 \hookrightarrow V_2$ and $\Delta_{V_1} : V_1 \to V_1^{G/H}$). Note that we have $\Gamma(I_2\text{-spr}_1(kG)) \subset I_2\text{-spr}_1(kG)$.

REMARK. (a) $V_1^{G/H} \sqcup_{V_1} V_2$ can be identified with the kG-module defined by the space $V_1^{G/H} \oplus \underline{V}_2$ with the G-action given by the matrices

$$\underline{\mu}^{G/H}(g) = \begin{bmatrix} (\mu_{1,1})^{G/H}(g) & \mu^{G/H}(g)_{1,2} \\ 0 & \mu(g)_{2,2} \end{bmatrix}, \quad g \in G,$$

where $\underline{V}_2$ is a fixed complementary direct summand for V_1 in V ($V_2 = V_1 \oplus \underline{V}_2$), $\mu_{1,1} = (\mu(g)_{1,1})_{g\in G}$ and $\mu^{G/H}(g)_{1,2} : \underline{V}_2 \to \bigoplus_{\gamma\in G/H} \gamma \otimes V_1$ is given by $v \mapsto \sum_{\gamma\in G/H} \gamma \otimes \mu(g)_{1,2}(v)$, $v \in \underline{V}_2$. The identification is induced by the canonical embeddings $w_1 : V_1^{G/H} \to V_1^{G/H} \oplus \underline{V}_2$ and $\Delta_{V_1} \oplus \mathrm{id}_{\underline{V}_2} : V_2 \to V_1^{G/H} \oplus \underline{V}_2$. Under this identification, i corresponds to w_1 and the second embedding $i' = i'(V) : V_2 \to V_1^{G/H} \sqcup_{V_1} V_2$ to $\Delta_{V_1} \oplus \mathrm{id}_{\underline{V}_2}$.

(b) The family $i'(V) : V_2 \to V_1^{G/H} \sqcup_{V_1} V_2$, V in I_2-spr(kG), of kG-homomorphisms defines the natural transformation $i' : \mathrm{id}_{I_2\text{-spr}(kG)} \to \Gamma$ of functors.

PROPOSITION. $\Psi^B \circ \Phi^B \simeq \Gamma$.

Proof. Fix $V = (V_1 \subseteq V_2)$ in I_2-spr(kG) together with a complementary direct summand $\underline{V}_2 \subseteq V_2$ for V_1 in V_2 ($V_2 = V_2 \oplus \underline{V}_2$). We construct an isomorphism $\eta(V) : \widetilde{\Psi}^B\widetilde{\Phi}^B(V) \to \Gamma(V)$ in I_2-spr(kG).

Fix bases of the spaces V_1, $\underline{V}_2$ and denote by $\psi_g : V_1 \otimes_k {}^gB \to {}^gB^{d_1}$, $g \in S_H$, and by $\psi_2 : \underline{V}_2 \otimes_k \widetilde{B} \to \widetilde{B}^{d_2}$ the isomorphisms induced by the selection of bases, where $d_1 = \dim_k V_1$ and $d_2 = \dim_k \underline{V}_2$. For any $g \in G$ we denote by $\underline{\mu(g)}_{i,j}$ the matrices of the k-linear maps $\mu(g)_{i,j}$, $i,j = 1,2$, in the fixed bases above (cf. 2.1).

For any $f \in \mathrm{Hom}_R(\widetilde{B}, \widetilde{\Phi}^B(V))$ we denote by $f_1 \in \mathrm{Hom}_R(\widetilde{B}, V_1 \otimes_k B^G)$ (resp. $f_2 \in \mathrm{Hom}_R(\widetilde{B}, \underline{V}_2 \otimes_k \widetilde{B})$) the components of f under the identification induced by the equality $\widetilde{\Phi}^B(V) = V_1 \otimes_k B^G \oplus \underline{V}_2 \otimes_k \widetilde{B}$, and by $f_{1,g} \in \mathrm{Hom}_R(\widetilde{B}, V_1 \otimes_k {}^gB)$, $g \in S_H$, the components of f_1 under the identification given by the canonical isomorphism $V_1 \otimes_k B^G \simeq \bigoplus_{g\in S_H} V_1 \otimes_k {}^gB$.

For any $f \in \mathrm{Hom}_R(\widetilde{B}, \underline{V}_2 \otimes_k \widetilde{B})$ we denote by $\overline{f} \in \bigoplus_{g\in S_H} (k\,\mathrm{id}_{\widetilde{B}})^{d_2}$ and $f' \in \mathcal{N}(\widetilde{B}, \widetilde{B})^{d_2}$ the components of f under the identification $\mathrm{Hom}_R(\widetilde{B}, \underline{V}_2 \otimes_k \widetilde{B}) \simeq (k\,\mathrm{id}_{\widetilde{B}})^{d_2} \oplus \mathcal{N}(\widetilde{B}, \widetilde{B})^{d_2}$ induced by ψ_2 (cf. 5.1(ii)).

For any $f \in \mathrm{Hom}_R(\widetilde{B}, V_1 \otimes_k {}^gB)$, $g \in S_H$, (resp. $f \in \mathrm{Hom}_R(\widetilde{B}, V_1 \otimes_k B^G)$ with components $f_g \in \mathrm{Hom}_R(\widetilde{B}, V_1 \otimes_k {}^gB)$, $g \in S_H$) we denote by $\overline{f} \in (k\beta_g)^{d_1}$ and $f' \in \mathcal{N}(\widetilde{B}, {}^gB)^{d_1}$ (resp. $\overline{f} = (\overline{f}_g) \in \bigoplus_{g\in S_H} (k\beta_g)^{d_1}$ and $f' = (f'_g) \in \bigoplus_{g\in S_H} \mathcal{N}(\widetilde{B}, {}^gB)^{d_1}$) the components of f under the identification $\mathrm{Hom}_R(\widetilde{B}, V_1 \otimes_k {}^gB) \simeq (k\beta_g)^{d_1} \oplus \mathcal{N}(\widetilde{B}, {}^gB)^{d_1}$, induced by ψ_g (cf. 5.1(ii)).

To construct the isomorphism $\eta(V)$ we first compute $\overline{(g*f)_i}$, $g\in G$, $i=1,2$, for any $f\in \operatorname{Hom}_R(\widetilde{B}, V_1\otimes_k B^G)$ and $f\in \operatorname{Hom}_R(\widetilde{B}, \underline{V}_2\otimes_k \widetilde{B})$.

Fix $f\in \operatorname{Hom}_R(\widetilde{B}, V_1\otimes_k B^G)$. Since $\mu(g)_{2,1}=0$, we have $(g*f)_2=0$, and consequently

(i) $$\overline{(g*f)_2}=0.$$

To compute $\overline{(g*f)_1}$ observe that

$$(g*f)_1={}^g(\mu(g)_{1,1}\otimes_k \nu_g^G)\cdot {}^gf\cdot \widetilde{\nu}_{g^{-1}}$$

and

$$((g*f)_1)_{g_2}=(\mu(g)_{1,1}\otimes_k {}^{gg_1}\nu_h)\cdot {}^gf\cdot \widetilde{\nu}_{g^{-1}}$$

for any fixed $g_2\in S_H$, where $gg_1=g_2h$, $g_1\in S_H$, $h\in H$ (see 2.5 and 5.1). Then

$$\begin{aligned}\psi_{g_2}\cdot((g*f)_1)_{g_2}&=(\underline{\mu(g)}_{1,1}\cdot {}^{gg_1}\nu_h)\,({}^g(\overline{f}_{g_1}+f'_{g_1})\cdot\widetilde{\nu}_{g^{-1}})\\&=(\underline{\mu(g)}_{1,1}\,a_{g_1})({}^{gg_1}\nu_h\cdot {}^g\beta_{g_1}\cdot\widetilde{\nu}_{g^{-1}})\\&\quad+(\underline{\mu(g)}_{1,1}\cdot {}^{gg_1}\nu_h)({}^gf'_{g_1}\cdot\widetilde{\nu}_{g^{-1}})\end{aligned}$$

where $\overline{f}_{g_1}=a_{g_1}\cdot\beta_{g_1}$, $a_{g_1}\in k^{d_1}$. The second summand belongs to $\mathcal{N}$ (see the proof of Lemma 5.2), the first is equal to $(\underline{\mu(g)}_{1,1}a_{g_1})\cdot\beta_{g_2}$ since

$$\begin{aligned}{}^{gg_1}\nu_h\cdot {}^g\beta_{g_1}\cdot\widetilde{\nu}_{g^{-1}}&={}^{gg_1}\nu_h\cdot {}^{gg_1}\beta\cdot {}^g\widetilde{\nu}_{g_1^{-1}}\cdot\widetilde{\nu}_{g^{-1}}\\&={}^{gg_1h^{-1}}\beta\cdot {}^{gg_1}\widetilde{\nu}_h\cdot\widetilde{\nu}_{(gg_1)^{-1}}={}^{g_2}\beta\cdot {}^g\widetilde{\nu}_{g_2^{-1}}\end{aligned}$$

(β is a morphism in $\operatorname{Mod}_{\mathrm{f}}^G R$). Consequently,

(ii) $$\overline{((g*f)_1)_{g_2}}=(\underline{\mu(g)}_{1,1}\,a_{g_1})\cdot\beta_{g_2}.$$

Fix $f\in \operatorname{Hom}_R(\widetilde{B}, \underline{V}_2\otimes_k \widetilde{B})$. Then by definition we have

$$(g*f)_2={}^g(\mu(g)_{2,2}\otimes_k\widetilde{\nu}_g)\cdot {}^gf\cdot\widetilde{\nu}_{g^{-1}}$$

and

$$\begin{aligned}\psi_2\,(g*f)_2&=(\underline{\mu(g)}_{2,2}\cdot {}^g\widetilde{\nu}_g)({}^g(\overline{f}_2+f'_2)\cdot\widetilde{\nu}_{g^{-1}})\\&=(\underline{\mu(g)}_{2,2}\widetilde{a})({}^g\widetilde{\nu}_g\cdot {}^g\operatorname{id}_{\widetilde{B}}\cdot\widetilde{\nu}_{g^{-1}})+(\underline{\mu(g)}_{2,2}\cdot {}^g\widetilde{\nu}_g)({}^gf'_2\cdot\widetilde{\nu}_{g^{-1}}),\end{aligned}$$

where $\overline{f}=\widetilde{a}\cdot\operatorname{id}_{\widetilde{B}}$, $\widetilde{a}\in k^{d_2}$. The second summand belongs to $\mathcal{N}$, the first is equal to $(\underline{\mu(g)}_{2,2}\widetilde{a})\cdot\operatorname{id}_{\widetilde{B}}$, and hence

(iii) $$\overline{(g*f)_2}=(\underline{\mu(g)}_{2,2}\widetilde{a})\cdot\operatorname{id}_{\widetilde{B}}.$$

Analogously we have

$$(g*f)_1={}^g(\mu(g)_{1,2}\otimes_k(\nu_g^G\cdot\beta^G))\cdot {}^gf\cdot\widetilde{\nu}_{g^{-1}}$$

and

$$((g*f)_1)_{g_2} = (\mu(g)_{1,2} \otimes_k ({}^{gg_1}\nu_h \cdot {}^{gg_1}\beta \cdot {}^{g}\widetilde{\nu}_{g_1^{-1}}) \cdot {}^{g}f \cdot \widetilde{\nu}_{g^{-1}}$$

for any fixed $g_2 \in S_H$, where $gg_1 = g_2 h$, $g_1 \in S_H$, $h \in H$ (see 2.5 and 3.1), and then

$$\begin{aligned}\psi_{1,g_2} \cdot ((g*f)_1)_{g_2} &= (\underline{\mu(g)}_{1,2} \cdot ({}^{gg_1}\nu_h \cdot {}^{gg_1}\beta \cdot {}^{g}\widetilde{\nu}_{g_1^{-1}})) \cdot {}^{g}(\overline{f} + f') \cdot \widetilde{\nu}_{g^{-1}} \\ &= (\underline{\mu(g)}_{1,2}\widetilde{a})({}^{gg_1}\nu_h \cdot {}^{gg_1}\beta \cdot {}^{g}\widetilde{\nu}_{g_1^{-1}} \cdot {}^{g}\,\mathrm{id}_{\widetilde{B}} \cdot \widetilde{\nu}_{g^{-1}}) \\ &\quad + (\underline{\mu(g)}_{1,2} \cdot ({}^{gg_1}\nu_h \cdot {}^{gg_1}\beta \cdot {}^{g}\widetilde{\nu}_{g_1^{-1}})) \cdot {}^{g}f' \cdot \widetilde{\nu}_{g^{-1}}.\end{aligned}$$

Again the second summand belongs to $\mathcal{N}$, the first is equal to $(\underline{\mu(g)}_{1,2}\,\widetilde{a}) \cdot \beta_{g_2}$ since

$${}^{gg_1}\nu_h \cdot {}^{gg_1}\beta \cdot {}^{g}\widetilde{\nu}_{g_1^{-1}} \cdot \widetilde{\nu}_{g^{-1}} = {}^{gg_1h^{-1}}\beta \cdot {}^{gg_1}\widetilde{\nu}_h \cdot \widetilde{\nu}_{(gg_1)^{-1}} = {}^{g_2}\beta \cdot \widetilde{\nu}_{h(gg_1)^{-1}} = \beta_{g_2}.$$

Consequently,

(iv) $$\overline{((g*f)_1)_{g_2}} = (\underline{\mu(g)}_{1,2}\,\widetilde{a}) \cdot \beta_{g_2}.$$

Note that we have the k-linear isomorphisms

(v) $$\overline{\mathcal{H}}_2(\widetilde{\Phi}^B(V)) \simeq \bigoplus_{g_1 \in S_H} (k\beta_{g_1})^{d_1} \oplus (k\,\mathrm{id}_{\widetilde{B}})^{d_2}$$

induced by $f \mapsto (((\overline{f}_1)_{g_1}), \overline{f}_2)$, $f \in \mathrm{Hom}_R(\widetilde{B}, \widetilde{\Phi}^B(V))$ (cf. Remark 5.2(a)), and

(vi) $$V_1^{G/H} \oplus \underline{V}_2 \simeq \bigoplus_{g_1 \in S_H} (k\beta_{g_1})^{d_1} \oplus (k\,\mathrm{id}_{\widetilde{B}})^{d_2}$$

given by $g_1H \otimes v_1 \mapsto a_1 \cdot \beta_{g_1}$, $g_1 \in S_H$, $v_1 \in V_1$, where $a_1 \in k^{d_1}$ is the coordinate vector of v_1 (resp. $v_2 \mapsto a_2 \cdot \mathrm{id}_{\widetilde{B}}$, $v_2 \in \underline{V}_2$, where $a_2 \in k^{d_2}$ is the coordinate vector of v_2). Then by (i)–(iv) the composition of the k-isomorphisms (v) and (vi) yields the kG-module isomorphism

(vii) $$\overline{\mathcal{H}}_2(\widetilde{\Phi}^B(V)) \simeq V_1^{G/H} \oplus \underline{V}_2$$

(see Remark 5.3 for the kG-module structure on $V_1^{G/H} \oplus \underline{V}_2$). It is easily seen that, under the above isomorphism, $\iota(\widetilde{\Phi}^B(V))$ corresponds to $V_1^{G/H}$ (see Remark 5.2(b)). Hence, defining

$$\eta(V) : \widetilde{\Psi}^B\widetilde{\Phi}^B(V) \to \Gamma(V)$$

as the composition of the isomorphism from Remark 5.3 and the isomorphism (vii) we obtain an isomorphism in I_2-spr(kG).

One can show that the family $\eta = (\eta(V))_{V \in I_2\text{-spr}(kG)}$ is natural with respect to V. Consequently, η defines an isomorphism $\widetilde{\Psi}^B\widetilde{\Phi}^B \simeq \Gamma$, and the proof is complete, since $\Psi^B\Phi^B \simeq \widetilde{\Psi}^B\widetilde{\Phi}^B$. ■

Let

$$\bar{\Phi}^B : I_2\text{-spr}(kG)/[I_2\text{-spr}_1(kG)] \to \text{mod}_{\{B,\widetilde{B}\}}(R/G)/[\text{mod}_{\{B\}}(R/G)],$$
$$\overline{\Psi}^B : \text{mod}_{\{B,\widetilde{B}\}}(R/G)/[\text{mod}_{\{B\}}(R/G)] \to I_2\text{-spr}(kG)/[I_2\text{-spr}_1(kG)],$$
$$\bar{\Gamma} : I_2\text{-spr}(kG)/[I_2\text{-spr}_1(kG)] \to I_2\text{-spr}(kG)/[I_2\text{-spr}_1(kG)]$$

be the functors induced by Φ^B, Ψ^B and Γ, respectively (see 5.1–5.3).

COROLLARY. $\overline{\Psi}^B \circ \bar{\Phi}^B \simeq \bar{\Gamma}$.

5.4. From now on we assume that $\text{char}(k)$ does not divide the index $[G : H]$.

LEMMA. *For any V in I_2-spr(kG) there exists a kG-isomorphism $\Gamma(V) \simeq V \oplus V^1$, where V^1 is in I_2-spr$_1(kG)$ ($V^1 = \varepsilon^2_{(1)}(\kappa(V_1)) = (\kappa(V_1) \subseteq \kappa(V_1))$ for $\kappa = \text{id}_{V_1^{G/H}} - \frac{1}{[G:H]} \cdot \Delta_{V_1} \nabla_{V_1}$).*

Proof. Consider the following commutative diagram with exact rows in the category $\text{mod}(kG)^{\text{op}}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1 & \hookrightarrow & V_2 & \xrightarrow{\pi} & V_2/V_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Delta_{V_1}} & & \downarrow{\scriptstyle i'(V)} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & V_1^{G/H} & \xrightarrow{i} & V_1^{G/H} \sqcup_{V_1} V_2 & \xrightarrow{p} & V_2/V_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla_{V_1}} & & \downarrow{\scriptstyle \alpha'(V)} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & V_1 & \xrightarrow{\alpha} & V_2' & \xrightarrow{\pi'} & V_2/V_1 & \longrightarrow & 0
\end{array}$$

Here the middle exact sequence is the standard exact sequence induced by Δ_{V_1} from the upper one, the lower one is the standard exact sequence induced by ∇_{V_1} from the middle one. The composition $\nabla_{V_1} \cdot \Delta_{V_1} = [G : H] \cdot \text{id}_{V_1}$ is an isomorphism, hence by the Five Lemma so is $\alpha'(V) \cdot i'(V)$. Now the assertion follows easily. ∎

We denote by I_2-spr$'_1(kG)$ the additive closure of the subcategory formed by all indecomposables in I_2-spr(kG) off I_2-spr$_1(kG)$ (I_2-spr$'_1(kG) = I_2$-spr$(kG) \setminus I_2$-spr$_1(kG)$).

COROLLARY. (a) $\overline{\Psi}^B \circ \bar{\Phi}^B \simeq \text{id}_{I_2\text{-spr}(kG)/[I_2\text{-spr}_1(kG)]}$.

(b) *The functor Φ^B yields an injection between the set of isoclasses in I_2-spr$'_1(kG)$ and the set of all isoclasses in* $\text{mod}_{\{B,\widetilde{B}\}}(R/G)$.

Proof. (a) By the above lemma, i' (see Remark 5.3(b)) induces an isomorphism $\text{id}_{I_2\text{-spr}(kG)/[I_2\text{-spr}_1(kG)]} \simeq \bar{\Gamma}$ and (a) follows directly from Corollary 5.3.

(b) If $\Phi^B(V) \simeq \Phi^B(V')$ for V, V' in I_2-spr$'(kG)$, then by Proposition 5.3 and Lemma 5.4 we have $V \oplus V^1 \simeq V' \oplus V'^1$, where V^1, V'^1 are in I_2-spr$_1(kG)$.

Consequently, by the uniqueness of decomposition into a direct sum of indecomposables we have $V \simeq V'$. ■

5.5. Finally we analyse decompositions of modules in $\operatorname{Im} \Phi^B_{|I_2\text{-spr}'_1(kG)^{\mathrm{op}}}$.

LEMMA. *Let $V = (V_1 \subseteq V_2)$ be an indecomposable object in I_2-$\mathrm{spr}'_1(kG)$. Then there exists an indecomposable direct summand X of $\Phi^B(V)$ with the following properties*:

(a) $\mathrm{dsc}(X)_{\widetilde{B}} = \mathrm{dsc}(\Phi^B(V))_{\widetilde{B}}$ $(= \dim_k(V_2/V_1))$,
(b) $\Psi^B(X) \simeq V \oplus \check{V}$ *for some* $\check{V}$ *in* I_2-$\mathrm{spr}_1(kG)$.

In particular, $X = X_V$ *as above is uniquely determined by* V *up to isomorphism, and* $\Phi^B(V) \simeq X \oplus Y$ *for some* Y *in* $\mathrm{mod}_{\{B\}}(R/G)$.

Proof. Since V is in I_2-$\mathrm{spr}'_1(kG)$, there exists an indecomposable direct summand X of $\Phi^B(V)$ such that $\widetilde{B} \in \mathrm{dss}(X)$. We show that X satisfies (a) and (b). By Proposition 5.3 and Lemma 5.4, $\Psi^B(X)$ is a direct summand of $V \oplus V^1$, where V^1 is in I_2-$\mathrm{spr}_1(kG)$. Moreover, $\Psi^B(X)$ does not belong to I_2-$\mathrm{spr}_1(kG)$ since $\widetilde{B} \in \mathrm{dss}(X)$ (see Remark 5.2(c)). This immediately implies (b). Consequently, by Remark 5.2(c), we have $\mathrm{dsc}(X)_{\widetilde{B}} = \dim_k(V_2/V_1)$, and (a) holds since $\mathrm{dsc}(\Phi^B(V))_{\widetilde{B}} = \dim_k(V_2/V_1)$ (see 5.1). The last assertion follows immediately from (a). ■

THEOREM. *Let $G \subseteq \mathrm{Aut}_k(R)$ be a group of k-linear automorphisms acting freely on R. Suppose that B is a G-atom which admits an R-action ν of G_B, and satisfies the following conditions*:

(a) $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$,
(b) $\widetilde{B} \not\simeq B$,
(c) $\widetilde{\nu}$ *can be extended to an R-action of* $G_{\widetilde{B}}$ *on* $\widetilde{B}$,
(d) $\mathrm{char}(k)$ *does not divide* $[G_{\widetilde{B}} : G_B]$,
(e) $G_{\widetilde{B}} = G$.

Then the functor $\bar{\Phi}^B$ *is a representation embedding and the mapping* $V \mapsto X_V$ (see Lemma 5.5) *yields an injection between the set of isoclasses of indecomposables in* I_2-$\mathrm{spr}'_1(kG)$ (*resp.* I_2-$\mathrm{spr}'(kG)$) *and the set of isoclasses of indecomposables in* $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G) \setminus \mathrm{mod}_{\{B\}}(R/G)$ (*resp. indecomposable non-orbicular modules in* $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)$) (cf. Lemma 3.7).

Proof. The first assertion follows from Corollary 5.4 since Lemma 5.5 shows that $\Phi^B(V)$ $(\simeq X_V)$ is an indecomposable object in the category $\mathrm{mod}_{\{B,\widetilde{B}\}}(R/G)/\,[\mathrm{mod}_{\{B\}}(R/G)]$ for any V in I_2-$\mathrm{spr}'_1(kG)$.

If now $X_V \simeq X_{V'}$ for indecomposable V, V' in I_2-$\mathrm{spr}'_1(kG)$, then by Lemma 5.5(b) we have $V \oplus \check{V} \simeq V' \oplus \check{V}'$, where $\check{V}, \check{V}'$ are in I_2-$\mathrm{spr}_1(kG)$. Consequently, $V \simeq V'$. Finally, note that if an indecomposable object V belongs to I_2-$\mathrm{spr}'_1(kG)$ then by Lemma 5.5(a) the indecomposable module

X_V does not belong to $\mathrm{mod}_{\{B\}}(R/G)$, and if V is in I_2-spr$'(kG)$ then X_V is non-orbicular by Remark 5.2(c) and Lemma 5.5(b). ■

5.6. We end this section by showing that as far as constructing indecomposable non-orbicular R/G-modules is concerned, one should expect different behaviour of Φ^B in the case char(k) is positive and divides $[G:H]$.

From now on we assume that $G = G_{\widetilde{B}}$ is an infinite cyclic group with a fixed generator g.

Let $V = (V_1 \subseteq V_2)$ be an indecomposable object in I_2-spr$'(kG)$ which is given by $V_2 = k^2$, $V_1 = k\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ where the kG-module structure on V_2 is defined by the action $\mu(g) = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]\cdot$ of the generator g on k^2. Clearly, we can take $\underline{V}_2 = k\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

Lemma. *If* char$(k) = 2 = [G:H]$ *then*

$$\widetilde{\Phi}^B(V) \simeq B^G \oplus \widetilde{B} \quad \textit{in} \ \mathrm{Mod}^G_{\mathrm{f},\{B,\widetilde{B}\}}(R/G)$$

(cf. Example 4.3(ii)).

Proof. We show that the exact sequence

$$0 \to V_1 \otimes_k B^G \xrightarrow{w} V \otimes_k B \xrightarrow{p} \underline{V}_2 \otimes_k \widetilde{B} \to 0$$

splits in $\mathrm{Mod}^G_{\mathrm{f}} R$. For simplicity we set $B_e = B$ and $B_g = {}^gB$ ($S_G = \{e,g\}$). Then under the standard identifications $V_1 \otimes_k B^G \simeq B_e \oplus B_g$, $\underline{V}_2 \otimes_k \widetilde{B} \simeq \widetilde{B}$ and $V \otimes_k B \simeq (B_e \oplus B_g) \oplus \widetilde{B}$, the R-actions of the generator g on these R-modules are given respectively by the R-homomorphisms

$$\nu_g^G = \begin{bmatrix} 0 & {}^g\nu_{g^2} \\ \mathrm{id}_{B_e} & 0 \end{bmatrix}, \qquad \widetilde{\nu}_g, \qquad \begin{bmatrix} \nu_g^G & (\nu_{1,2})_g \\ 0 & \widetilde{\nu}_g \end{bmatrix},$$

where

$$(\nu_{1,2})_g = \begin{bmatrix} {}^g\nu_{g^2} \cdot {}^g\beta \cdot \widetilde{\nu}_{g^{-1}} \\ \beta \end{bmatrix} : \widetilde{B} \to {}^{g^{-1}}B_e \oplus {}^{g^{-1}}B_g.$$

To prove our claim we show that the R-homomorphism

$$\begin{bmatrix} s \\ \mathrm{id}_{\widetilde{B}} \end{bmatrix} : \widetilde{B} \to (B_e \oplus B_g) \oplus \widetilde{B}, \quad \text{where} \quad s = \begin{bmatrix} \beta \\ 0 \end{bmatrix},$$

splits p in $\mathrm{Mod}^G_{\mathrm{f}} R$. It suffices to check that $\left[\begin{smallmatrix} s \\ \mathrm{id}_{\widetilde{B}} \end{smallmatrix}\right]$ is a morphism in $\mathrm{Mod}^G R$, or equivalently to verify the formula

$$\text{(i)} \qquad (\nu_{1,2})_g = {}^{g^{-1}}s \cdot \widetilde{\nu}_g - \nu_g^G \cdot s$$

($G = \langle g \rangle$!). It is easily seen that

$${}^{g^{-1}}s \cdot \widetilde{\nu}_g - \nu_g^G \cdot s = \begin{bmatrix} {}^{g^{-1}}\beta \cdot \widetilde{\nu}_g \\ -\beta \end{bmatrix}.$$

We also have ${}^{g}\nu_{g^2} \cdot {}^{g}\beta \cdot \widetilde{\nu}_{g^{-1}} = {}^{g}({}^{g^{-2}}\beta \cdot \widetilde{\nu}_{g^2}) \cdot \widetilde{\nu}_{g^{-1}} = {}^{g^{-1}}\beta \cdot \widetilde{\nu}_g$. Now (i) follows from the assumption $\operatorname{char}(k) = 2$. ■

6. Extension embeddings for matrix rings. In this section we develop the extension embedding technique (see Theorem 6.3), used later in the proof of Theorem 7.1.

6.1. Let A_0, A' be k-algebras, ${}_{A_0}M_{A'}$ be an A_0-A'-bimodule and ${}_{A'}N_{A_0}$ be an A'-A_0-bimodule. Assume that the field k acts centrally on both bimodules M and N. Suppose we are given two bimodule homomorphisms $\gamma_0 : {}_{A_0}M \otimes_{A'} N_{A_0} \to {}_{A_0}A_{0\,A_0}$ and $\gamma' : {}_{A'}N \otimes_{A_0} M_{A'} \to {}_{A'}A'_{A'}$ such that $\gamma_0(m \otimes n) \cdot m_1 = m \cdot \gamma'(n \otimes m_1)$ and $n_1 \cdot \gamma_0(m \otimes n) = \gamma'(n_1 \otimes m) \cdot n$ for all $m, m_1 \in M$, $n, n_1 \in N$. These data define a k-algebra structure on the k-vector space

$$A = \begin{pmatrix} A_0 & M \\ N & A' \end{pmatrix}.$$

The space A equipped with this structure is called a *matrix algebra.*

A right module over the matrix algebra A can be viewed as a quadruple $X = (X_0, X', \varphi, \psi)$, which consists of X_0 in $\operatorname{MOD} A_0$, X' in $\operatorname{MOD} A'$, an A'-homomorphism $\varphi : X_0 \otimes {}_{A_0}M_{A'} \to X'_{A'}$, and an A_0-homomorphism $\psi : X' \otimes_{A'} N_{A_0} \to X_{0\,A_0}$, satisfying the equalities $\psi(\varphi(x_0 \otimes m) \otimes n) = x_0 \cdot \gamma_0(m \otimes n)$ and $\varphi(\psi(x' \otimes n) \otimes m) = x' \cdot \gamma'(n \otimes m)$ for all $x_0 \in X_0$, $x' \in X'$, $m \in M$, $n \in N$. Under the above interpretation of A-modules, an A-homomorphism from $X = (X_0, X', \varphi_X, \psi_X)$ to $Y = (Y_0, Y', \varphi_Y, \psi_Y)$ is a pair $c = (c_0, c')$ where $c_0 : X_0 \to Y_0$ is an A_0-homomorphism and $c' : X' \to Y'$ is an A'-homomorphism such that $\varphi_Y \circ (c_0 \otimes \mathrm{id}_M) = c' \circ \varphi_X$ and $\psi_Y \circ (c' \otimes \mathrm{id}_N) = c_0 \circ \psi_X$.

Denote by

$$\overline{A} = \begin{pmatrix} A_0 & M \\ 0 & A' \end{pmatrix}$$

the upper triangular matrix algebra associated with A. Then $\overline{A}$-modules can be regarded as triples $X = (X_0, X', \varphi)$, where X_0 is in $\operatorname{MOD} A_0$, X' is in $\operatorname{MOD} A'$ and $\varphi : X_0 \otimes {}_{A_0}M_{A'} \to X'_{A'}$ is an A'-homomorphism. Morphisms from X to Y are pairs $c = (c_0, c')$ of homomorphisms (as above) satisfying the equality $\varphi_Y \circ (c_0 \otimes \mathrm{id}_M) = c' \circ \varphi_X$.

Observe that the mapping $X = (X_0, X', \varphi, \psi) \mapsto \overline{X} = (X_0, X', \varphi)$ defines a faithful k-linear functor $\zeta : \operatorname{MOD} A \to \operatorname{MOD} \overline{A}$.

Denote by $\operatorname{MOD}^0 A$ (resp. $\operatorname{mod}^0 A$) the full subcategory of $\operatorname{MOD} A$ (resp. $\operatorname{mod} A$) formed by all $X = (X_0, X', \varphi_X, \psi_X)$ such that $\psi = 0$, and by $\operatorname{MOD}^A \overline{A}$ (resp. $\operatorname{mod}^A \overline{A}$) the full subcategory of $\operatorname{MOD} \overline{A}$ (resp. $\operatorname{mod} \overline{A}$) formed by all $Y = (Y_0, Y', \varphi_Y)$ such that $\operatorname{Im} \gamma_0 \subseteq \operatorname{ann}(Y_{0\,A_0})$ and $\operatorname{Im} \gamma' \subseteq \operatorname{ann}(Y'_{A'})$.

Lemma. *The functor ζ yields an equivalence*

$$\mathrm{MOD}^0 A \simeq \mathrm{MOD}^A \overline{A} \quad (\textit{resp.}\ \mathrm{mod}^0 A \simeq \mathrm{mod}^A \overline{A}).$$

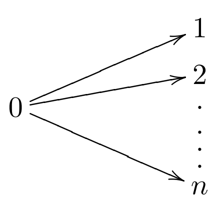
Proof. Note that an $\overline{A}$-module $Y = (Y_0, Y', \varphi_Y)$ belongs to $\mathrm{MOD}^A \overline{A}$ if and only if $Y = (Y_0, Y', \varphi_Y, 0)$ defines an A-module. ■

6.2. Denote by $e_\lambda : \mathrm{MOD}\, A_0 \to \mathrm{MOD}\, A$ the left adjoint functor to the restriction functor $e_\bullet : \mathrm{MOD}\, A \to \mathrm{MOD}\, A_0$, where $e_\bullet(X) = X_0$ for $X = (X_0, X', \varphi, \psi)$ in $\mathrm{MOD}\, A$. Recall that for Z in $\mathrm{MOD}\, A_0$, $e_\lambda(Z) = (Z, Z \otimes_{A_0} M, \mathrm{id}_{Z \otimes_{A_0} M}, \mathrm{id}_Z \cdot \gamma_0)$, where $(\mathrm{id}_Z \cdot \gamma_0)(z \otimes m \otimes n) = z \cdot \gamma_0(m \otimes n)$ for all $z \in Z$, $m \in M$ and $n \in N$. It is clear that if $\dim_k A$ is finite then $e_\lambda(\mathrm{mod}\, A_0) \subset \mathrm{mod}\, A$.

Remark. We say that Z in $\mathrm{MOD}\, A_0$ is a *module over* A provided $(Z, 0, 0, 0)$ is in $\mathrm{MOD}\, A$, or equivalently $\mathrm{Im}\, \gamma_0 \subseteq \mathrm{ann}(Z_{A_0})$ (see Lemma 6.1). If Z is as above then the A-module $e_\lambda(Z)$ belongs to $\mathrm{MOD}^0 A$.

Suppose that $A' = A_1 \times \ldots \times A_r$, $r \in \mathbb{N}$, is a product of rings. Consequently, the A_0-A'-bimodule M decomposes into a direct sum of bimodules $M = \bigoplus_{i=1}^r M_i$, where each M_i is an A_0-A'-bimodule. Then an $\overline{A}$-module X is given by a tuple $(X_0, (X_i)_{i=1,\ldots,r}, (\varphi_i)_{i=1,\ldots,r})$, where each X_i is in $\mathrm{MOD}\, A_i$ and each $\varphi_i : X_0 \otimes_{A_0} M_i \to X_i$ is an A_i-homomorphism. Accordingly, an $\overline{A}$-homomorphism from X to $X' = (X'_0, (X'_i)_{i=1,\ldots,r}, (\varphi'_i)_{i=1,\ldots,r})$ is a family $c = (c_0, (c_i)_{i=1,\ldots,r})$ of A_i-homomorphisms $c_i : X_i \to X'_i$ such that $\varphi \circ (c_0 \otimes \mathrm{id}_{M_i}) = c_i \circ \varphi_i$ for every $i = 1, \ldots, r$. From now on we assume that $\dim_k A$ is finite and that $A' = A_1 \times \ldots \times A_r$. We fix a module Z in $\mathrm{mod}\, A_0$ which is also an A-module, i.e. $\mathrm{Im}\, \gamma_0 \subseteq \mathrm{ann}(Z_{A_0})$. Then the A-module $\widetilde{Z} = e_\lambda(Z)$ regarded as an object of $\mathrm{mod}^A \overline{A}$ is defined by the collection $(Z, (Z \otimes_{A_0} M_i)_{i=1,\ldots,r}, (\mathrm{id}_{Z \otimes_{A_0} M_i})_{i=1,\ldots,r})$ (see Lemma 6.1 and Remark 6.2).

Let $\Sigma_r = kQ_r^{\mathrm{op}}$ be the path k-category of the quiver Q_r^{op} opposite to the following one:



and by $\mathrm{mod}^e \Sigma_r$ the full cofinite subcategory of $\mathrm{mod}\, \Sigma_r$ formed by all representations $V = (f_i : V_0 \to V_i)_{i=1,\ldots,r}$ of Q_r such that all f_i's are surjective. We define a functor

$$\mathcal{E} : \mathrm{mod}^e \Sigma_r \to \mathrm{mod}\, \overline{A}$$

as follows. Given an object $V = (f_i : V_0 \to V_i)_{i=1,\ldots,r}$ in $\mathrm{mod}^e \Sigma_r$ we set

$$\mathcal{E}(V) = (V_0 \otimes_{A_0} Z, (V_i \otimes_k Z \otimes_{A_0} M_i)_{i=1,\ldots,r}, (f_i \otimes \mathrm{id}_{Z \otimes_{A_0} M_i})_{i=1,\ldots,r}),$$

where $f_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i} : V_0 \otimes_k Z \otimes {}_{A_0}M_i \to V_i \otimes_k Z \otimes {}_{A_0}M_i$. If $\alpha : V \to V'$ is a morphism in $\mathrm{mod}^e \Sigma_r$ given by the family $(\alpha_i : V_i \to V_i')_{i=0,1,\ldots,r}$ of k-linear maps, where $V = (f_i : V_0 \to V_i)_{i=1,\ldots,r}$, $V' = (f_i' : V_0' \to V_i')_{i=1,\ldots,r}$, we set

$$\mathcal{E}(\alpha) = (\alpha_0 \otimes \mathrm{id}_Z, (\alpha_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i})_{i=1,\ldots,r}).$$

LEMMA. *The mapping $\mathcal{E}$ as above defines a k-linear functor $\mathcal{E} : \mathrm{mod}^e \Sigma_r \to \mathrm{mod}^A \overline{A}$.*

Proof. It is clear that $\mathcal{E}$ defines a k-linear functor $\mathcal{E} : \mathrm{mod}^e \Sigma_r \to \mathrm{mod}\, \overline{A}$. To prove that $\mathrm{Im}\, \mathcal{E} \subset \mathrm{mod}^A \overline{A}$ observe that $\mathrm{ann}(V_0 \otimes_k Z_{A_0}) = \mathrm{ann}(Z_{A_0})$ and $\mathrm{ann}((\bigoplus_{i=1}^r V_i \otimes_k \otimes Z \otimes {}_{A_0}M_i)_{A'}) = \prod_{i=1}^r \mathrm{ann}(Z \otimes {}_{A_0}M_i) = \mathrm{ann}(Z \otimes {}_{A_0}M_{A'})$ for V in $\mathrm{mod}^e \Sigma_r$. Consequently, $\mathcal{E}(V)$ belongs to $\mathrm{mod}^A \overline{A}$ since $\widetilde{Z}$ does. ■

6.3. Our main result of this section is the following.

THEOREM. *Let Z be an indecomposable A_0-module such that $\mathrm{Im}\, \gamma_0 \subseteq \mathrm{ann}(Z_{A_0})$. Assume that $\mathrm{End}_{A_0}(Z)/J(\mathrm{End}_{A_0}(Z)) \simeq k$ and that all modules $Z \otimes {}_{A_0}M_i$, $i = 1, \ldots, r$, are non-zero. Then the functor $\mathcal{E} : \mathrm{mod}^e \Sigma_r \to \mathrm{mod}^A \overline{A}$ is a faithful embedding (in the sense of [27]). In particular the algebras $\overline{A}$ and A are wild provided $r \geq 5$.*

To prove the above theorem we show that the restriction $\mathcal{E}_0$ of $\mathcal{E}$ to the dense full subcategory $\mathrm{mod}_0^e \Sigma_r$ consisting of all matrix representations of Q_r is a representation embedding. Recall that by a matrix representation of Q_r we mean a Σ_r-module (= representation of Q_r) of the form $V = (f_i : k^{n_0} \to k^{n_i})_{i=1,\ldots,r}$ ($f_i = F_i\cdot$, where $F_i \in \mathrm{M}_{n_i \times n_0}(k)$, for every i). For this purpose we construct a left inverse functor $\pi : \mathbb{E} \to \mathrm{mod}_0^e \Sigma_r$ for $\mathcal{E}_0$, where $\mathbb{E}$ is the full subcategory of $\mathrm{mod}\, \Sigma_r$ formed by all $\mathcal{E}_0(V)$, V in $\mathrm{mod}_0^e \Sigma_r$ (see Proposition 6.6).

6.4. LEMMA. *Let $V = (f_i : V_0 \to V_i)_{i=1,\ldots,r}$, $V' = (f_i' : V_0' \to V_i')_{i=1,\ldots,r}$ be objects in $\mathrm{mod}^e \Sigma_r$, and $c = (c_0, (c_i)_{i=1,\ldots,r}) : \mathcal{E}(V) \to \mathcal{E}(V')$ be a Σ_r-homomorphism. Then each $c_i : V_i \otimes_k Z \otimes {}_{A_0}M_i \to V_i' \otimes_k Z \otimes {}_{A_0}M_i$ has the form $c_i' \otimes \mathrm{id}_{M_i}$ where $c_i' \in \mathrm{Hom}_{A_0}(V_i \otimes_k Z, V_i' \otimes_k Z)$, $i = 1, \ldots, r$.*

Proof. Fix $i \in \{1, \ldots, r\}$. The k-epimorphism $f_i : V_0 \to V_i$ admits a section. Fix a k-linear map $s_i : V_i \to V_0$ such that $f_i \circ s_i = \mathrm{id}_{V_i}$, consequently $(f_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}) \circ (s_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}) = \mathrm{id}_{V_i \otimes_k Z \otimes_{A_0} M_i}$. Then multiplying the equality

$$(f_i' \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}) \circ (c_0 \otimes \mathrm{id}_{M_i}) = c_i \circ (f_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i})$$

by $s_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}$ on the right, we obtain $c_i = c_i' \otimes \mathrm{id}_{M_i}$ where $c_i' = (f_i' \otimes \mathrm{id}_Z) \circ c_0 \circ (s_i \otimes \mathrm{id}_Z)$. ■

For any $m, n \in \mathbb{N}$ and a module X over an algebra Λ we have at our disposal the standard isomorphisms

(i) $$k^m \otimes_k X_\Lambda \simeq (X_\Lambda)^m$$

and

(ii) $$\mathrm{Hom}_\Lambda(X^n, X^m) \simeq \mathrm{M}_{m\times n}(\mathrm{End}_\Lambda(X)).$$

We set $E_0 = \mathrm{End}_{A_0}(Z)$ for simplicity. For any $i = 1, \ldots, r$, we denote by E_i the image $\mathrm{Im}\, p_i$, where $p_i : E_0 \to \mathrm{End}_{A_i}(Z \otimes {}_{A_0}M_i)$ is the k-algebra homomorphism given by $h \mapsto h \otimes \mathrm{id}_{M_i}$ for $h \in E_0$.

Applying now the identifications (i) and (ii), we can rephrase the lemma as follows.

Corollary. *Let* $c = (c_0, (c_i)_{i=1,\ldots,r}) : \mathcal{E}_0(V) \to \mathcal{E}(V')$ *be a morphism in* $\mathbb{E}$, *where* $V = (f_i = A_i\cdot : V_0 \to V_i)_{i=1,\ldots,r}$ *and* $V' = (f_i' = A_i'\cdot : V_0' \to V_i')_{i=1,\ldots,r}$ *are in* $\mathrm{mod}_0^e \Sigma_r$. *Then each* $c_i \in \mathrm{Hom}_{A_i}(V_i \otimes_k Z \otimes {}_{A_0}M_i, V_i' \otimes_k Z \otimes {}_{A_0}M_i)$ *belongs to* $\mathrm{M}_{n_i'\times n_i}(E_i)$ *for* $i = 1, \ldots, r$.

6.5. From now on we assume that all modules $Z \otimes {}_{A_0}M_i$, $i = 1, \ldots, r$, are non-zero and the A_0-module Z is indecomposable with $E_0/J_0 \simeq k$, where $J_0 = J(E_0)$. Observe that then each E_i is a local k-algebra with Jacobson radical $J_i = p_i(J_0)$, and $E_i/J_i \simeq k$.

Let E be a local k-algebra such that $E/J \simeq k$, where $J = J(E)$. For any $m, n \in \mathbb{N}$ and $c \in \mathrm{M}_{m\times n}(E)$, we denote by $\overline{c}$ and c' the matrices $\overline{c} \in \mathrm{M}_{m\times n}(k)$ and $c' \in \mathrm{M}_{m\times n}(J)$ corresponding to c under the canonical identification

(i) $$\mathrm{M}_{m\times n}(E) = \mathrm{M}_{m\times n}(k)\cdot 1_E \oplus \mathrm{M}_{m\times n}(J)$$

induced by the equality $E = k\cdot 1_E \oplus J$. It is easily seen that

(ii) $$\overline{cd} = \overline{c}\,\overline{d}$$

for all $c \in \mathrm{M}_{m\times n}(E)$, $d \in \mathrm{M}_{n\times p}(E)$, $m, n, p \in \mathbb{N}$.

Let $V = (f_i : k^{n_0} \to k^{n_i})_{i=1,\ldots,r}$ and $V' = (f_i' : k^{n_0'} \to k^{n_i'})_{i=1,\ldots,r}$ be objects in $\mathrm{mod}_0^e \Sigma_r$, where $f_i = F_i\cdot$, $f_i' = F_i'\cdot$ for some $F_i \in \mathrm{M}_{n_i\times n_0}(k)$, $F_i' \in \mathrm{M}_{n_i'\times n_0'}(k)$, and let $c = (c_0, (c_i)_{i=1,\ldots,r}) : \mathcal{E}_0(V) \to \mathcal{E}_0(V')$ be a morphism in $\mathbb{E}$. We denote by $\overline{c}$ the collection $\overline{c} = (\overline{c}_i\cdot : k^{n_i} \to k^{n_i'})_{i=0,\ldots,r}$ of k-linear maps, where each c_i is now regarded as an element of $\mathrm{M}_{n_i'\times n_i}(E_i)$ (cf. 6.4(i), 6.4(ii) and Corollary 6.4).

Lemma. (a) *The collection* $\overline{c}$ *is a morphism from* V *to* V' *in* $\mathrm{mod}_0^e \Sigma_r$.
(b) $V = V'$ *provided* $\mathcal{E}_0(V) = \mathcal{E}_0(V')$.

Proof. (a) Fix $i \in \{1, \ldots, r\}$. To show that $F_i'\overline{c}_0 = \overline{c}_i F_i$ we treat the map $f_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}$ (resp. $f_i' \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}$) as an element of $\mathrm{M}_{n_i\times n_0}(E_i)$ (resp. $\mathrm{M}_{n_i'\times n_0'}(E_i)$), $c_0 \otimes \mathrm{id}_{M_i}$ as an element of $\mathrm{M}_{n_0'\times n_0}(E_i)$ (cf. 6.4(i), 6.4(ii)), and c_i as an element of $\mathrm{M}_{n_i'\times n_i}(E_i)$ (cf. Corollary 6.4). Note that we have $\overline{f_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}} = F_i$, $\overline{f_i' \otimes \mathrm{id}_{Z\otimes_{A_0}M_i}} = F_i'$ (in fact, $f_i \otimes \mathrm{id}_{Z\otimes_{A_0}M_i} = F_i \cdot \mathrm{id}_{Z\otimes_{A_0}M_i}$, $f_i' \otimes \mathrm{id}_{Z\otimes_{A_0}M_i} = F_i' \cdot \mathrm{id}_{Z\otimes_{A_0}M_i}$) and $\overline{c_0 \otimes \mathrm{id}_{M_i}} = \overline{c}_0$ (see definition of p_i and 6.5(i)). Now the equality

$$c_i \circ (f_i \otimes \mathrm{id}_{Z\otimes_{A_0} M_i}) = (f_i' \otimes \mathrm{id}_{Z\otimes_{A_0} M_i}) \circ (c_0 \otimes \mathrm{id}_{M_i})$$

immediately implies the required assertion by 6.5(ii).

(b) Suppose that $\mathcal{E}_0(V) = \mathcal{E}_0(V')$. Clearly, $n_i = n_i'$ for every $i = 0, \ldots, r$. Since $f_i \otimes \mathrm{id}_{Z\otimes_{A_0} M_i} = F_i \cdot \mathrm{id}_{Z\otimes_{A_0} M_i}$ and $f_i' \otimes \mathrm{id}_{Z\otimes_{A_0} M_i} = F_i' \cdot \mathrm{id}_{Z\otimes_{A_0} M_i}$, the equality $f_i \otimes \mathrm{id}_{Z\otimes_{A_0} M_i} = f_i' \otimes \mathrm{id}_{Z\otimes_{A_0} M_i}$ implies $F_i = F_i'$ for all $i = 1, \ldots, r$. ■

6.6. We now define a functor $\pi : \mathbb{E} \to \mathrm{mod}_0^e \Sigma_r$ (cf. 6.3). For any object $X = \mathcal{E}_0(V)$ in $\mathbb{E}$ we set

$$\pi(X) = V.$$

For any morphism $c = (c_0, (c_i)_{i=1,\ldots,r}) : X \to X'$ in $\mathbb{E}$, where $X = \mathcal{E}_0(V)$, $X' = \mathcal{E}_0(V')$ and V, V' are in $\mathrm{mod}_0^e \Sigma_r$, we set

$$\pi(c) = \overline{c}.$$

The following fact immediately implies Theorem 6.3.

PROPOSITION. *The mapping π as above defines a k-linear functor $\pi : \mathbb{E} \to \mathrm{mod}_0^e \Sigma_r$ which has the following properties*:

(a) $\pi \mathcal{E}_0 = \mathrm{id}_{\mathrm{mod}_0^e \Sigma_r}$,

(b) $\operatorname{Ker} \pi$ *contains no non-zero idempotent.*

Proof. By Lemma 6.5 the mapping π yields well defined functions from $\mathrm{ob}\, \mathbb{E}$ to $\mathrm{ob}\, \mathrm{mod}_0^e \Sigma_r$ and from $\mathrm{Hom}_A(X, X')$ to $\mathrm{Hom}_{\Sigma_r}(\pi(X), \pi(X'))$ for any X, X' in $\mathbb{E}$. The functoriality of π follows immediately from 6.5(ii). The property (a) is satisfied by construction. To show (b), it suffices to observe that $\operatorname{Ker} \pi$ is a nilpotent ideal, since each ideal J_i, $i = 0, \ldots, r$, is nilpotent with nilpotency degree bounded by $\dim_k E_0$. ■

7. Embedding induced by the left Kan extension of an infinite G-atom

7.1. The main aim of this section is to prove the following result.

THEOREM. *Let R be a tame locally bounded k-category and G be a group of k-linear automorphisms acting freely on R. Then for any infinite G-atom B with* $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$ *the counit map $\beta(B)$ yields an R-isomorphism* $\widetilde{B} \simeq B$ (see 4.1 and 1.5 for definition of $\widetilde{B}$ and $\beta(B)$).

7.2. We recall from [7] and [8] a notion which is essential for our study of the indecomposable objects of the category $\mathrm{Mod}\, R$.

DEFINITION. Let M be in $\mathrm{Ind}\, R$ and C a full subcategory of R. The full subcategory U of R containing C is called an *M-neighbourhood* of C provided there exists an indecomposable U-module M^U satisfying the following two conditions:

(N1) M^U is isomorphic to a direct summand of $M_{|U}$,
(N2) ${M^U}_{|C} = M_{|C}$.

An M-neighbourhood U of C is called *finite* (resp. *connected*) if the category U is finite (resp. connected). An M-neighbourhood U of C is called *sincere* provided there exists M^U as above which is sincere. A subcategory U is said to be an M-neighbourhood provided U is an M-neighbourhood of some subcategory C which intersects $\operatorname{supp} M$ non-trivially.

REMARK. (a) If U is an M-neighbourhood then $M_{|U} \neq 0$. If $M_{|C} = 0$ then any subcategory U containing C such that $M_{|U} \neq 0$ is an M-neighbourhood of C.

(b) If C is contained in $\operatorname{supp} M$ and U is an M-neighbourhood of C, then $U \cap \operatorname{supp} M$ (resp. $\operatorname{supp} M^U$) is an M-neighbourhood (resp. a sincere M-neighbourhood, hence connected) of C, contained in $\operatorname{supp} M$.

(c) If U is an M-neighbourhood and V is a full subcategory of R containing U then V is also an M-neighbourhood (by the uniqueness of decomposition into a direct sum of indecomposables in $\operatorname{Mod} S$ for any S). Moreover, U is then an M^V-neighbourhood, where M^V is as in the definition.

The following fact is crucial for the remaining part of the paper.

THEOREM. *Let R be a connected locally bounded k-category and M be an R-module in* $\operatorname{Ind} R$. *Then any finite full subcategory of R* (*resp. which in addition is contained in* $\operatorname{supp} M$) *admits a finite*, *connected M-neighbourhood* (*resp. which in addition is sincere*).

7.3. We present the proof of the above result (see [12] for k algebraically closed). The basic role is played by the following fact.

PROPOSITION (cf. [12, Proposition 4.2]). *Let $\{C_n\}_{n\in\mathbb{N}}$ be a family of finite full subcategories of R such that $\bigcup_{n\in\mathbb{N}} C_n = R$ and $C_n \subset C_{n+1}$ for every $n \in \mathbb{N}$. Then for any M, N in* $\operatorname{Mod} R$, *N is isomorphic to a direct summand of M if and only if $N_{|C_n}$ is isomorphic to a direct summand of $M_{|C_n}$ for all $n \in \mathbb{N}$.*

Proof. We can repeat all the arguments from the proof of [12, Proposition 4.2] which use only the fact that $\dim_k \operatorname{Hom}_{C_n}(V_{|C_n}, W_{C_n})$ is finite for V, W in $\operatorname{Mod} R$ and does not use the general assumption of [12] (that k is algebraically closed). The only part of that proof which need to be proved in the more general setting is the lemma below (cf. also [7, Proof of Proposition 2.6]); in fact the full proof of the original version of the lemma for k algebraically closed was not presented in [12] and differed from the one given here).

LEMMA. *Let $\varrho : A \to A'$ be a surjective homomorphism of artinian rings*, *and e and f two* (*orthogonal*) *idempotents of A. Suppose that there* *exist elements $x \in fAe$ and $y \in eAf$ such that $yx = e$. Then for all $a' \in \varrho(f)A'\varrho(e)$ and $b' \in \varrho(e)A'\varrho(f)$ such that $b'a' = \varrho(e)$ there exist elements $a \in fAe$ and $b \in eAf$ such that $\varrho(a) = a'$, $\varrho(b) = b'$ and $ba = e$.*

Proof. Fix a', b' as above. We start by observing that if the element $z = e - b_1a_1$ ($\in \operatorname{Ker}\varrho$) is nilpotent for $a_1 \in fAe \cap \varrho^{-1}(a')$ and $b_1 \in eAf \cap \varrho^{-1}(b')$ (it is the case for all a_1, b_1 as above provided $\operatorname{Ker}\varrho \subseteq J(A)$), then a, b satisfying the assertion exist. Indeed, setting $a = \sum_{i=0}^{\infty} a_1 z^i$ and $b = b_1$ ($z^0 = e$), we have $\varrho(a) = a'$, $\varrho(b) = b'$ and $e - ba = e - b_1a_1 \sum_{i=0}^{\infty} z^i = e - (e - z)\sum_{i=0}^{\infty} z^i = 0$.

Next we show the existence of a and b under the extra assumption that A is a semisimple ring. In this case $A'' = \operatorname{Ker}\varrho$ is a direct factor of A (as a ring), therefore we may assume that $A = A' \times A''$ and that ϱ is the canonical projection on the first component. It is easy to check that now the elements $a = (a', x'')$ and $b = (b', y'')$, where $x = (x', x'')$ and $y = (y', y'')$, satisfy the required condition.

Consider the general case. Since $\varrho(J) \subseteq J'$, ϱ induces a (surjective) homomorphism $\overline{\varrho} : A/J \to A'/J'$ such that $\pi'\varrho = \overline{\varrho}\pi$, where $J = J(A)$, $J' = J(A')$ and $\pi : A \to A/J$, $\pi' : A' \to A'/J'$ are the canonical projections. Note that $\varrho(J) = J'$ since $\operatorname{Im}\varrho = A'$ and $A'/\varrho(J)$ as a factor of A/J is a semisimple ring. The semisimple ring A/J and $\overline{\varrho}$, $\pi(e)$ and $\pi(f)$ satisfy the assumption of the lemma. Therefore by the previous observation there exist $\overline{a} \in \pi(f)(A'/J')\pi(e)$ and $\overline{b} \in \pi(e)(A'/J')\pi(f)$ such that $\overline{\varrho}(\overline{a}) = \pi'(a')$, $\overline{\varrho}(\overline{b}) = \pi'(b')$ and $\overline{b}\overline{a} = \pi(e)$. Then by the first remark there exist $a_0 \in fAe \cap \pi^{-1}(\overline{a})$ and $b_0 \in eAf \cap \pi^{-1}(\overline{b})$ such that $b_0a_0 = e$. Since $\varrho(a_0) - a', \varrho(b_0) - b' \in J'$ and $\varrho(J) = J'$, there exist $c \in fJe$ and $d \in eJf$ such that $a_1 = a_0 + c \in \varrho^{-1}(a')$ and $b_1 = b_0 + d \in \varrho^{-1}(b')$. Then $e - b_1a_1$ belongs to J and hence is a nilpotent element. Consequently, the first remark implies the existence of a, b satisfying the required conditions. ■

7.4. For the benefit of the reader, we complete the proof of Theorem 7.2, slightly reordering and simplifying arguments from [12].

Proof of Theorem 7.2. Fix a full finite subcategory C of R. We can assume that $C \cap \operatorname{supp} M$ is non-trivial (see Remark 7.2(a)). Denote by $\{C_n\}_{n\in\mathbb{N}}$ the family of finite full subcategories of R defined inductively by setting $C_0 = C$ and $C_{n+1} = \widehat{C}_n$ for $n \in \mathbb{N}$. Since R is connected, we have $R = \bigcup_{n\in\mathbb{N}} C_n$ and C_n is connected for almost all n. Fix a sequence of indecomposable direct summands M^n of $M_{|C_n}$, $n \in \mathbb{N}$, such that M^n is isomorphic to a direct summand of ${M^{n+1}}_{|C_n}$. Fix also a sequence of splittable C_n-monomorphisms $u_n : M^n \to {M^{n+1}}_{|C_n}$, $n \in \mathbb{N}$. For simplicity set $e_\lambda^n = e_\lambda^{C_{n+1},C_n}$ and $\varepsilon_\lambda^n = e_\lambda^{C_n}$ for $n \in \mathbb{N}$. The functors ε_λ^n and $\varepsilon_\lambda^{n+1}e_\lambda^n$ are isomorphic (both are left adjoint to the restriction functor $e_\bullet^{C_n}$). Fix isomorphisms $\theta_n : \varepsilon_\lambda^n \to \varepsilon_\lambda^{n+1} e_\lambda^n$, $n \in \mathbb{N}$. For every n we denote by w_n the composite R-homomorphism

$$\varepsilon_\lambda^n(M^n) \xrightarrow{\theta_n(M^n)} \varepsilon_\lambda^{n+1} e_\lambda^n(M^n) \xrightarrow{\varepsilon_\lambda^{n+1}(v_n)} \varepsilon_\lambda^{n+1}(M^{n+1}),$$

where $v_n : e_\lambda^n(M^n) \to M^{n+1}$ is the C_{n+1}-homomorphism adjoint to $u_n : M^n \to M^{n+1}{}_{|C_n}$. We set

$$M' = \lim\,(\varepsilon_\lambda^n(M^n), w_n)_{n\in\mathbb{N}}.$$

Note that for each $n \in \mathbb{N}$ there exists $p = p(n) \geq n$ such that $M'{}_{|C_n} \simeq M^m{}_{|C_n}$ for all $m \geq p$. Indeed, $\{u_{n|C_n}\}_{m\geq n}$ is a sequence of monomorphisms between finite-dimensional C_n-modules whose dimensions are bounded by $\dim_k M_{|C_n}$ so it stabilizes at some p, and then

$$M'{}_{|C_n} \simeq \lim\,(M^n{}_{|C_n}, u_{n|C_n})_{m\geq n} \simeq M^p{}_{|C_n} \quad \text{for } m \geq p.$$

Consequently, $M'{}_{|C_n}$ is isomorphic to a direct summand of $M_{|C_n}$ for all $n \in \mathbb{N}$ ($M^{p(n)}{}_{|C_n}$ is a direct summand of $M_{|C_n}$) and then by Proposition 7.3, $M' \simeq M$ (M is indecomposable). It is now clear that C_m is a finite (resp. finite connected) M-neighbourhood of C for all (resp. almost all) $m \geq p(0)$. ∎

Remark. If C is a finite full subcategory of R then for any finite full subcategory V containing C there exists a finite connected M-neighbourhood U of C such that $V \subset U$. Moreover, if additionally C and V are contained in $\operatorname{supp} M$ then one can find U as above which is also sincere and contained in $\operatorname{supp} M$.

7.5. Proposition. *Let M be in* $\operatorname{Mod} R$. *Then* $\operatorname{End}_R(M)/J(\operatorname{End}_R(M))$ *is isomorphic to k if and only if so is* $\operatorname{End}_U(M^U)/J(\operatorname{End}_U(M^U))$ *for some finite M-neighbourhood U, where M^U is as in Definition* 5.2.

Proof. Fix any M in $\operatorname{Ind} R$. Suppose that we are given a full subcategory U of R (finite for simplicity) and an indecomposable direct summand M^U of $M_{|U}$ such that M^U is not isomorphic to a direct summand of M', where M' is a (fixed) direct summand of $M_{|U}$ such that $M_{|U} = M^U \oplus M'$. Denote by $\varrho : \operatorname{End}_R(M) \to \operatorname{End}_U(M_{|U})$ the homomorphism given by the restriction functor $e_\bullet^U$ and by h_{11} the component of any $h \in \operatorname{End}_U(M_{|U})$ in $\operatorname{End}_U(M^U)$, under the canonical identification

$$\operatorname{End}_U(M_{|U}) = \begin{pmatrix} \operatorname{Hom}_U(M^U, M^U) & \operatorname{Hom}_U(M', M^U) \\ \operatorname{Hom}_U(M', M^U) & \operatorname{Hom}_U(M', M') \end{pmatrix}.$$

Then the mapping $f \mapsto (\varrho(f))_{11}$ induces a k-algebra homomorphism

$$\sigma = \sigma(M, M^U) : \operatorname{End}_R(M) \to \operatorname{End}_U(M^U)/J(\operatorname{End}_U(M^U))$$

(in fact σ does not depend on M'). Note that this holds in particular if U

is an M-neighbourhood and M^U satisfies the conditions of Definition 7.2. Then σ induces a division k-algebra homomorphism (= embedding)

$$\tau = \tau(M, M^U) : \mathrm{End}_R(M)/J(\mathrm{End}_R(M)) \to \mathrm{End}_U(M^U)/J(\mathrm{End}_U(M^U)),$$

since there exists x in U such that $f(x) = f_{11}(x)$ for all $f \in \mathrm{End}_R(M)$ (cf. [6, Lemma 2.2]). Hence, one implication: $\mathrm{End}_R(M)/J(\mathrm{End}_R(M)) \simeq k$ provided $\mathrm{End}_U(M^U)/J(\mathrm{End}_U(M^U)) \simeq k$.

Assume now that $\mathrm{End}_R(M)/J(\mathrm{End}_R(M)) \simeq k$. Fix a family $\{C_n\}_{n\in\mathbb{N}}$ of finite full connected subcategories of R such that $C_0 = \{x\}$ for some fixed x in $\operatorname{supp} M$ and C_{n+1} is an M-neighbourhood of C_n containing $\widehat{C}_n$ for every $n \in \mathbb{N}$. Note that by Theorem 7.2 and Remark 7.4 one can inductively construct such a family. For simplicity set $E = \mathrm{End}_R(M)$ and $E_n = \mathrm{End}_{C_n}(M_n)$, where $M_n = M_{|C_n}$, for every $n \in \mathbb{N}$.

For all $m, n \in \mathbb{N}$, $m \geq n$, denote by $\varrho_n^m : E_m \to E_n$ the k-algebra homomorphism given by the restriction functor $e_\bullet^{C_m, C_n} : \mathrm{MOD}\, C_m \to \mathrm{MOD}\, C_n$, and by $\varrho_n : E_m \to E_n$ the k-algebra homomorphism given by the restriction functor $e_\bullet^{C_n} : \mathrm{MOD}\, R \to \mathrm{MOD}\, C_n$. Clearly, we have $\varrho_p^n \varrho_n^m = \varrho_p^m$ and $\varrho_n \varrho_n^m = \varrho_m$ for all $m \geq n \geq p$.

We show (cf. [12, 4.2]) that, for each $n \in \mathbb{N}$, there exists $m = m(n) \geq n$ such that $f_n \in E_n$ can be extended to an R-endomorphism of M (i.e. $f_n \in \operatorname{Im} \varrho_n$) if and only if f_n can be extended to an C_n-endomorphism of M_n (i.e. $f_n \in \operatorname{Im} \varrho_m$), briefly that $\operatorname{Im} \varrho_n = \operatorname{Im} \varrho_n^m$. Recall that, for every $i \in \mathbb{N}$, the decreasing sequence $\{\operatorname{Im} \varrho_i^j\}_{j\geq i}$ of k-subalgebras of E_i stabilizes at some $m = m(i)$, since $\dim_k E_i$ is finite. Consequently, for $f_i \in E_i$, $f_i = \varrho_i^m(f_m)$ for some $f_m \in E_m$ if and only if for every $j \geq i$, $f_i = \varrho_i^j(f_j)$ for some $f_j \in E_j$. Suppose now that we are given $f_n \in \operatorname{Im} \varrho_n^{m(n)}$. Then there exists $f_{m(n_1)} \in E_{m(n_1)}$ such that $\varrho_n^{m(n_1)}(f_{m(n_1)}) = f_n$, where $n_1 = \max(m(n), n+1)$. Consequently, we have $f_{n_1} = \varrho_{n_1}^{m(n_1)}(f_{m(n_1)}) \in \operatorname{Im} \varrho_{n_1}^{m(n_1)}$ and $f_n = \varrho_n^{n_1}(f_{n_1})$. Repeating this procedure we can inductively construct $f \in E$ such that $\varrho_n(f) = f_n$.

For every $n \geq 1$, we fix a module $M^n = M^{C_n}$ in $\operatorname{ind} C_n$ satisfying the conditions of Definition 7.2 and a C_n-submodule M_n' of M_n such that $M_n = M^n \oplus M_n'$. For simplicity set $\bar{E} = E/J(E)$ and $\bar{E}^n = E/J(E)$, where $E^n = \mathrm{End}_{C_n}(M^n)$, for $n \geq 1$. Note that each C_n (regarded as a subcategory of C_m) is an M^m-neighbourhood (of C_{n-1}) for $m \geq n$, and that each E^n can be identified, under the canonical embedding $h \mapsto \left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with a k-subspace of E_n.

For simplicity we denote by τ_n^m the homomorphism $\tau(M^{C_m}, M^{C_n}) : \bar{E}^m \to \bar{E}^n$ for all $m \geq n$, and by τ_n the homomorphism $\tau(M, M^{C_n}) : \bar{E} \to \bar{E}^n$. Note that

$$\tau_n^m(f_m + J(E^m)) = \varrho_n^m(f_m)_{11} + J(E^n)$$

for $f_m \in E^m \subseteq E_m$, and

$$\tau_n(f + J(E)) = \varrho_n(f)_{11} + J(E^n)$$

for $f \in E$. Just as for ϱ's we have

$$\tau_p^n \tau_n^m = \tau_p^m \quad \text{and} \quad \tau_n^m \tau_m = \tau_n$$

for all $m \geq n \geq p$. The first formula follows from $\varrho_p^m(f_m)_{11} = \varrho_p^n(\varrho_n^m(f_m)_{11})$ for $f_m \in E^m \subseteq E_m$ ($M'_{n|C_p} = 0$ and $\varrho_p^n(\varrho_n^m(f_m)) = k\varrho_p^n(\varrho_n^m(f_m)_{11})$ if $n > p$), the second from $\varrho_n(f)_{11} = \varrho_n^m(\varrho_m(f)_{11})_{11}$ for $f \in E$ ($M'_{m|C_n} = 0$ and $\varrho_n^m(\varrho_m(f)_{11}) = \varrho_n^m(\varrho_m(f))$ if $m > n$). Consequently, we can assume that all τ_n^m's and τ_n's are now inclusions in the following infinite decreasing sequence of finite-dimensional division k-algebras:

$$\overline{E}_1 \supseteq \ldots \supseteq \overline{E}_n \supseteq \overline{E}_{n+1} \supseteq \ldots \supseteq \overline{E} = k.$$

Then there exists $p \in \mathbb{N}$ such that $\overline{E}^i = \overline{E}^p$ for all $i \geq p$. We show that $\overline{E}^p = \overline{E}$. For every $f_p \in E^p$, we have $f_p + J(E^p) = \tau_p^m(f_m + J(E^m))$ for some $f_m \in E^m \subseteq E_m$, where $m = m(p)$. Consequently, $f_p - \varrho_p^m(f_m)_{11} \in J(E^p)$ and $\varrho_p^m(f_m) \in \operatorname{Im} \varrho_p$; therefore $f_p - \varrho_p(f)_{11} \in J(E^p)$ for some $f \in E$. In this way we have shown that τ_p is surjective, $\overline{E}^p \simeq \overline{E} \simeq k$, and C_p is an M-neighbourhood with the required property. ∎

Remark. (a) If U is an M-neighbourhood (not necessarily finite) such that $\operatorname{End}_U(M^U)/J(\operatorname{End}_U(M^U)) \simeq k$, then each V containing U is an M-neighbourhood (see Remark 7.2(c)) such that for any M^V satisfying the conditions of Definition 7.2, $\operatorname{End}_V(M^V)/J(\operatorname{End}_V(M^V)) \simeq k$.

(b) $\operatorname{End}_R(M)/J(\operatorname{End}_R(M)) \simeq k$ if and only if there exists a (finite) M-neighbourhood U such that $\operatorname{End}_V(M^V)/J(\operatorname{End}_V(M^V)) \simeq k$ for any V containing U, where M^V satisfies the conditions of Definition 7.2.

As a consequence of the above considerations we obtain the following result which is essential for the proof of Theorem 7.1.

Corollary. *If* $\operatorname{End}_R(M)/J(\operatorname{End}_R(M)) \simeq k$, *then for any finite full subcategory* C *of* $\operatorname{supp} M$, *there exists a sincere* M*-neighbourhood* U *of* C, *contained in* $\operatorname{supp} M$, *such that* $\operatorname{End}_U(M^U)/J(\operatorname{End}_U(M^U)) \simeq k$, *where* M^U *is as in Definition* 7.2.

Proof. Note that $\operatorname{End}_V(X) \simeq \operatorname{End}_{\operatorname{supp} X}(X_{|\operatorname{supp} X})$ for any X in $\operatorname{mod} V$. ∎

7.6. It is clear that in order to prove Theorem 7.1 it suffices to show the following result.

Theorem. *Let* R *be a locally bounded* k*-category and* G *be a group of* k*-linear automorphisms acting freely on* R. *Suppose that* R *admits an* *infinite G-atom B such that* $\operatorname{End}_R(B)/J(\operatorname{End}_R(B)) \simeq k$ *and* $\widetilde{B} \not\simeq B$. *Then R is wild.*

Before the proof, fix B in $\operatorname{Mod} R$ (not necessarily a G-atom). Then the R-module $\widetilde{B} = e^S_\lambda(B_{|S})$ ($S = \operatorname{supp} B$ and $\widetilde{S} = \operatorname{supp} \widetilde{B}$ as in 4.1) is given by

$$\widetilde{B}(x) = B_{|S} \otimes R(x,-)_{|S} = \bigoplus_{y \in \operatorname{ob} S} B(y) \otimes_k R(x,y)/I_x \quad \text{for } x \in \operatorname{ob} R,$$

where $I_x = I_x(B_{|S}, R(x,-)_{|S})$ is the k-subspace of $\bigoplus_{y\in \operatorname{ob} S} B(y) \otimes_k R(x,y)$ generated by the set N_x of all non-zero elements of the form $n_{s,b,r} = B(s)(b) \otimes r - b \otimes sr$, $s \in S(y,z)$, $r \in R(x,y)$, $b \in B(z)$, $y,z \in \operatorname{ob} S$ (cf. 1.5). Now it is clear that $\widetilde{S} \subset \widehat{S}$. Note that for any subcategory S' containing $\widehat{\{x\}} \cap S$, I_x can be regarded as a k-subspace of $\bigoplus_{y\in \operatorname{ob} S'} B(y) \otimes_k R(x,y)$. To understand I_x properly we consider the subcategory

$$S_x = (\widehat{\widehat{\{x\}} \cap S}) \cap S$$

of S, which is usually strictly larger than $\widehat{\{x\}} \cap S$. Observe that $y, z \in \operatorname{ob} S_x$ provided $n_{y,z,r} \neq 0$.

LEMMA. *Let x be a fixed object in $R \setminus S$. Then*

$$e^{R',S'}_\lambda(B_{|S'})(x) = e^{R,S}_\lambda(B_{|S})(x)$$

for any full subcategories R', S' of R such that $x \in \operatorname{ob} R'$ and $S_x \subset S' \subset S \cap R'$.

Proof. We can identify $\bigoplus_{y\in \operatorname{ob} S'} B_{|S'}(y) \otimes_k R'(x,y)$ and $\bigoplus_{y\in \operatorname{ob} S} B(y) \otimes_k R(x,y)$, since $\widehat{\{x\}} \cap S \subset S'$. Moreover, by the assumptions, the sets N_x and N'_x of k-generators of $I_x = I_x(B_{|S}, R(x,-))$ and $I'_x = I_x(B_{|S'}, R'(x,-))$ respectively, coincide (under the above identification). Consequently, $I_x = I'_x$ and $e^{R',S'}_\lambda(B_{|S'})(x) = e^{R,S}_\lambda(B_{|S})(x)$. ∎

7.7. *Proof of Theorem 7.6.* Suppose that B is an infinite G-atom such that $B \not\simeq \widetilde{B}$, equivalently $S \subsetneq \widetilde{S}$ (we keep the notation from 7.6). Fix an object x in $\widetilde{S} \setminus S$ and a finite connected subcategory R_x containing $S_x \cup \{x\}$ (S_x is finite since R is locally bounded). Since G_B is an infinite group acting freely on R, we can inductively construct $g_1 = e, g_2, \ldots, g_5 \in G_B$ such that the subcategories $\{g_i R_x\}_{i=1,\ldots,5}$ are pairwise orthogonal. Fix a finite sincere B-neighbourhood U_0 of $C = \bigcup_{i=1}^5 g_i R_x \cap S$ contained in S, for which the module $B_0 = B^{U_0}$ in $\operatorname{ind} U_0$ satisfying the conditions of Definition 7.2 has the property $\operatorname{End}_U(B)/J(\operatorname{End}_U(B)) \simeq k$ (it exists by Corollary 7.5). For simplicity set $U = U_0 \cup \bigcup_{i=1}^5 g_i R_x$ and $U_i = g_i R_x \setminus U_0$, $i = 1, \ldots, 5$; then $U = U_0 \vee (U_1 \sqcup \ldots \sqcup U_5)$. Moreover, $e^{U,U_0}_\lambda(B_0)(g_i x) \neq 0$ for every $i = 1, \ldots, 5$.

Indeed, by Lemma 7.6 and Definition 7.2,

$$\widetilde{B}(x) = e_\lambda^{R,C}(B_{|C})(x) = e_\lambda^{R,U_0}(B_{0|C})(x) = e_\lambda^{U,U_0}(B_{0|U_0})(x)$$

(the cases $g_i \neq e$ follow analogously since $G_B \subset G_{\widetilde{B}}$).

Observe that in this situation the finite-dimensional k-algebra $A = A(U)$ can be viewed as a matrix algebra with $A_0 = A(U_0)$, $A' = A(U_1 \sqcup \ldots \sqcup U_5) \simeq A(U_1) \times \ldots \times A(U_5)$, and that the A_0-module Z corresponding to the U-module B_0 under the standard equivalence $\operatorname{mod} A_0 \simeq \operatorname{mod} U_0$ satisfies the assumptions of Theorem 6.3 (the A-module $\widetilde{Z} = e_\lambda(Z)$ corresponds to $e_\lambda^{U,U_0}(B_0)$ via $\operatorname{mod} A \simeq \operatorname{mod} U$). Consequently, A and R are wild. ∎

REFERENCES

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Grad. Texts in Math. 13, Springer, 1992.

[2] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.

[3] P. Dowbor, *On modules of the second kind for Galois coverings*, Fund. Math. 149 (1996), 31–54.

[4] —, *Galois covering reduction to stabilizers*, Bull. Polish Acad. Sci. Math. 44 (1996), 341–352.

[5] —, *The pure projective ideal of a module category*, Colloq. Math. 71 (1996), 203–214.

[6] —, *On stabilizers of G-atoms of representation-tame categories*, Bull. Polish Acad. Sci. Math. 46 (1998), 304–315.

[7] —, *Properties of G-atoms and full Galois covering reduction to stabilizers*, Colloq. Math. 83 (2000), 231–265.

[8] —, *Stabilizer conjecture for representation-tame Galois coverings of algebras*, J. Algebra, to appear.

[9] P. Dowbor and S. Kasjan, *Galois covering technique and non-simply connected posets of polynomial growth*, J. Pure Appl. Algebra 147 (2000), 1–24.

[10] P. Dowbor, H. Lenzing and A. Skowroński, *Galois coverings of algebras by locally support-finite categories*, in: Lecture Notes in Math. 1177, Springer, 1986, 91–93.

[11] P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras*, Arch. Math. (Basel) 44 (1985), 522–529.

[12] —, —, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. 62 (1987), 311–337.

[13] Yu. A. Drozd and S. A. Ovsienko, *Coverings of tame boxes*, preprint, Max-Planck Institute, Bonn, 2000.

[14] Yu. A. Drozd, S. A. Ovsienko and B. Yu. Furtsin, *Categorical constructions in representation theory*, in: Algebraic Structures and their Applications, University of Kiev, Kiev UMK VO, 1988, 43–73 (in Russian).

[15] P. Gabriel, *The universal cover of a representation-finite algebra*, in: Lecture Notes in Math. 903, Springer, 1982, 68–105.

[16] C. Geiss and J. A. de la Peña, *An interesting family of algebras*, Arch. Math. (Basel) 60 (1993), 25–35.

[17] E. L. Green, *Group-graded algebras and the zero relation problem*, in: Lecture Notes in Math. 903, Springer, 1982, 106–115.

[18] S. Mac Lane, *Categories for the Working Mathematician*, Grad. Texts in Math. 5, Springer, 1971.
[19] Z. Leszczyński and A. Skowroński, *Tame triangular matrix rings*, Colloq. Math. 86 (2000), 259–303.
[20] B. Mitchell, *Rings with several objects*, Adv. Math. 8 (1972), 1–162.
[21] Z. Pogorzały, *Regularly biserial algebras*, in: Topics in Algebra, Banach Center Publ. 26, PWN, Warszawa, 1990, 371–384.
[22] C. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. 55 (1980), 199–224.
[23] D. Simson, *Socle reduction and socle projective modules*, J. Algebra 108 (1986), 18–68.
[24] —, *Representations of bounded stratified posets, coverings and socle projective modules*, in: Topics in Algebra, Banach Center Publ. 26, PWN, Warszawa, 1990, 499–533.
[25] —, *Right peak algebras of two-separate stratified posets, their Galois coverings and socle projective modules*, Comm. Algebra 20 (1992), 3541–3591.
[26] —, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon & Breach, 1992.
[27] —, *On representation types of module categories and orders*, Bull. Polish Acad. Sci. Math. 41 (1993), 77–93.
[28] —, *Posets of finite prinjective type and a class of orders*, J. Pure Appl. Algebra 90 (1993), 77–103.
[29] A. Skowroński, *Tame triangular matrix algebras over Nakayama algebras*, J. London Math. Soc. 34 (1986), 245–264.
[30] —, *Selfinjective algebras of polynomial growth*, Math. Ann. 285 (1989), 177–193.
[31] —, *Criteria for polynomial growth of algebras*, Bull. Polish Acad. Sci. Math. 42 (1994), 173–183.
[32] —, *Tame algebras with strongly simply connected Galois coverings*, Colloq. Math. 72 (1997), 335–351.

Faculty of Mathematics and Computer Science
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: dowbor@mat.uni.torun.pl

Received 24 January 2001;
revised 5 April 2001 (4025)