COLLOQUIUM MATHEMATICUM

VOL. 87 2001 NO. 1

LYAPUNOV FUNCTIONS AND L^p-ESTIMATES FOR A CLASS OF REACTION-DIFFUSION SYSTEMS

BY

DIRK HORSTMANN (Köln)

Abstract. We give a sufficient condition for the existence of a Lyapunov function for the system

$$\begin{aligned} a_t &= \nabla(k(a,c)\nabla a - h(a,c)\nabla c), \quad && x \in \Omega,\ t>0,\\ \varepsilon c_t &= k_c \Delta c - f(c)c + g(a,c), && x\in\Omega,\ t>0, \end{aligned}$$

for $\Omega \subset \mathbb{R}^N$, completed with either $a = c = 0$, or

$$\frac{\partial a}{\partial n} = \frac{\partial c}{\partial n} = 0, \quad \text{or} \quad k(a,c)\frac{\partial a}{\partial n} = h(a,c)\frac{\partial c}{\partial n},\ c = 0 \quad \text{on } \partial\Omega \times \{t>0\}.$$

Furthermore we study the asymptotic behaviour of the solution and give some uniform L^p-estimates.

1. Introduction. In this paper we study the following system of two nonlinear parabolic partial differential equations:

$$(1) \qquad \begin{cases} a_t = \nabla(k(a,c)\nabla a - h(a,c)\nabla c), & x\in\Omega,\ t>0,\\ \varepsilon c_t = k_c\Delta c - f(c)c + g(a,c), & x\in\Omega,\ t>0,\end{cases}$$

for $\Omega \subset \mathbb{R}^N$, completed with either

$$(2) \qquad \frac{\partial a}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega\times\{t>0\},$$

or

$$(3) \qquad a = 0, \quad c = 0 \quad \text{on } \partial\Omega\times\{t>0\},$$

or

$$(4) \qquad k(a,c)\frac{\partial a}{\partial n} = h(a,c)\frac{\partial c}{\partial n}, \quad c=0 \quad \text{on } \partial\Omega\times\{t>0\}$$

as boundary conditions, and with initial data

$$(5) \qquad a(x,0) = a_0(x) \quad \text{and} \quad c(x,0) = c_0(x), \quad x\in\Omega.$$

2000 *Mathematics Subject Classification*: 35A07, 35B40, 35K40, 35K50, 35K55, 35K57.

Key words and phrases: local existence of solution, Lyapunov functions, asymptotic behaviour of solution, uniform L^p-estimates.

Here k_c is a positive constant and $\varepsilon \in \{0, 1\}$. For the functions appearing in the model the following conditions have been considered to be reasonable:

- $k(a, c) > 0$ for all $(a, c) \in \mathbb{R} \times \mathbb{R}$,
- $f(c) \geq \text{const}$ for all $c \in \mathbb{R}$,
- $\frac{\partial}{\partial a} g(a, c) \neq 0$ for all $(a, c) \in \mathbb{R} \times \mathbb{R}$.

Problems of this kind are known in many mathematical fields. They can be written in the form

$$\frac{\partial u}{\partial t} - \nabla \cdot (k(u)\nabla u) = f(u, \nabla u) \tag{6}$$

with $u = (a, c)$. They appear in population dynamics, phase transition models, flows in porous media, models of gravitational interaction of particles and other fields (see [3] for further examples).

To give some explicit examples where systems like (1) appear, we mention the following:

1. If $h(a, c) = a\phi(c)$ with $\phi(c) > 0$ for all $c \in \mathbb{R}_+$ and $g(a, c) > 0$ for all $(a, c) \in \mathbb{R}_+ \times \mathbb{R}_+$, then the equations (1) are known as the so-called Keller–Segel model, which describes the aggregation of the cellular slime mold Dictyostelium discoideum (see for instance [9, 12–15, 19]).

2. These equations also appear in describing animal coat pattern mechanisms (see for example [16]). In [16] system (1) appears with

$$k(a, c) = \text{const}_1 > 0, \qquad h(a, c) = a\phi(c) = a \cdot \text{const}_2 > 0,$$
$$f(c) = \text{const}_3 > 0, \qquad g(a, c) = \frac{\text{const}_4\, a}{\text{const}_5 + a}$$

together with boundary data (2).

3. If $h(a, c) = a\phi(c)$ with $\phi(c) < 0$ for all $c \in \mathbb{R}_+$ and $\varepsilon = 0$, then the equations (1) have some similarities with the Debye system (see for instance [5], [6]).

In this paper we give a sufficient condition for the existence of a Lyapunov function for system (1) in Section 3. Our assumptions are more general than those made in [10, 18, 20, 22]. The results given are also true for the systems studied there. But the conditions assumed here allow a larger variety of nonlinearities than in those papers. This is especially true for the Keller–Segel case (see [20, p. 3]).

In Section 4 we will study the asymptotic behaviour of the solution of those systems which have a Lyapunov function.

In Section 5 we give some L^p-estimates for the solution of system (1). The results stated there are also true for systems which do not have a Lyapunov function.

2. Local existence. In this section we assume that $\varepsilon = 1$. For $\varepsilon = 0$ the existence of a solution can be shown by other methods.

Before we introduce our new results, we will refer to the existence results of A. Yagi [23] for the local (in time) existence of a positive solution of problem (1) in the chemotaxis case with homogeneous Neumann boundary data. In [23], $\Omega \subset \mathbb{R}^2$ is a C^2-smooth domain. For the local existence of a solution with other boundary data and in higher dimensions we use a result of H. Amann [2].

Let $\Omega \subset \mathbb{R}^N$, $p \in (N, \infty)$, $\delta_i \in C(\partial\Omega, \{0,1\})$ for $1 \le i \le N$, $\delta = \operatorname{diag}[\delta_i]_{1 \le i \le N}$ and

$$H^{1,p}_{\mathcal{B}} \equiv \{w \in H^{1,p}(\Omega, \mathbb{R}^2) \mid (I - \delta)w|_{\partial\Omega} = 0\}.$$

Now we set

$$V \equiv \{v \in H^{1,p}_{\mathcal{B}} \mid v(\overline{\Omega}) \subset \mathcal{G}\},$$

where $\mathcal{G}$ is an open subset of $\mathbb{R}^2$. Furthermore we will assume that $k(a,c)$, $h(a,c)$, $g(a,c) \in C^\infty(\mathcal{G}, \mathbb{R})$ and $f(c) \in C^\infty(\mathbb{R}, \mathbb{R})$.

Now we can state the following existence theorem:

THEOREM 1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain with boundary $\partial\Omega$. Furthermore let $w_0 \in V$. Then there exists a unique maximal solution*

$$w(\cdot, w_0) \in C([0, t^+(w_0)), V) \cap C^{2,1}(\overline{\Omega} \times (0, t^+(w_0)), \mathbb{R}^2)$$

of (1) *with boundary condition* (2), (3) *or* (4), *where* $0 < t^+(w_0) \le \infty$.

Proof. We set $w \equiv (a,c) \in \mathbb{R}^2$. Now we can write (1) together with one of the boundary conditions as

$$\begin{aligned} w_t + \mathcal{A}(w)w &= \mathcal{F}(w) && \text{in } \Omega \times (0,T), \\ \mathcal{B}(w)w &= 0, && \text{on } \partial\Omega \times (0,T), \\ w(\cdot, 0) &= (a_0, c_0) && \text{in } \Omega. \end{aligned}$$

The operators $\mathcal{A}(\eta)$ and $\mathcal{B}(\eta)$ are defined by

$$\mathcal{A}(\eta)w \equiv -\partial_j(a_{jk}(\eta)\partial_k w)$$

with

$$\begin{aligned} a_{11}(\eta)(w_1, w_2) &\equiv (k(\eta_1,\eta_2)w_1 - h(\eta_1,\eta_2)w_2, k_c w_2)^T, \\ a_{22}(\eta)(w_1, w_2) &\equiv (k(\eta_1,\eta_2)w_1 - h(\eta_1,\eta_2)w_2, k_c w_2)^T, \\ a_{12}(\eta)(w_1, w_2) &\equiv (0,0), \\ a_{21}(\eta)(w_1, w_2) &\equiv (0,0), \end{aligned}$$

and

$$\mathcal{B}(\eta)w \equiv \delta(a_{jk}(\cdot, \eta)n^j \partial_k w) + (I - \delta)w = 0 \quad \text{on } \partial\Omega,\ t \ge 0,$$

where $n = (n^1, \ldots, n^N)$ is the outer unit normal vector field on $\partial\Omega$. The function $\mathcal{F}(\eta)$ is given by

$$\mathcal{F}(\eta) \equiv (0, g(\eta_1, \eta_2) - f(\eta_2)\eta_2).$$

If we now apply the existence result of H. Amann [2, p. 17] resp. [3, Theorem 14.6, p. 93], we get the assertion of the theorem. ∎

3. A Lyapunov function. Again we set $\varepsilon = 1$, but the results can also be applied to the case $\varepsilon = 0$. For the rest of the paper we use the following notations:

$$F(c) \equiv \int_0^c f(s)s\, ds, \tag{7}$$

$$G(a,c) \equiv -\int_0^c g(a,s)\, ds. \tag{8}$$

Occasionally we assume that

$$\int_\Omega F(c)\, dx \geq k_1 \int_\Omega c^2\, dx, \tag{9}$$

where k_1 is a nonnegative constant (if we have homogeneous Neumann boundary data we assume $k_1 > 0$!).

THEOREM 2. *If there exists a function $R(a)$ such that*

$$\frac{h(a,c)}{k(a,c)}(G_{aa}(a,c) + R''(a)) + G_{ac}(a,c) = 0, \tag{10}$$

then there exists a Lyapunov function for system (1), *provided*

$$G_{aa}(a,c) + R''(a) \geq 0 \tag{11}$$

holds for the solution of (1). *In the case of boundary condition* (3) *we have to assume additionally that $G_a(a,c) = 0 = R'(a)$ on $\partial\Omega \times \{t > 0\}$. A Lyapunov function for system* (1) *is then given by*

$$\mathcal{H}(a(t), c(t)) \equiv \frac{k_c}{2} \int_\Omega |\nabla c(t)|^2\, dx + \int_\Omega F(c(t))\, dx + \int_\Omega R(a(t))\, dx + \int_\Omega G(a(t), c(t))\, dx. \tag{12}$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(a,c) &= -\int_\Omega c_t^2\, dx + \int_\Omega a_t R'(a)\, dx + \int_\Omega a_t G_a(a,c)\, dx \\ &= -\int_\Omega (R''(a) + G_{aa}(a,c))\nabla a(k(a,c)\nabla a - h(a,c)\nabla c) dx\end{aligned}$$

$$\begin{aligned}
&-\int_\Omega G_{ac}(a,c)\nabla c(k(a,c)\nabla a - h(a,c)\nabla c)dx - \int_\Omega c_t^2\,dx\\
&= -\int_\Omega \frac{(R''(a)+G_{aa}(a,c))}{k(a,c)}|k(a,c)\nabla a - h(a,c)\nabla c|^2\,dx\\
&\quad -\int_\Omega c_t^2\,dx \le 0. \blacksquare
\end{aligned}$$

We will explain the differences between our results and those of [10, 18, 20] at the end of the paper.

Example 1. Let us give some examples of functions appearing in system (1), for which our theory holds true and a Lyapunov function $\mathcal{H}(a,c)$ exists.

1. $h(a,c) = a$, $g(a,c) = a^2\exp(-c)/2$, $k(a,c) = 1$, $f(c)$ arbitrary and $R(a) = a^2/2$.

2. $h(a,c) = a/(2c+2)$, $g(a,c) = a^3(c+1)^{-2}/3$, $k(a,c) = 1$, $f(c)$ arbitrary, $R(a) = a^3/3$.

3. In the case of a model of gravitational interaction of particles (see [7, 8] or [4]) we have $\varepsilon = 0$, $h(a,c) = -a$, $g(a,c) = -a$, $k(a,c) = 1$, $f(c) = 0$, $R(a) = a\log(a)$ together with boundary condition (4).

4. $h(a,c) = k_2$, $g(a,c) = k_3 a$, $k(a,c) = k_4$, $f(c)$ arbitrary, and $R(a) = k_3k_4a^2/(2k_2)$.

5. $h(a,c) = a(c+2)^{-2}$, $g(a,c) = a(c+2)^{-2}$,

$$k(a,c) = k(a) = (a+1)^2+1,$$
$$R(a) = \frac{1}{6}a^3 + a^2 + 2a\log(a) - \frac{9}{2}a$$

and $f(c)$ arbitrary.

6. $h(a,c) = a\phi_1(c)$, $k(a,c) = k_5 + k_6 c$, $g(a,c) = a\phi_2'(c)$ with

$$\phi_2(c) = \int_0^c \frac{\phi_1(s)}{k_5+k_6 s}\,ds,$$

$f(c)$ arbitrary, $R(a) = a\log(a)$.

7. One can also find a whole class of other examples where a Lyapunov function exists. Suppose that we study system (1) together with (2). Let $h(a,c) = h_2(a)\phi(c)$ and $g(a,c) = \phi(c)\int_0^a h_1(s)\,ds$,

$$k(a,c) = \widetilde{k}(a) + \frac{h_2(a)h_1'(a)}{h_1(a)}\int_0^c \phi(s)\,ds$$

and let $f(c)$ be arbitrary. We see that there is a function $R(a)$ such that

$$R''(a) = \frac{k(a,c)h_1(a)}{h_2(a)} + h_1'(a)\int_0^c \phi(s)\,ds.$$

Thus there exists a Lyapunov function $\mathcal{H}(a,c)$ of the type given above, which is possibly unbounded from below. This example includes the systems studied in [10] and [20]. In [20] we have $h(a,c) = a\phi(c)$ (with $\phi(c) > 0$), $g(a,c) = a\phi(c)$, $k(a,c) = 1$ and $f(c) = \text{const} > 0$. Finally in this case we get $R(a) = a\log(a)$. For further results concerning some special cases of this type of systems see [20].

For the rest of the paper we make the following assumption.

MAIN ASSUMPTION.

$$\int_\Omega G(a,c) + R(a) \geq k_7 \int_\Omega |\nabla c|^2\,dx + \text{const} \quad \text{with } \frac{k_c}{2} + k_7 > 0. \tag{13}$$

In fact our observations allow us to work with a larger class of nonlinearities in $g(a,c)$ than those studied before for system (1) (as far as the author knows). We formulate the following three propositions.

PROPOSITION 1. *Suppose that* (9) *and our main assumption* (13) *hold. Then*

$$\mathcal{H}(a_0,c_0) \geq \mathcal{H}(a(t),c(t)) \geq k_8 \quad \textit{for all } t \geq 0.$$

PROPOSITION 2. *Suppose that* (9) *and* (13) *hold. Then*

$$\int_\Omega |\nabla c(t)|^2\,dx \leq k_9 \quad \textit{for all } t \geq 0.$$

The proofs of Propositions 1 and 2 are trivial.

PROPOSITION 3. *Suppose that* (9) *and* (13) *hold. Then*

$$\int_0^T \int_\Omega c_t^2\,dx \leq \mathcal{H}(a_0,c_0) + k_{10}.$$

Proof. We have

$$\int_\Omega c_t^2\,dx + \frac{d}{dt}\mathcal{H}(a,c) \leq 0.$$

Thus

$$\int_0^T \int_\Omega c_t^2\,dx \leq \mathcal{H}(a_0,c_0) - \mathcal{H}(a(T),c(T)) \leq \mathcal{H}(a_0,c_0) + k_{10}. \ \blacksquare$$

4. The asymptotic behaviour of the solution of system (1). In some special cases of (1) (for example for some special forms of the Keller–Segel model) one can show that the solution of (1) converges to a possibly nontrivial steady state as $t \to \infty$ (see [10, 14, 20]). The results of R. Schaaf [21] and K. Post [20] concerning the Keller–Segel model in chemotaxis give hope that one can expect such a behaviour also in a more general setting.

We summarize our results concerning this aspect in the following theorem.

THEOREM 3. *Suppose that* $(a(t), c(t))$ *is a weak solution of* (1) *and that* (9) *and* (13) *hold. Furthermore let the solution* $(a(t), c(t))$ *of* (1) *satisfy either*

(i) $(G_{aa}(a,c) + R''(a))/k(a,c) \le k_{11}$ *and* $G_a(a,c) + R'(a) > k_{12}$ *or*
(ii) $0 < (G_{aa}(a,c) + R''(a)) \exp(G_a(a,c) + R'(a)) \le k_{13} k(a,c)$ *and*

$$\sqrt{\frac{k(a(t),c(t))}{G_{aa}(a,c) + R''(a(t))}} \in L^2(\Omega) \quad \text{for all } t \ge 0.$$

Let additionally f *be Hölder continuous with exponent* $\beta \le 1$ *such that*

$$0 < \beta \le 1 \text{ if } N \le 3 \quad \text{or} \quad \beta < 2/N \text{ if } N > 3.$$

Finally assume that $|f(c)| \le K_f$ *for all* $c \in \mathbb{R}$. *Then there exist a sequence* $(t_k)_{k\in\mathbb{N}}$ *and two functions* c^* *and* g^* *such that*

$$\begin{aligned} c(t_k) &\rightharpoonup c^* && \text{in } H^1(\Omega) \text{ (resp. in } H_0^1(\Omega)),\\ f(c(t_k))c(t_k) &\to f(c^*)c^* && \text{in } L^2(\Omega) \end{aligned}$$

and

$$g(a(t_k), c(t_k)) \rightharpoonup g^* \quad \text{in } L^2(\Omega).$$

Furthermore

$$\int_\Omega (k_c \nabla c^* \nabla \varphi + f(c^*)c^*\varphi)\, dx = \int_\Omega g^* \varphi\, dx$$

for all $\varphi \in H^1(\Omega)$ *(resp.* $\varphi \in H_0^1(\Omega)$*). Finally,*

$$\exp\left(\frac{-(G_a(a(t_k), c(t_k)) + R'(a(t_k)))}{2}\right) \to \text{const}$$

in $L^2(\Omega)$ *if* (i) *holds, and*

$$\exp\left(\frac{G_a(a(t_k), c(t_k)) + R'(a(t_k))}{2}\right) \to \text{const}$$

in $L^2(\Omega)$ *if* (ii) *holds.*

REMARK 1. Assumption (i) of Theorem 3 is satisfied by Example 1.1, while the systems studied in [10] and [20] satisfy (ii).

Proof. We first assume that (i) is true. We set

$$W \equiv \exp(-(G_a(a,c) + R'(a))).$$

We note that assuming $G_a(a,c) + R'(a) > k_{12}$ implies that

$$\|\sqrt{W(t)}\|_{L^2(\Omega)} \le \text{const}$$

for all $t \ge 0$. Now there exists a constant k_{14}/k_{11} such that

$$
\begin{aligned}
&\frac{k_{14}}{k_{11}}|\nabla\sqrt{W}|^2\\
&\quad= \frac{k_{14}}{4k_{11}} e^{-(G_a(a,c)+R'(a))}|(G_{aa}(a,c)+R''(a))\nabla a + G_{ac}(a,c)\nabla c|^2\\
&\quad\le \frac{k_{14}}{4k_{11}}\cdot\frac{(G_{aa}(a,c)+R''(a))^2 e^{-(G_a(a,c)+R'(a))}}{k^2(a,c)}|k(a,c)\nabla a - h(a,c)\nabla c|^2\\
&\quad\le \frac{G_{aa}(a,c)+R''(a)}{k(a,c)}|k(a,c)\nabla a - h(a,c)\nabla c|^2.
\end{aligned}
$$

As in [20] we set

$$
\mathcal{I}(t) \equiv t\biggl(\|c_t(t)\|^2_{L^2(\Omega)} + \frac{k_{14}}{k_{11}}\|\nabla\sqrt{W(t)}\|^2_{L^2(\Omega)}\biggr).
$$

From the estimate above and the fact that

$$
\begin{aligned}
&\int_0^t \biggl(\|c_s(s)\|^2_{L^2(\Omega)} + \frac{k_{14}}{k_{11}}\|\nabla\sqrt{W(s)}\|^2_{L^2(\Omega)}\biggr)\,ds\\
&\qquad\qquad \le -\int_0^t \frac{d}{ds}\mathcal{H}(a(s),c(s))\,ds = \mathcal{H}(a_0,c_0) - \mathcal{H}(a(t),c(t)) \le \text{const}
\end{aligned}
$$

for all $t \ge 0$, we see that

$$
\int_0^t \mathcal{I}(s)\,ds \le k_{15}\cdot t.
$$

Therefore there exists a sequence $t_k \to \infty$ such that $\mathcal{I}(t_k) \le 2k_{16}$ for all t_k. So we get

$$
\|c_t(t_k)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \to 0 \quad \text{as } k\to\infty.
$$

From our main assumption and assumption (9) we see that

$$
\|\nabla c(t_k)\|^2_{L^2(\Omega)} + \|c(t_k)\|^2_{L^2(\Omega)} \le k_{17}.
$$

Thus there exists a function c^* with $c(t_k) \rightharpoonup c^*$ in $H^1(\Omega)$ (resp. in $H^1_0(\Omega)$ depending on the boundary data!).

In view of the compact imbedding of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) into the appropriate L^p-space we get

$$
\begin{aligned}
&\|f(c(t_k))c(t_k) - f(c^*)c^*\|_{L^2(\Omega)}\\
&\quad \le \|f(c(t_k))c(t_k) - f(c(t_k))c^*\|_{L^2(\Omega)} + \|f(c(t_k))c^* - f(c^*)c^*\|_{L^2(\Omega)}.
\end{aligned}
$$

CASE 1: $N \le 3$ and $0 < \beta \le 2/3$. Then

$$\begin{aligned}
&\|f(c(t_k))c(t_k) - f(c^*)c^*\|_{L^2(\Omega)} \\
&\qquad \le \|f(c(t_k))c(t_k) - f(c(t_k))c^*\|_{L^2(\Omega)} \\
&\qquad\quad + k_{\text{Hölder}}\|c^*\|_{L^{2/(1-\beta)}(\Omega)}\|c(t_k) - c^*\|^{\beta}_{L^2(\Omega)} \\
&\qquad \le K_f\|c(t_k) - c^*\|_{L^2(\Omega)} + k_{\text{Hölder}}\|c^*\|_{L^{2/(1-\beta)}(\Omega)}\|c(t_k) - c^*\|^{\beta}_{L^2(\Omega)} \\
&\qquad \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

CASE 2: $N \le 3$ and $2/3 < \beta \le 1$. Then

$$\begin{aligned}
&\|f(c(t_k))c(t_k) - f(c^*)c^*\|_{L^2(\Omega)} \\
&\qquad \le \|f(c(t_k))c(t_k) - f(c(t_k))c^*\|_{L^2(\Omega)} \\
&\qquad\quad + k_{\text{Hölder}}\|c^*\|_{L^{4/(2-\beta)}(\Omega)}\|c(t_k) - c^*\|^{\beta}_{L^4(\Omega)} \\
&\qquad \le K_f\|c(t_k) - c^*\|_{L^2(\Omega)} + k_{\text{Hölder}}\|c^*\|_{L^{4/(2-\beta)}(\Omega)}\|c(t_k) - c^*\|^{\beta}_{L^4(\Omega)} \\
&\qquad \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

CASE 3: $N > 3$ arbitrary and $\beta < 2/N$. Then

$$\begin{aligned}
&\|f(c(t_k))c(t_k) - f(c^*)c^*\|_{L^2(\Omega)} \\
&\qquad \le \|f(c(t_k))c(t_k) - f(c(t_k))c^*\|_{L^2(\Omega)} \\
&\qquad\quad + k_{\text{Hölder}}\|c^*\|_{L^{2/(1-\beta)}(\Omega)}\|c(t_k) - c^*\|^{\beta}_{L^2(\Omega)} \\
&\qquad \le K_f\|c(t_k) - c^*\|_{L^2(\Omega)} + k_{\text{Hölder}}\|c^*\|_{L^{2/(1-\beta)}(\Omega)}\|c(t_k) - c^*\|^{\beta}_{L^2(\Omega)} \\
&\qquad \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Thus

$$f(c(t_k))c(t_k) \to f(c^*)c^* \quad \text{in } L^2(\Omega) \text{ as } k \to \infty. \tag{14}$$

Since $W(t_k)$ is uniformly bounded in $H^1(\Omega)$ and $\|\nabla\sqrt{W(t_k)}\|_{L^2(\Omega)} \to 0$ we see that

$$\sqrt{W(t_k)} \to \text{const} \quad \text{in } L^2(\Omega) \text{ as } k \to \infty.$$

To prove the statement of the theorem if (ii) holds, we set

$$V \equiv \exp(G_a(a,c) + R'(a)).$$

Now we see that

$$\frac{1}{k_{13}}|\nabla\sqrt{V}|^2 = \frac{1}{4k_{13}}e^{G_a(a,c)+R'(a)}|(G_{aa}(a,c) + R''(a))\nabla a + G_{ac}(a,c)\nabla c|^2$$

$$\leq \frac{(G_{aa}(a,c)+R''(a))^2 e^{G_a(a,c)+R'(a)}}{4k_{13}k^2(a,c)}|k(a,c)\nabla a - h(a,c)\nabla c|^2$$
$$\leq \frac{G_{aa}(a,c)+R''(a)}{k(a,c)}|k(a,c)\nabla a - h(a,c)\nabla c|^2.$$

We now define

$$\mathcal{J}(t) \equiv t\biggl(\|c_t(t)\|^2_{L^2(\Omega)} + \frac{1}{k_{13}}\|\nabla\sqrt{V(t)}\|^2_{L^2(\Omega)}\biggr)$$

and use the same argument as in the first case. ∎

REMARK 2. If

$$V(t) = a(t)\exp(-\Psi(c(t))) \tag{15}$$

and $0 \leq \Psi(s) \leq \text{const}$ for all $s \in \mathbb{R}_+$, then the mass conservation for the function $a(t)$ in the case of boundary condition (2) or (4) (the first equation of (1)) implies that

$$V(t_k) \to \frac{\int_\Omega a_0(x)\,dx}{\int_\Omega \exp(\Psi(c^*(x)))\,dx} \quad \text{in } L^2(\Omega) \text{ (see [20])}.$$

Since no further results concerning the asymptotic behaviour of the solution are available in the general setting studied in the present paper, we turn to some L^p-estimates for the solution.

5. Uniform L^p-bounds of the solution for $1 \leq p \leq \infty$. If we look at (1) with either (2) or (4) as boundary conditions we see that

$$\int_\Omega a_t\,dx = 0 \quad \text{for all } t > 0,$$

as mentioned in Remark 2.

LEMMA 1. *Let $(a(t), c(t))$ be the solution of (1) with boundary conditions (2) or (4). Then*

$$\|a(t)\|_{L^1(\Omega)} = \|a_0\|_{L^1(\Omega)} \quad \textit{for all } t > 0. \tag{16}$$

For the rest of this section we assume that

$$h(a,c) = a\phi(c) \quad \text{with } |\phi(c)| \leq k_{18} \quad \text{for all } c \in \mathbb{R}.$$

Even if we do not prove the positivity of $a(t,x)$ in this case, we will always assume it. This property can be derived for classical solutions from the maximum principle for parabolic equations, provided the initial data is strictly positive. Under weaker assumptions on the solution, one can use the results of A. Yagi [23] to prove the nonnegativity of $a(t,x)$ in the maximal existence interval under reasonable assumptions on the initial data.

Independently of the choice of the boundary conditions for (1) we have the following L^p-estimate for the function $a(t)$.

THEOREM 4. *Let* (a,c) *be the solution of* (1) *with boundary conditions* (2) *or* (4). *Furthermore let* $k(a,c) \geq k_{19} > 0$ *for all* $(a,c) \in \mathbb{R}_+ \times \mathbb{R}$ *and* $|\phi(c)| \leq k_{18}$ *for all* $c \in \mathbb{R}_+$. *If* $c(t) \in W^{1,\infty}(\Omega)$ (*resp.* $c(t) \in W_0^{1,\infty}(\Omega)$) *for all* $t \in [0,\infty)$ *and if there exists a constant* $k_{\nabla c}$ *such that*

$$|\nabla c(t,x)| \leq k_{\nabla c} \quad \text{for all } t \in [0,\infty) \text{ and all } x \in \Omega,$$

then

$$\|a(t,x)\|_{L^\infty(\Omega)} \leq k_{20} \quad \text{for all } t \in [0,\infty).$$

For solutions of (1) *with boundary conditions* (3), *the above holds true if we additionally assume that*

$$\sup_{t\geq 0} \|a(t)\|_{L^1(\Omega)} \leq \text{const}\,.$$

Proof. The assertion can be proved in the same way as in [17] for a special case of (1) with homogeneous Neumann boundary data.

Let $1 \leq p < \infty$. We multiply the first equation of (1) with a^p and integrate over Ω to derive

$$\begin{aligned}
\frac{1}{p+1}\frac{d}{dt}\int_\Omega a^{p+1}\,dx &= -\frac{4p}{(p+1)^2}\int_\Omega k(a,c)|\nabla a^{(p+1)/2}|^2\,dx + p\int_\Omega a^p\phi(c)\nabla a\nabla c\,dx \\
&\leq -\frac{4pk_{19}}{(p+1)^2}\int_\Omega |\nabla a^{(p+1)/2}|^2\,dx + pk_{18}k_{\nabla c}\int_\Omega a^p|\nabla a|\,dx \\
&\leq -\frac{2pk_{19}}{(p+1)^2}\int_\Omega |\nabla a^{(p+1)/2}|^2\,dx + \frac{p\widetilde{k}_{18}k_{\nabla c}}{2}\int_\Omega a^{p+1}\,dx.
\end{aligned}$$

If we now use Moser's technique from [1] we get the assertion. ∎

By (9) we have the uniform boundedness of the L^2-norm of $c(t)$ for all $t > 0$ if system (1) has a Lyapunov function.

LEMMA 2. *Let* $(a(t),c(t))$ *be the solution of* (1). *Furthermore suppose that* $\Omega \subset \mathbb{R}^2$ *and that there is a Lyapunov function for system* (1), *and* (9) *as well as our main assumption hold. Then*

$$\|c(t)\|_{L^p(\Omega)} \leq k(p) \quad (1 \leq p < \infty) \quad \text{for all } t > 0. \tag{17}$$

Proof. By the assumption (9), the boundedness of the Lyapunov function $\mathcal{H}(a,c)$ from below yields a uniform bound of the H^1-norm of $c(t)$ for all $t > 0$. Applying the Sobolev embedding theorem we derive uniform L^p-bounds for all $t \geq 0$. ∎

THEOREM 5. *Let* (a,c) *be the solution of* (1), *and assume that* $g(a,c) > 0$ *for all* $(a,c) \in \mathbb{R}_+ \times \mathbb{R}$, $\varepsilon = 1$ *and* $|f(c)| \leq K_f < \infty$ *in* (1). *If*

$$\sup_{t\geq 0}\int_\Omega |c(t)|\,dx = k_{21} < \infty$$

and

$$\int_\Omega (g(a,c)c^p - f(c)c^{p+1})\,dx \le k_{22}\int_\Omega c^{p+1}\,dx \quad \textit{for all } 1\le p<\infty, \tag{18}$$

then

$$\|c(t,x)\|_{L^\infty(\Omega)} \le k_{23} \quad \textit{for all } t\in[0,\infty).$$

Since our assumption on $g(a,c)$ implies that $c(t,x)\ge 0$ in Ω for all $t\in[0,T_{\max})$ provided $c_0(x)\ge 0$ in Ω, we see that the statement of the theorem can be shown in the same way as Theorem 4. Therefore we leave the proof to the reader.

EXAMPLE 2. We give some examples of functions appearing in system (1) for which Theorems 4 and 5 hold true.

1. Let $\Omega\subset\mathbb{R}^2$ be smooth. Consider the problem

$$\begin{aligned}
a_t &= \nabla(\nabla a - \chi a\nabla c), && x\in\Omega,\ t>0,\\
c_t &= k_c\Delta c - \gamma c + \frac{ac}{a+\delta} \quad (\gamma+\lambda_1>1), && x\in\Omega,\ t>0,\\
\partial a/\partial n &= \chi a\partial c/\partial n,\ c=0, && x\in\partial\Omega,\ t>0,\\
a(0,x) &= a_0(x)>0, && x\in\Omega,\\
c(0,x) &= c_0(x)>0, && x\in\Omega.
\end{aligned}$$

First of all let us mention that this system does not have a Lyapunov function of the form presented in the previous sections. Secondly we see from Theorem 5 that

$$\sup_{t\ge0}\|c(t)\|_{L^\infty(\Omega)}<\infty.$$

Furthermore we see that $g(a(t),c(t))\in L^\infty(\Omega)$ for all $t\ge0$ and

$$\sup_{t\ge0}\|g(a(t),c(t))\|_{L^\infty(\Omega)} \le \sup_{t\ge0}\|c(t)\|_{L^\infty(\Omega)} < k_{24}.$$

We now follow an idea from [18]. If we consider the second equation of our system as an abstract evolution equation, we see that

$$c(t) = T_p(t)c_0 + \int_0^t T_p(t-s)\frac{a(s)c(s)}{a(s)+\delta}\,ds,$$

where $\{T_p(t)\}$ is the analytic semigroup of the sectorial (in $L^p(\Omega)$) operator $A_p\equiv -\Delta+\gamma$ with domain $\mathcal{D}\equiv\{u\in W^{2,p}(\Omega)\mid u=0 \text{ on } \partial\Omega\}$. We know from [11] that we can define fractional powers A_p^μ for $\mu\ge0$ of A_p with domain $X_p^\mu=\mathcal{D}(A_p^\mu)$. We also know that X_p^μ is a Banach space with

$$\|u\|_{X_p^\mu} = \|A_p^\mu u\|_{L^p(\Omega)}.$$

The imbedding $X_p^\mu\subset C^\nu(\Omega)$ is continuous for $0\le\mu\le1$ and $0\le\nu<2\mu-N/p$ ($N=2$ in our case!) (see [11, Theorem 1.6.1, p. 39]).

If we now choose $3/4 < \mu < 1$ and consider

$$A_4^\mu c(t) = A_4^\mu T_4(t)c_0 + \int_0^t A_4^\mu T_4(t-s)\frac{a(s)c(s)}{a(s)+\delta}\,ds,$$

we see that there exist constants $k_{21} > 0$ and θ such that

$$\|A_4^\mu c(t)\|_{L^4(\Omega)} \le k_{21}\left\{\frac{e^{-\theta t}}{t^\mu}\|c_0\|_{L^4(\Omega)} + k_{25}\int_0^t \frac{e^{-\theta t}}{t^\mu}\,ds\right\}.$$

Thus we can derive a uniform bound of $\|A_4^\mu c(t)\|_{L^4(\Omega)}$ for all $t \ge 0$ and—in view of the continuous imbedding—a uniform bound of the C^1-norm of $c(t)$ for all $t \ge 0$.

Applying now Theorem 4 we also get a uniform bound of the L^∞-norm of $a(t)$ for all $t \ge 0$.

2. As a second example we consider a parabolic-elliptic problem. Let us assume that there is a positive solution of the following problem:

$$\begin{aligned}
a_t &= \nabla(\nabla a - \chi a \nabla c), && x \in \Omega \subset \mathbb{R}^2,\ t > 0,\\
0 &= k_c \Delta c - \gamma c + \frac{a}{a+\delta}, && x \in \Omega,\ t > 0,\\
\partial a/\partial n &= \partial c/\partial n = 0, && x \in \partial\Omega,\ t > 0,\\
a(0,x) &= a_0(x) > 0, && x \in \Omega,
\end{aligned}$$

where k_c, χ, γ and δ are positive constants. Again there is no Lyapunov function for this problem, but we see that

$$\frac{a(t)}{a(t)+\delta} \in L^p(\Omega)$$

for $1 \le p \le \infty$ with a uniform bound for all $t > 0$. Using elliptic regularity theory we get $c(t) \in H^{2,p}(\Omega)$ with a uniform bound for all $t > 0$. For p large enough the Sobolev imbedding theorems imply $c(t) \in C^1(\overline{\Omega})$ provided $\partial\Omega$ is smooth enough, where we have a uniform C^1-bound for all $t > 0$. Thus Theorem 4 implies that

$$\sup_{t\ge 0} \|a(t)\|_{L^\infty(\Omega)} < \infty.$$

REMARK 3. Example 2.1 shows that, if the second equation can be written as an abstract evolution equation with a sectorial (in $L^p(\Omega)$, for $\Omega \subset \mathbb{R}^2$) operator and a right-hand side which is uniformly bounded in $L^4(\Omega)$ for all $t \ge 0$, then one can derive uniform L^∞-bounds for $a(t)$ and $c(t)$ for all $t \ge 0$.

6. Final remarks. We now indicate the differences between our results and those in [10, 18, 20, 22].

Those papers consider the so-called *Keller–Segel model*

$$\begin{aligned}
a_t &= \nabla(\nabla a - \chi a \nabla c), && x \in \Omega \subset \mathbb{R}^2,\ t > 0,\\
\varepsilon c_t &= k_c \Delta c - \gamma c + \alpha a, && x \in \Omega,\ t > 0,\\
a(0,x) &= a_0(x) > 0, && x \in \Omega,\\
c(0,x) &= c_0(x) > 0, && x \in \Omega,
\end{aligned}$$

with either boundary condition (2) (see [10, 18, 20]) or (4) (see [22]).

Our results include the results in [10, 18, 20, 22] but they are also true for higher space dimensions, other boundary conditions and further nonlinearities in the system (1). As far as I know, K. Post of [20] was the only one to study the case of higher space dimensions. She considered the system

$$\begin{aligned}
a_t &= \nabla(\nabla a - \chi a \nabla \Phi(c)), && x \in \Omega \subset \mathbb{R}^N,\ t > 0,\\
\varepsilon c_t &= k_c \Delta c - \gamma c + \alpha a \Phi'(c), && x \in \Omega,\ t > 0,\\
\partial a/\partial n &= \partial c/\partial n = 0, && x \in \partial\Omega,\ t > 0,\\
a(0,x) &= a_0(x) > 0, && x \in \Omega,\\
c(0,x) &= c_0(x) > 0, && x \in \Omega,
\end{aligned}$$

with $\Phi(s) \in \{\Phi \in C^1(\mathbb{R},\mathbb{R}) \mid 0 \le \Phi(s),\ 0 \le \Phi'(s) \le \text{const for all } s \ge 0\}$.

Our results also cover this system, but again one can allow further nonlinearities and still have the existence of a Lyapunov function.

To end the paper let us mention that our results have been strongly inspired by those of K. Post [20] and by the question of whether one can show the existence of a Lyapunov function for a larger class of systems than those studied in [20], and if the asymptotic behaviour of the solution of such a system is similar.

REFERENCES

[1] N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations 4 (1979), 827–868.

[2] H. Amann, *Dynamic theory of quasilinear parabolic equations II. Reaction-diffusion systems*, Differential Integral Equations 3 (1990), 13–75.

[3] —, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in: Function Spaces, Differential Operators and Nonlinear Analysis, Teubner Texte Math. 133, Teubner, 1993, 9–126.

[4] P. Biler, *Existence and nonexistence of solutions for a model of gravitational interaction of particles*, *III*, Colloq. Math. 68 (1995), 229–239.

[5] —, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. 8 (1998), 715–743.

[6] P. Biler, W. Hebisch and T. Nadzieja, *The Debye system: existence and large time behaviour of solutions*, Nonlinear Anal. 23 (1994), 1189–1209.

[7] P. Biler, D. Hilhorst and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles*, *II*, Colloq. Math. 67 (1994), 287–308.

[8] P. Biler and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles*, *I*, Colloq. Math. 66 (1994), 319–334.

[9] S. Childress and J. K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci. 56 (1981), 217–237.

[10] H. Gajewski and K. Zacharias, *Global behavior of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. 195 (1998), 77–114.

[11] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer, Berlin, 1981.

[12] D. Horstmann, *Aspekte positiver Chemotaxis*, Dissertation, Universität zu Köln, 1999.

[13] —, *On the existence of radially symmetric blowup-solutions for the so-called Keller–Segel model*, NoDEA Nonlinear Differential Equations Appl., to appear.

[14] —, *The nonsymmetric case of the Keller–Segel model in chemotaxis*: *some recent results*, preprint, 1999, submitted.

[15] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. 26 (1970), 399–415.

[16] J. D. Murray, *Modelling the pattern generating mechanism in the formation of stripes on alligators*, in: Mathematical Physics, 9th Int. Congr. Swansea/UK 1988, (1989), 208–213.

[17] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. 5 (1995), 581–601.

[18] T. Nagai, T. Senba and K. Yoshida, *Application of the Moser–Trudinger inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. 40 (1997), 411–433.

[19] V. Nanjundiah, *Chemotaxis*, *signal relaying and aggregation morphology*, J. Theor. Biol. 42 (1973), 63–105.

[20] K. Post, *A non-linear parabolic system modeling chemotaxis with sensitivity functions*, preprint no. 99–12, Humboldt-Universität zu Berlin, Institut für Mathematik, 1999.

[21] R. Schaaf, *Stationary solutions of chemotaxis systems*, Trans. Amer. Math. Soc. 292 (1985), 531–556.

[22] G. Wolansky, *A critical parabolic estimate and application to nonlocal equations arising in chemotaxis*, Appl. Anal. 66 (1997), 291–321.

[23] A. Yagi, *Norm behavior of solutions to a parabolic system of chemotaxis*, Math. Japon. 45 (1997), 241–265.

Mathematisches Institut
Universität zu Köln
Weyertal 86–90
D-50931 Köln, Germany
E-mail: dhorst@mi.uni-koeln.de

Received 14 December 1999;
revised 15 February 2000 (3860)