COLLOQUIUM MATHEMATICUM

VOL. 87 2001 NO. 1

AN ANALOGUE OF HARDY'S THEOREM FOR THE HEISENBERG GROUP

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Abstract. We observe that the classical theorem of Hardy on Fourier transform pairs can be reformulated in terms of the heat kernel associated with the Laplacian on the Euclidean space. This leads to an interesting version of Hardy's theorem for the sublaplacian on the Heisenberg group. We also consider certain Rockland operators on the Heisenberg group and Schrödinger operators on $\mathbb{R}^n$ related to them.

1. Introduction and the main results. A classical theorem of Hardy [9] on Fourier transform pairs says that a function f on the real line and its Fourier transform $\widehat{f}$ cannot both be very rapidly decreasing. To be more precise, let

$$\widehat{f}(y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dx$$

be the Fourier transform of a function f defined on the real line $\mathbb{R}$. Hardy's theorem says that if $|f(x)| \le Ce^{-a|x|^2}$ and $|\widehat{f}(y)| \le Ce^{-b|y|^2}$ with $ab > 1/4$ then $f = 0$. There are infinitely many linearly independent functions satisfying the above estimates when $ab < 1/4$ and when $ab = 1/4$ there is essentially one function, viz. $f(x) = Ce^{-ax^2}$.

Recently, considerable attention has been paid to discover analogues of Hardy's theorem in the context of Lie groups. Generalisations of this result have been established for semisimple Lie groups [3], [14], [15], for the n-dimensional motion group [18], for the Heisenberg group [16], for general nilpotent Lie groups [12], for solvable extensions of H-type groups [1], and also for some eigenfunction expansions [13]. In all these generalisations, the formulations of the results involve various group Fourier transforms. However, a close examination of the proofs reveal that, in almost all the cases, the results follow, after some initial reduction, from the classical Hardy's theorem. The aim of this article is to show that in almost all the known

2000 *Mathematics Subject Classification*: Primary 43A30; Secondary 43A10, 22E25.

Key words and phrases: Fourier transform, Heisenberg group, heat equation, Rockland operators, unitary representations, Schrödinger operators, Hermite functions.

cases, Hardy's theorem can be stated in terms of the heat kernels of the Laplacians on the groups concerned. This substantiates a remark made by V. S. Varadarajan some years ago on the connection between Hardy's theorem and heat kernels.

Consider the heat equation $\partial_t u(x,t) = \Delta u(x,t)$ associated with the standard Laplacian on $\mathbb{R}^n$. The solution with the initial condition $u(x,0) = f(x)$ is given by $u(x,t) = f * p_t(x)$ where $p_t(x)$ is the heat kernel

$$p_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}.$$

Taking the Fourier transform of $u_t(x) = u(x,t)$ in the x variable and noting that $\widehat{p}_t(\xi) = e^{-t|\xi|^2}$ we have the estimate

$$|\widehat{u}_t(\xi)| \le Ce^{-t|\xi|^2}$$

whenever $\widehat{f}$ is bounded. Suppose now we also have the estimate

$$|u_t(x)| \le C_t e^{-a|x|^2}.$$

If f is non-trivial, then by Hardy's theorem the above estimate is possible only when $4at \le 1$. In other words, $u(x,t)$ cannot have more decay than the heat kernel $p_t(x)$ unless $f = 0$.

Our aim in this paper is to show that a similar result is true for solutions of the heat equation associated with the sublaplacian $\mathcal{L}$ on the Heisenberg group. More generally, we consider certain Rockland operators on the Heisenberg group. Before stating the main theorem, let us recall some relevant definitions. We refer to the monographs [6] and [21] for an introduction to the representation theory of the Heisenberg group. The *Heisenberg group* H^n is simply $\mathbb{C}^n \times \mathbb{R}$ with group law given by

$$(z,s)(w,\tau) = \big(z + w, s + \tau + \tfrac{1}{2}\,\mathrm{Im}(z\overline{w})\big)$$

which makes H^n into a step-two nilpotent Lie group.

On H^n we have certain left invariant vector fields X_j, Y_j, $j = 1, \dots, n$, which together with $T = \partial/\partial s$ form a basis for the Heisenberg Lie algebra. The *sublaplacian*, denoted by $\mathcal{L}$, is the second order differential operator defined by

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

This operator, which plays the role of the Laplacian on H^n, is hypoelliptic, self-adjoint and non-negative. It generates a diffusion semigroup with kernel $p_t(z,s)$. The kernel is explicitly known and satisfies the estimate

$$|p_t(z,s)| \le Ct^{-n-1} e^{-A|(z,s)|^2/t}$$

where $|(z,s)|$ is the norm defined by $|(z,s)|^4 = |z|^4 + s^2$, which is homogeneous of degree one with respect to the non-isotropic dilations $\delta_r(z,s) = (rz, r^2 s)$.

The solution of the heat equation $\partial_t u(z,s;t) = -\mathcal{L}u(z,s;t)$ with the initial condition $u(z,s;0) = f(z,s)$ is given by $u(z,s;t) = f * p_t(z,s)$ where the convolution is on the group H^n. Concerning the solutions of this equation we have the following

THEOREM 1.1. *Suppose $f \in L^2(H^n)$ satisfies the estimate $|f * p_t(z,s)| \leq C_t p_{t'}(z,s)$. Then $f = 0$ whenever $0 < t' < t/2$.*

Thus we see that the solution $u(z,s;t)$ of the heat equation cannot decay faster than the heat kernel $p_{t/2}(z,s)$. An examination of the proof of the above theorem given in the next section shows that we have the following version of Hardy's theorem for the group Fourier transform on the Heisenberg group. First we recall some definitions. The group Fourier transform on H^n is defined using the infinite-dimensional irreducible unitary representations π_λ which are parametrised by non-zero reals and realised on the same Hilbert space, namely $L^2(\mathbb{R}^n)$. Given a function $f \in L^1(H^n)$ the *group Fourier transform* is defined to be the operator valued function

$$\widehat{f}(\lambda) = \pi_\lambda(f) = \int_{H^n} f(z,s)\pi_\lambda(z,s)\,dz\,ds.$$

In the proof of the above theorem, as well as in the formulation of the following result, the operator $H(\lambda) = -\Delta + \lambda^2|x|^2$ which is the rescaled Hermite operator plays an important role. By $e^{bH(\lambda)}$ we mean the unbounded operator defined on finite linear combinations of the Hermite functions which are dense in $L^2(\mathbb{R}^n)$. See [21] for more about this operator.

THEOREM 1.2. *Suppose f is a measurable function on the Heisenberg group which satisfies the estimate $|f(z,s)| \leq Ce^{-a(|z|^2+|s|)}$ for some $a > 0$. Further assume that for some $b > 0$ the operator $\widehat{f}(\lambda)e^{bH(\lambda)}$ is Hilbert–Schmidt for every $\lambda \neq 0$. Then $f = 0$ whenever $ab > 1/2$.*

There is an analogue of Hardy's theorem for the Fourier transform on H^n proved in [16]. If f satisfies the estimate $|f(z,s)| \leq g(z)e^{-as^2}$ with $g \in L^2(\mathbb{C}^n)$ and if the Fourier transform of f satisfies $\|\widehat{f}(\lambda)\|_{\mathrm{HS}} \leq Ce^{-b\lambda^2}$ then $f = 0$ whenever $ab > 1/4$. As we have observed elsewhere, this is really a theorem for the central variable. Note that neither of these conditions are assumed in our version of the Hardy theorem.

As we have mentioned earlier there is an analogue of Theorem 1.1 for certain Rockland operators. A left invariant differential operator L on H^n is said to be a *Rockland operator* if it is homogeneous with respect to the non-isotropic dilations and $\pi(L)$ is injective on the space of C^∞ vectors for every non-trivial irreducible, unitary representation π of H^n. Under these conditions it is known that L is hypoelliptic. If we further assume that L is non-negative and homogeneous of degree 2 then it generates a diffusion semigroup and the kernel $q_t(z,s)$ satisfies an estimate similar to the one satisfied by the kernel p_t associated with the sublaplacian (see [5]). For a class of such operators we can establish the following result.

THEOREM 1.3. *Let L be a non-negative Rockland operator which is homogeneous of degree two. Assume that L commutes with the sublaplacian $\mathcal{L}$ and there is a positive constant c such that $\pi_\lambda(L) \geq cH(\lambda)$ for all non-zero λ. Let $f \in L^2(H^n)$ be non-trivial and let $q_t(z,s)$ be the heat kernel associated with L. Then there is a constant B such that the estimate $|f * q_t(z,s)| \leq C_t q_{t'}(z,s)$ is not possible for any $t' < Bt$.*

It would be interesting to see if analogues of the above theorem can be established for general second order elliptic operators on Euclidean spaces. The Hardy theorem for Hermite expansions proved in [13] can be restated in terms of the heat kernel for the Hermite operator. Similarly, the Hardy theorem for the Weyl transform proved in [21] can be written in terms of the heat kernel for the special Hermite operator. It is also possible to restate the result of [1] in terms of the heat kernel for the Laplace–Beltrami operator on the solvable group $S = NA$. For a class of elliptic operators on $\mathbb{R}^n$ we have the following result.

THEOREM 1.4. *Let $P = P(x,D)$ be a second order uniformly elliptic differential operator on $\mathbb{R}^n$ which commutes with the Hermite operator and satisfies $P \geq cH$ for some $c > 0$. Let $K_t(x,y)$ be the heat kernel associated with P. Let $f \in L^2(\mathbb{R}^n)$ be non-trivial and let $u(x,t) = e^{-tP}f(x)$. Then there is a constant B such that the estimate $|u(x,t)| \leq CK_{t'}(x,0)$ is not possible for any $t' < B\tanh(ct)$.*

In Section 3 we will give some examples of operators for which the above theorems are true. We prove our main results in the next section. Finally, we remark that Hardy's theorem can be viewed as an uncertainty principle and there are several versions of this principle. We refer the reader to the excellent survey [7].

2. Proofs of the main results. We first take up the proof of Theorem 1.1. The heat kernel $p_t(z,s)$ associated with the sublaplacian is a Schwartz class function whose Fourier transform in the s variable is explicitly known. In fact,

$$\int_{-\infty}^{\infty} e^{is\lambda} p_t(z,s)\,ds = c_n \lambda^n (\sinh(t\lambda))^{-n} e^{-(1/4)(\lambda\coth(t\lambda))|z|^2}.$$

This result has been proved in [8] and [10] by using Brownian motion on the Heisenberg group. As the sublaplacian is homogeneous of degree two it satisfies $p_t(z,s) = t^{-n-1}p_1(t^{-1/2}z, t^{-1}s)$. It is also known that (see e.g. [5])

p_t satisfies the estimate

$$p_t(z,s) \leq Ct^{-n-1}e^{-A(|z|^2+|s|)/t}$$

for a constant $A > 0$. Note that, as opposed to the Euclidean case, the heat kernel decays only like $e^{-A|s|}$ in the central variable. This is a reflection of the fact that the natural dilation structure on H^n is non-isotropic and the sublaplacian is homogeneous with respect to this dilation.

For a function g on H^n we let

$$g^\lambda(z) = \int\limits_{-\infty}^{\infty} g(z,s)e^{is\lambda}\,ds.$$

If $u_t(z,s) = f * p_t(z,s)$ a simple calculation shows that $u_t^\lambda(z) = f^\lambda *_\lambda p_t^\lambda(z)$ where the λ-twisted convolution of two functions φ and ψ on $\mathbb{C}^n$ is defined by

$$\varphi *_\lambda \psi(z) = \int\limits_{\mathbb{C}^n} e^{i\lambda\,\mathrm{Im}(z\overline{w})/2}\varphi(z-w)\psi(w)\,dw.$$

The estimate on u_t given in the hypothesis of Theorem 1.1 leads to the estimate

$$|u_t^\lambda(z)| \leq Cp_{t'}^0(z) \leq C'e^{-|z|^2/(4t')}.$$

We also have the estimate

$$|u_t(z,s)| \leq Ce^{-A(|z|^2+|s|)/t'}$$

and therefore, the function $u_t^\lambda(z)$ extends to a holomorphic function of $\lambda \in \mathbb{C}$ in a strip $|\mathrm{Im}\,\lambda| < A/t'$. Given $0 < t' < t/2$ we can choose $\delta > 0$ such that $t' < t/(e^{t\delta} + e^{-t\delta})$. We will show that $u_t^\lambda(z) = 0$ for $0 < \lambda < 2\delta$, which will force $u_t(z,s) = 0$ and hence $f(z,s) = 0$, thus proving the theorem.

In order to prove that $u_t^\lambda(z) = 0$ for $0 < \lambda < 2\delta$ we use the following variant of a theorem proved in [21] (see Theorem 1.6.5). For $\lambda > 0$ set $\pi_\lambda(z) = \pi_\lambda(z,0)$ where $\pi_\lambda(z,s)$ are the Schrödinger representations of H^n and define

$$\pi_\lambda(g) = \int\limits_{\mathbb{C}^n} g(z)\pi_\lambda(z)\,dz.$$

Note that $\pi_\lambda(f^\lambda) = \widehat{f}(\lambda)$ and when $\lambda = 1$, then $\pi_\lambda(g)$ is called the *Weyl transform* of g. Hence we have the following version of Hardy's theorem for $\pi_\lambda(g)$.

THEOREM 2.1. *Let g be a measurable function on $\mathbb{C}^n$ which satisfies the estimate $|g(z)| \leq Ce^{-a\lambda|z|^2}$. Further assume that $\pi_\lambda(g)e^{bH(\lambda)}$ is Hilbert–Schmidt for some $b > 0$. Then $g = 0$ whenever $a\tanh(b\lambda/2) > 1/4$.*

When $\lambda = 1$ this theorem was announced in [22] and a proof is given in [21] (see Theorem 1.6.5). The proof in the general case is similar and we omit the details.

Once we have the above theorem we can easily complete the proof of Theorem 1.1. A simple calculation shows that

$$\pi_\lambda(u_t^\lambda) = \pi_\lambda(f^\lambda)\pi_\lambda(p_t^\lambda) = \widehat{f}(\lambda)e^{-tH(\lambda)},$$

which implies that $\pi_\lambda(u_t^\lambda)e^{tH(\lambda)}$ is Hilbert–Schmidt. Since we also have the estimate

$$|u_t^\lambda(z)| \le Ce^{-\lambda|z|^2/(4t'\lambda)}$$

we conclude that $u_t^\lambda = 0$ whenever $\tanh(t\lambda/2) > t'\lambda$. But now for $0<\lambda<2\delta$,

$$0 < t'\lambda < \frac{t\lambda}{e^{t\delta}+e^{-t\delta}} < \frac{e^{t\lambda/2}-e^{-t\lambda/2}}{e^{t\lambda/2}+e^{-t\lambda/2}} = \tanh\left(\frac{t\lambda}{2}\right)$$

so that the above condition is satisfied.

This completes the proof of Theorem 1.1, and Theorem 1.2 is proved similarly. We now take up the proof of Theorem 1.3 which is very similar to that of Theorem 1.1. As proved in [5] the heat kernel $q_t(z,s)$ associated with the Rockland operator L satisfies the estimate

$$|q_t(z,s)| \le Ct^{-n-1}e^{-A(|z|^2+|s|)/t}$$

using which we show that $u_t^\lambda(z)$ extends to a holomorphic function of λ in a strip. We will show that for every $0 < b < c$ the operator $\pi_\lambda(u_t^\lambda)e^{btH(\lambda)}$ is Hilbert–Schmidt. Since $|u_t^\lambda(z)| \le Ce^{-A|z|^2/t'}$ we can conclude that $u_t^\lambda = 0$ whenever $\lambda t' < 4A\tanh(bt\lambda/2)$. If $t' < 2Act$ then for some $b < c$ we also have $t' < 2Abt$ and choosing δ small so that

$$\frac{t'}{4A} < \frac{bt}{e^{bt\delta}+e^{-bt\delta}}$$

we can assure that the above condition is satisfied for all $0 < \lambda < 2\delta$. Thus with $B = 2Ac$ we obtain Theorem 1.3.

It remains to be shown that $\pi_\lambda(u_t^\lambda)e^{tbH(\lambda)}$ is Hilbert–Schmidt. As $\widehat{f}(\lambda)$ is a bounded operator it is enough to show that $e^{-t\pi_\lambda(L)}e^{tbH(\lambda)}$ is Hilbert–Schmidt. We can estimate the Hilbert–Schmidt norm of this using the Hermite basis. Let Φ_α be the normalised Hermite functions on $\mathbb{R}^n$. For $\lambda > 0$ define $\Phi_\alpha^\lambda(x) = \lambda^{n/4}\Phi_\alpha(\lambda^{1/2}x)$. Then Φ_α^λ is an eigenfunction of $H(\lambda)$ with eigenvalue $(2|\alpha|+n)\lambda$ and they form an orthonormal basis for $L^2(\mathbb{R}^n)$. Since L commutes with $\mathcal{L}$ so does $\pi_\lambda(L)$ with $H(\lambda)$ because $H(\lambda) = \pi_\lambda(\mathcal{L})$. As $\pi_\lambda(L) \ge cH(\lambda)$ we have

$$(e^{-t(\pi_\lambda(L)-cH(\lambda))}\varphi,\varphi) \le (\varphi,\varphi)$$

for every $\varphi \in L^2(\mathbb{R}^n)$. Applying this to $e^{-ctH(\lambda)/2}\varphi$ and noting that $\pi_\lambda(L)$ commutes with $H(\lambda)$ we get the inequality $\|e^{-t\pi_\lambda(L)}\varphi\|_2 \le \|e^{-ctH(\lambda)}\varphi\|_2$. Taking $\varphi = e^{btH(\lambda)}\Phi_\alpha^\lambda$ we obtain

$$\|e^{-t\pi_\lambda(L)}e^{btH(\lambda)}\Phi_\alpha^\lambda\|_2 \le e^{-(c-b)t(2|\alpha|+n)\lambda}.$$

This estimate proves that $e^{-t\pi_\lambda(L)}e^{btH(\lambda)}$ is Hilbert–Schmidt whenever $b < c$.

This completes the proof of Theorem 1.3. Finally, in order to prove Theorem 1.4 we need the following Hardy's theorem for Hermite expansions (see [13] for a proof).

THEOREM 2.2. *Let f be a measurable function on $\mathbb{R}^n$ which satisfies the estimate $|f(x)| \le Ce^{-a|x|^2}$. Further assume that $|(f, \Phi_\alpha)| \le Ce^{-b(2|\alpha|+n)}$ for every multi-index α. Then $f = 0$ whenever $a\tanh(b) > 1/2$.*

Coming to the proof of Theorem 1.4 we note that the kernel $K_t(x, y)$ of the semigroup e^{-tP} satisfies the estimate

$$|K_t(x, y)| \le Ct^{-n/2}e^{-A|x-y|^2/t}$$

for some constant $A > 0$. This has been proved in [4] as $P(x, D)$ is assumed to be uniformly elliptic. Thus the solution $u_t(x) = u(x, t)$ satisfies the estimate $|u_t(x)| \le Ce^{-A|x|^2/t'}$. In order to estimate (u_t, Φ_α) we note that

$$|(u_t, \Phi_\alpha)| = |(f, e^{-tP}\Phi_\alpha)| \le \|f\|_2\|e^{-tP}\Phi_\alpha\|_2.$$

Since P commutes with H and $P \ge cH$ we conclude that $\|e^{-tP}\Phi_\alpha\|_2 \le e^{-ct(2|\alpha|+n)}$. Consequently, $|(u_t, \Phi_\alpha)| \le Ce^{-ct(2|\alpha|+n)}$ and therefore by Theorem 2.2 we infer that $u_t = 0$ whenever $t' < 2A\tanh(ct)$. This completes the proof of Theorem 1.4.

3. Some examples. In this section we give some examples of differential operators for which Theorems 1.3 and 1.4 are applicable. Consider the Folland–Stein operators $\mathcal{L}_\gamma = \mathcal{L} + i\gamma T$. If γ is admissible, that is, if $|\gamma| \ne 2k + n$ for $k = 0, 1, \ldots,$ then $\mathcal{L}_\gamma$ is a Rockland operator which is homogeneous of degree two. If $|\gamma| < n$ then $\pi_\lambda(\mathcal{L}_\gamma) \ge cH(\lambda)$ holds with $c = 1 - |\gamma|/n$. Consequently, Theorem 1.3 is true for these operators.

We can obtain other examples by considering K-invariant operators where K is a compact subgroup of the unitary group $U(n)$. Recall that the $U(n)$-invariant (also called *radial*) integrable functions on H^n form a commutative Banach algebra under convolution. There are other subgroups K of $U(n)$ for which the K-invariant functions in $L^1(H^n)$ form a commutative subalgebra. The action of K on H^n is defined by $\sigma(z, s) = (\sigma.z, s)$ for $\sigma \in K$. For such K the pair (H^n, K) is called a *Gelfand pair*, and harmonic analysis of K-invariant functions is relatively simpler and can be studied using the so-called K-spherical functions (see [2]). The most well known examples of Gelfand pairs are $(H^n, U(n))$ and $(H^n, T(n))$ where $T(n)$ is the subgroup of $U(n)$ consisting only of diagonal matrices.

The operator $\mathcal{L}$ is $U(n)$-invariant and hence commutes with any K-invariant differential operator. Let L be a K-invariant Rockland operator which is homogeneous of degree two. If the operator L also satisfies the condition $\pi_\lambda(L) \geq cH(\lambda)$ then we have an analogue of Theorem 1.3. For example, when $K = T(n)$, any $T(n)$-invariant operator which is homogeneous of degree two will be of the form $p(\mathcal{L}_1, \ldots, \mathcal{L}_n, T)$ where p is a first degree polynomial and $\mathcal{L}_j = -(X_j^2 + Y_j^2)$ are the partial sublaplacians. The operator $L = \sum_{j=1}^n c_j \mathcal{L}_j$ will satisfy the conditions of Theorem 1.3 with c being the minimum of c_j, $j = 1, \ldots, n$.

It would be interesting to see if the second condition, namely that $\pi_\lambda(L) \geq cH(\lambda)$, automatically holds for Rockland operators of homogeneity two. This is the case in the Euclidean setup: a homogeneous, constant coefficient differential operator is hypoelliptic if and only if it is elliptic and consequently if it is of degree two then the symbol satisfies the estimate $|p(i\xi)| \geq c|\xi|^2$. We do not know if something similar holds for Rockland operators.

By transferring Rockland operators on H^n by a unitary representation π we can obtain operators on $\mathbb{R}^n$. To be precise, let us rename the vector fields Y_j by calling them X_{j+n} so that X_j, $j = 1, \ldots, 2n$, together with T generate the Heisenberg Lie algebra. If $L = \sum_{i,j} a_{i,j} X_i X_j$ is a Rockland operator on H^n then $\pi_1(L)$, where $\pi_1(z, s)$ is the Schrödinger representation with parameter $\lambda = 1$, is an operator of the form

$$P(x, D) = \sum_{i,j} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j c_j(x) \frac{\partial}{\partial x_j} + a(x)$$

where c_j and a are polynomials.

If $p_t(z, s)$ and $K_t(\xi, \eta)$ are the heat kernels of L and $\pi_1(L)$ then they are related by

$$K_t(\xi, \eta) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} p_t(x + i(\eta - \xi), s) e^{is} e^{(i/2)x(\xi+\eta)} \, dx \, ds.$$

From this we see that whenever p_t has an exponential decay so does K_t. If L commutes with $\mathcal{L}$ then $\pi_1(L)$ commutes with H. If we further assume that $\pi_1(L) \geq cH$ then Theorem 1.4 holds for $\pi_1(L)$. It would be interesting to see for what other operators we have an analogue of Theorem 1.4.

Added in proof (September 2000). Recently the author has proved that Theorem 1.1 is valid for $0 < t' < t$ and Theorem 1.2 for all $ab > 1/4$.

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Received 15 March 2000 (3898)