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COMPACTNESS AND CONVERGENCE OF SET-VALUED MEASURES

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Abstract. We prove criteria for relative compactness in the space of set-valued measures whose values are compact convex sets in a Banach space, and we generalize to set-valued measures the famous theorem of Dieudonné on convergence of real non-negative regular measures.

Introduction. We expand and complete the two results presented in [9, Theorem 9] and [10, Theorem 4]. Let T be an abstract set, let $\mathcal{B}$ be a σ-field of subsets of T, and let $\mathcal{K}$ be a family of subsets of T closed under finite unions and finite intersections. Let $\widetilde{M_+}(T,\mathcal{K},\mathrm{ck}(E))$ be the set of all positive $\mathcal{K}$-inner regular set-valued measures defined on $\mathcal{B}$ with values in $\mathrm{ck}(E)$ where $\mathrm{ck}(E)$ is the set of all compact convex non-empty subsets of a Banach space E. We consider on $\mathrm{ck}(E)$ the Hausdorff distance and on $\widetilde{M_+}(T,\mathcal{K},\mathrm{ck}(E))$ the s-topology, that is, the weakest topology for which all mappings $M \mapsto M(A)$, $A \in \mathcal{B}$, are continuous. We prove criteria of compactness of subsets of $\widetilde{M_+}(T,\mathcal{K},\mathrm{ck}(E))$ (Theorems 1–3) and we generalize to set-valued measures (Theorem 4) the famous theorem of Dieudonné [4, Proposition 8] on convergence of real non-negative regular measures. Theorems 2 and 3 are known for real non-negative measures (see e.g. [12]). This paper is a continuation of [11].

1. Notations and preliminaries. Throughout this paper, T denotes an abstract set, and $\mathcal{G}$ and $\mathcal{K}$ denote families of subsets of T. We let $\mathcal{B}$ denote the smallest σ-field containing every set $A \subseteq T$ for which $K \cap A \in \mathcal{K}$ for all $K \in \mathcal{K}$. The family $\mathcal{K}$ is said to be *semicompact* if every countable family of sets in $\mathcal{K}$ with the finite intersection property has a non-empty intersection. We shall say that $\mathcal{G}$ *separates the sets* in $\mathcal{K}$ if for any pair K, K' of disjoint sets in $\mathcal{K}$ we can find a pair G, G' of disjoint sets in $\mathcal{G}$ such that $K \subset G$ and $K' \subset G'$.

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A set $\mathcal{F}$ of subsets of T is *filtering to the left* if for all $F \in \mathcal{F}$ and $F' \in \mathcal{F}$ there exists $F_1 \in \mathcal{F}$ such that $F_1 \subseteq F \cap F'$. We write $\mathcal{F} \downarrow F_0$ if $\mathcal{F}$ is filtering to the left and $F_0 = \bigcap\{F;\, F \in \mathcal{F}\}$.

1.1. *Nets on T.* Let X be a non-empty subset of T and $(x_i)_{i\in I}$ be a net on T. We say that $x_i \in X$ *eventually* if there exists $i \in I$ such that $x_j \in X$ for every $j \in I$ with $j \geq i$. A net $(x_i)_{i\in I}$ on T is *universal* if, for every subset $X \subset T$, either $x_i \in X$ eventually or $x_i \in T \setminus X$ eventually.

Note that a net corresponding to an ultrafilter is universal. Conversely, the filter corresponding to a universal net is an ultrafilter. Every net has a universal subnet. For a more detailed account of nets we refer to [7]. Let T' be a Hausdorff topological space. A subset X of T' is called *net-compact* if every net on X has a convergent subnet, i.e. if every universal net on X converges. In case T' is a regular topological space, a subset X of T' is net-compact if and only if X is relatively compact.

1.2. *Set-valued measures.* Let E be a Banach space and E' its dual space. We denote by $|\cdot|$ the norm on E and E'. The closed unit ball of E', denoted by $B'(0,1)$, is $\{y;\, y \in E',\, |y| \leq 1\}$. If F and G are two subsets of E, we shall denote by $F + G$ the family of all elements of the form $x + y$ with $x \in F$ and $y \in G$, and by $F \bar{+} G$ the closure of $F + G$. The closed convex hull of F is denoted by $\overline{\mathrm{co}}\, F$. The *support function* of F is the function $\delta^*(\cdot|F)$ from E' to $[-\infty, +\infty]$ defined by

$$\delta^*(y|F) = \sup\{y(x);\, x \in F\}.$$

We denote by $\mathrm{cf}(E)$ the set of all closed convex non-empty subsets of E, and by $\mathrm{ck}(E)$ the set of all compact convex non-empty subsets of E. We endow $\mathrm{ck}(E)$ with the Hausdorff distance, denoted by δ. Recall that for C and C' in $\mathrm{ck}(E)$, $\delta(C, C') = \sup\{|\delta^*(y|C) - \delta^*(y|C')|;\, y \in B'(0,1)\}$ and that $(\mathrm{ck}(E), \delta)$ is a complete metric space [2].

DEFINITION 1. Let $\mathcal{A}$ be a set of subsets of T. Assume that $\emptyset \in \mathcal{A}$ and that $\mathcal{A}$ is closed under finite unions and finite intersections. Let M be a map from $\mathcal{A}$ to $\mathrm{cf}(E)$. Then M is

(a) *additive* if for any disjoint sets A, B in $\mathcal{A}$ we have $M(A \cup B) = M(A) \bar{+} M(B)$; that is, $M(A) = \mathrm{closure}\{a + b;\, a \in M(A),\, b \in M(B)\}$;
(b) *monotone* if $A \subseteq B$ implies $M(A) \subseteq M(B)$, and $M(\emptyset) = \{0\}$;
(c) *subadditive* if $M(A \cup B) \subseteq M(A) \bar{+} M(B)$ for all A, B in $\mathcal{A}$;
(d) *positive* if $M(\emptyset) = \{0\}$ and if $\{0\} \subseteq M(A)$ for all $A \in \mathcal{A}$;
(e) *σ-smooth* with respect to $\mathcal{K}$ if M is monotone and for all countable subsets $\mathcal{K}^*$ of $\mathcal{K}$ the conditions $\mathcal{K}^* \downarrow A_0$ and $A_0 \in \mathcal{A}$ imply $M(A_0) = \bigcap\{M(A);\, \exists K \in \mathcal{K}^*, A \supseteq K\}$.

If we only require the last relation to hold when $A_0 = \emptyset$, then we shall say that M is *σ-smooth at $\emptyset$* with respect to $\mathcal{K}$. In the case of a real non-negative measure μ the previous equality is replaced by $\mu(A_0) = \inf\{\mu(A);\ \exists K \in \mathcal{K}^*,\ A \supseteq K\}$.

Definition 2. Let M be a map from $\mathcal{B}$ to $\mathrm{cf}(E)$. We say that M is a *weak set-valued measure* if for every $y \in E'$ the map $\delta^*(y|M(\cdot)) : \mathcal{B} \to]-\infty, +\infty]$ is a σ-additive measure.

Note that if M maps into $\mathrm{ck}(E)$ this condition is equivalent to the following: for any sequence (A_n) of pairwise disjoint sets in $\mathcal{B}$ of union A the series $\sum_n M(A_n)$ is convergent and $M(A) = \sum_n M(A_n)$; that is, $M(A) = \lim_{n\to\infty} \sum_{k=0}^{n} M(A_k)$ where the limit is taken with respect to the Hausdorff topology [3]. A weak set-valued measure from $\mathcal{B}$ to $\mathrm{ck}(E)$ will be called a *set-valued measure*.

Definition 3. A positive weak set-valued measure $M : \mathcal{B} \to \mathrm{cf}(E)$ is said to be *inner regular with respect to $\mathcal{K}$* or *$\mathcal{K}$-inner regular* if $M(A) = \overline{\mathrm{co}} \bigcup\{M(K);\ K \subseteq A,\ K \in \mathcal{K}\}$ for any $A \in \mathcal{B}$.

In the case of a real non-negative measure μ defined on $\mathcal{B}$ this equality is replaced by $\mu(A) = \sup\{\mu(K);\ K \subseteq A,\ K \in \mathcal{K}\}$.

Note that a positive additive map $M : \mathcal{B} \to \mathrm{cf}(E)$ is monotone. Indeed, let A and B be elements of $\mathcal{B}$ such that $A \subset B$. Then $M(B) = M(A) \bar{+} M(B \setminus A)$ and for all $x \in M(A)$, $x = x + 0 \in M(B)$ where $0 \in M(B \setminus A)$. Hence $M(A) \subset M(B)$.

1.3. *Topologies on $\widetilde{M}_+(T, \mathrm{ck}(E))$.* We denote by $\widetilde{M}_+(T, \mathrm{ck}(E))$ [resp. $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$] the set of all positive [resp. positive $\mathcal{K}$-inner regular] set-valued measures defined on $\mathcal{B}$. In the case of real non-negative measures we shall use the notations $M_+(T)$, $M_+(T, \mathcal{K})$ respectively.

Definition 4. The *narrow topology* on $\widetilde{M}_+(T, \mathrm{ck}(E))$ is the weakest topology on $\widetilde{M}_+(T, \mathrm{ck}(E))$ for which the map $M \mapsto M(T)$ is continuous and the maps $M \mapsto \delta^*(y|M(G))$ are lower semicontinuous for all $G \in \mathcal{G}$ and $y \in E'$.

Let $(M_i)_{i\in I}$ be a net on $\widetilde{M}_+(T, \mathrm{ck}(E))$ and $M \in \widetilde{M}_+(T, \mathrm{ck}(E))$. Then (M_i) *converges narrowly* to M, i.e. converges in the narrow topology, if and only if $(M_i(T))$ converges to $M(T)$ in $\mathrm{ck}(E)$ and $\liminf_i \delta^*(y|M_i(G)) \geq \delta^*(y|M(G))$ for all $y \in E'$ and $G \in \mathcal{G}$.

The narrow topology on $M_+(T)$ is the weakest topology on $M_+(T)$ for which the map $\mu \mapsto \mu(T)$ is continuous and the maps $\mu \mapsto \mu(G)$ are lower semicontinuous for all $G \in \mathcal{G}$.

Definition 5. The *s-topology* on $\widetilde{M}_+(T,\mathrm{ck}(E))$ is the weakest topology on $\widetilde{M}_+(T,\mathrm{ck}(E))$ for which all maps $M \mapsto M(A)$, $A \in \mathcal{B}$, are continuous.

$\widetilde{M}_+(T,\mathrm{ck}(E))$ endowed with this topology is a Hausdorff space. The s-topology is a uniform topology. The uniformity is generated by the family of pseudo-metrics $(h_A)_{A\in\mathcal{B}}$ where $h_A(M,M') = \delta(M(A),M'(A))$ for all M and M' in $\widetilde{M}_+(T,\mathrm{ck}(E))$.

The s-topology on $M_+(T)$ is defined analogously. $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ [resp. $M_+(T,\mathcal{K})$] will be endowed with the relative topology generated by the topology considered on $\widetilde{M}_+(T,\mathrm{ck}(E))$ [resp. $M_+(T)$].

Let $K_0 \in \mathrm{ck}(E)$ with $0 \in K_0$, and let $\mu \in M_+(T,\mathcal{K})$. Denote by $\mu \otimes K_0$ the set-valued measure defined by $\mu \otimes K_0(A) = \mu(A)K_0$ for all $A \in \mathcal{B}$. Consider $M_+(T,\mathcal{K})$ [resp. $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$] with the s-topology. By identifying $M_+(T,\mathcal{K})$ with the closed subspace $\{\mu\otimes K_0;\ \mu \in M_+(T,\mathcal{K})\}$ of $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$, the results of Topsøe ([12, Lemma 4, p. 208; Theorem 8, p. 209; Corollary 3, p. 211]) may be regarded as particular cases of our Theorems 2–4.

2. Preliminary results. Consider now the following axioms on $\mathcal{G}$ and $\mathcal{K}$. These are the same as those of Topsøe [12].

(I) $\mathcal{K}$ is closed under finite unions and countable intersections and $\emptyset \in \mathcal{K}$.
(II) $\mathcal{G}$ is closed under finite unions and finite intersections and $\emptyset \in \mathcal{G}$.
(III) $K \setminus G \in \mathcal{K}$ for all $K \in \mathcal{K}$ and $G \in \mathcal{G}$.
(IV) $\mathcal{G}$ separates the sets in $\mathcal{K}$.
(V) $\mathcal{K}$ is semicompact.

Note that axioms (I) and (IV) imply that "$\mathcal{G}$ dominates $\mathcal{K}$": for every $K \in \mathcal{K}$ there exists $G \in \mathcal{G}$ such that $G \supset K$. In the following, sets denoted by the letters K, G, A are elements of $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{B}$, respectively.

Lemma 1. *Let T be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be sets of subsets of T. Assume that axioms* (I)–(IV) *are satisfied, and consider a finite non-negative $\mathcal{K}$-inner regular measure μ defined on $\mathcal{B}$. Then*

$$\mu(K) = \inf\{\mu(G);\ G \supseteq K,\ G \in \mathcal{G}\} \quad \textit{for all } K \in \mathcal{K}.$$

Proof. Let $K \in \mathcal{K}$. We have $\mu(K) = \mu(T) - \mu(T\setminus K) = \mu(T) - \sup\{\mu(K');\ K' \subseteq T\setminus K,\ K' \in \mathcal{K}\}$. For a given $\varepsilon > 0$ choose K'_ε such that $-\mu(T\setminus K) \geq -\mu(K'_\varepsilon) - \varepsilon$. Since $\mathcal{G}$ separates the sets in $\mathcal{K}$ we may find a pair G, G' of disjoint sets in $\mathcal{G}$ such that $G \supset K$ and $G' \supset K'_\varepsilon$. We have $\mu(T) \geq \mu(G\cup G') = \mu(G) + \mu(G')$. On the other hand, $-\mu(T\setminus K) \geq -\mu(G') - \varepsilon$. Hence $\mu(K) = \mu(T) - \mu(T\setminus K) \geq \mu(G) - \varepsilon$. It follows that $\mu(K) \geq \inf\{\mu(G);\ G \supseteq K,\ G \in \mathcal{G}\}$. The converse inequality is obvious.

Lemma 2. *Let T be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be sets of subsets of T. Assume that axioms* (I)–(IV) *are satisfied, and consider a positive $\mathcal{K}$-inner regular set-valued measure M defined on $\mathcal{B}$. Then*

$$M(K) = \bigcap\{M(G);\, G \in \mathcal{G},\, G \supseteq K\} \quad \textit{for all } K \in \mathcal{K}.$$

Proof. Let $y \in E'$. By applying Lemma 1 to the measure $\delta^*(y|M(\cdot))$ we have $\delta^*(y|M(K)) = \inf\{\delta^*(y|M(G));\, G \in \mathcal{G},\, G \supseteq K\}$ for all $K \in \mathcal{K}$. By [11, Lemmas 2–3], $\delta^*(y|M(K)) = \delta^*(y|\bigcap_{G\supseteq K} M(G))$. Hence $M(K) = \bigcap\{M(G);\, G \supseteq K,\, G \in \mathcal{G}\}$ ([11, Lemmas, 1–2]).

Lemma 3. *Let $\mu \in M_+(T,\mathcal{K})$ let $(\mu_i)_{i\in I}$ be a net on $M_+(T,\mathcal{K})$, and let $(G_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{G}$ such that* $\lim_i \mu_i(G_k) = \mu(G_k)$ *for every k and* $\lim_i \mu_i(\bigcup_j G_j) = \mu(\bigcup_j G_j)$. *Then*

$$\lim_i \sum_{k=0}^{\infty} |\mu_i(G_k) - \mu(G_k)| = 0.$$

Proof. Fix $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that $\sum_{k\geq p} \mu(G_k) < \varepsilon/3$. Since $\lim_i \mu_i(G_k) = \mu(G_k)$ for each $k \in \mathbb{N}$ and $\lim_i \mu_i(\bigcup_i G_k) = \mu(\bigcup_k G_k)$ there is i_0 such that for all $i \in I$ and $i \geq i_0$,

$$\sum_{k=0}^{p} |\mu_i(G_k) - \mu(G_k)| \leq \varepsilon/3 \quad \text{and} \quad \Big|\sum_{k=0}^{\infty}(\mu_i(G_k) - \mu(G_k))\Big| \leq \varepsilon/3.$$

Then

$$\begin{aligned}\sum_{k\geq p} \mu_i(G_k) &= \sum_{k=0}^{\infty}[\mu_i(G_k) - \mu(G_k)] - \sum_{k=0}^{p-1}[\mu_i(G_k) - \mu(G_k)] + \sum_{k\geq p} \mu(G_k)\\ &\leq \Big|\sum_{k=0}^{\infty}[\mu_i(G_k) - \mu(G_k)]\Big| + \sum_{k=0}^{p} |\mu_i(G_k) - \mu(G_k)| + \sum_{k\geq p} \mu(G_k)\\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{\infty} |\mu_i(G_k) - \mu(G_k)| &= \sum_{k=0}^{p} |\mu_i(G_k) - \mu(G_k)| + \sum_{k\geq p+1} |\mu_i(G_k) - \mu(G_k)|\\ &\leq \sum_{k=0}^{p} |\mu_i(G_k) - \mu(G_k)| + \sum_{k\geq p+1} \mu_i(G_k) + \sum_{k\geq p+1} \mu(G_k)\\ &\leq \varepsilon/3 + \varepsilon + \varepsilon/3 < 2\varepsilon.\end{aligned}$$

Lemma 4. *Consider on $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ the narrow topology. Let H be a subset of $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ such that $\{M(G);\, M \in H\}$ is relatively compact in $\mathrm{ck}(E)$ for every $G \in \mathcal{G}$. If H is net-compact, then so is $\delta(H) =$*

$\{\delta^*(y|M(\cdot)); M \in H, y \in E', |y| \leq 1\}$ *in the space* $M_+(T,\mathcal{K})$ *endowed with the narrow topology.*

Proof. Let $(\delta^*(y_i|M_i(\cdot)))_{i\in I}$ be a net on $\delta(H)$. By assumption the net (M_i) has a subnet which converges to $M \in \widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$. The closed unit ball $B'(0,1)$ is a compact subset of E' endowed with the weak topology $\sigma(E',E)$. Hence the net (y_i) has a subnet which converges to $y \in B'(0,1)$. We may assume that these subnets have the same indices. Assume for simplicity that the nets (M_i) and (y_i) converge to M and y respectively. Consider now a universal subnet (M_{i_k}) of (M_i). We will prove that $(\delta^*(y_{i_k}|M_{i_k}(\cdot)))$ converges narrowly to $\delta^*(y|M(\cdot))$. We have

$$|\delta^*(y_{i_k}|M_{i_k}(T)) - \delta^*(y|M(T))| \leq \sup_{|y|\leq 1}(|\delta^*(y|M_{i_k}(T)) - \delta^*(y|M(T))|) + |\delta^*(y_{i_k}|M(T)) - \delta^*(y|M(T))|.$$

Since (M_{i_k}) converges narrowly to M, the subnet $(M_{i_k}(T))$ converges to $M(T)$ in $\mathrm{ck}(E)$. On the other hand, $(\delta^*(y_{i_k}|M(T)))$ converges to $\delta^*(y|M(T))$ because the map $\delta^*(\cdot|M(T)) : B'(0,1) \to \mathbb{R}$ is continuous for the restriction of $\sigma(E',E)$ to $B'(0,1)$. Hence it follows from the previous inequality that $(\delta^*(y_{i_k}|M_{i_k}(T)))$ converges to $\delta^*(y|M(T))$. Let $G \in \mathcal{G}$; since $\{M(G);\ M \in H\}$ is relatively compact, the universal subnet $(M_{i_k}(G))$ converges to an element C of $\mathrm{ck}(E)$. Using once again the previous arguments we infer that $(\delta^*(y_{i_k}|M_{i_k}(G)))$ converges to $\delta^*(y|C)$. Since (M_{i_k}) converges narrowly to M and $(M_{i_k}(G))$ converges to C we then have $\liminf_k \delta^*(y|M_{i_k}(G)) \geq \delta^*(y|M(G))$ and $\lim_k \delta^*(y_{i_k}|M_{i_k}(G)) = \delta^*(y|C)$. Hence $\liminf_k \delta^*(y_{i_k}|M_{i_k}(G)) \geq \delta^*(y|M(G))$. This ends the proof.

Lemma 5. *Let* E *be a Banach space, let* T *be an abstract set, and let* $\mathcal{G}$ *and* $\mathcal{K}$ *be sets of subsets of* T. *Assume that* $\mathcal{G}$ *and* $\mathcal{K}$ *satisfy axioms* (I)–(V). *Let* H *be a subset of* $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ *such that for any sequence* $(G_n)_{n\geq 1}$ *of pairwise disjoint sets in* $\mathcal{G}$ *we have* $\lim_{n\to\infty} M(G_n) = \{0\}$ *uniformly with respect to* $M \in H$. *Then*

$$\forall K \in \mathcal{K} \qquad \inf_{G\supseteq K} \sup\{\delta^*(y|M(G \setminus K));\ M \in H,\ y \in E',\ |y| \leq 1\} = 0.$$

Proof. Assume that there exist $K_0 \in \mathcal{K}$ and $\varepsilon > 0$ such that for every $G \in \mathcal{G}$ with $G \supset K_0$ we may find $M_G \in H$ and $y_G \in B'(0,1)$ which satisfy $\delta^*(y_G|M_G(G \setminus K_0)) > \varepsilon$. We will construct by induction a decreasing sequence (G_n), a sequence (G'_n) of pairwise disjoint sets, a sequence (M_n) in H, and a sequence (y_n) in $B'(0,1)$ such that for all $n \geq 1$ we have $K_0 \subset G_n$, $G'_n \subset G_n \setminus G_{n+1}$, $\delta^*(y_n|M_n(G_n \setminus K_0)) > \varepsilon$ and $\delta^*(y_n|M_n(G'_n)) > \varepsilon$. This last inequality contradicts the condition $\lim_{n\to\infty} M(G'_n) = \{0\}$ uniformly with respect to $M \in H$.

Assume that the construction is made up to rank n. Choose $G_{n+1} \supset K_0$ (G_{n+1} exists since $\mathcal{G}$ dominates $\mathcal{K}$), $M_{n+1} \in H$, and $y_{n+1} \in B'(0,1)$ such that $\delta^*(y_{n+1}|M_{n+1}(G_{n+1} \setminus K_0)) > \varepsilon$. Since M_{n+1} is $\mathcal{K}$-inner regular, so is $\delta^*(y_{n+1}|M_{n+1}(\cdot))$. Hence there exists $K_{n+1} \subset G_{n+1} \setminus K_0$ with $\delta^*(y_{n+1}|M_{n+1}(K_{n+1})) > \varepsilon$. By axiom (IV) there exist G_{n+2} and G'_{n+1} such that $K_0 \subset G_{n+2}$, $K_{n+1} \subset G'_{n+1}$ and $G_{n+2} \cap G'_{n+1} = \emptyset$. Clearly we may assume that $G_{n+2} \subset G_{n+1}$ and $G'_{n+1} \subset G_{n+1} \setminus G_{n+2}$. We have $\delta^*(y_{n+1}|M_{n+1}(G'_{n+1})) > \varepsilon$.

Proposition 1. *Let T be an abstract set, and let $\mathcal{K}$ and $\mathcal{G}$ be sets of subsets of T. Assume that axioms* (I)–(V) *hold, and consider the space $M_+(T,\mathcal{K})$ with the s-topology. Let H be a subset of $M_+(T,\mathcal{K})$ such that* $\sup\{m(T);\, m \in H\} < \infty$. *Then the following four conditions are equivalent*:

(1) *H is relatively compact.*

(2) (i) $\forall K \in \mathcal{K}\ \inf_{G \supseteq K} \sup_{m \in H} m(G \setminus K) = 0$,

(ii) $\forall A \in \mathcal{B}\ \inf_{K \subseteq A} \sup_{m \in H} m(A \setminus K) = 0$.

(3) (i) $\forall K \in \mathcal{K}\ \inf_{G \supseteq K} \sup_{m \in H} m(G \setminus K) = 0$,

(ii) $\inf_{K \in \mathcal{K}} \sup_{m \in H} m(T \setminus K) = 0$.

(4) (i) $\forall K \in \mathcal{K}\ \inf_{G \supseteq K} \sup_{m \in H} m(G \setminus K) = 0$,

(ii) *H is net-compact in the narrow topology.*

Proof. The proposition results from [12, Theorem 7, p. 207]. Indeed, we consider the net on $M_+(T,\mathcal{K})$ defined by the identity map id : $H \to H$ where the domain of id is given the "diffuse" ordering: $m \leq m'$ for any pair of measures in H. (It is the reasoning that is used in [12] for the proof of Corollary 2, pp. 203–204.)

3. Main results

Theorem 1 ([9]). *Let E be a Banach space, let T be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be sets of subsets of T. Assume that axioms* (I)–(V) *hold. Consider the space $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ of positive $\mathcal{K}$-inner regular set-valued measures defined on $\mathcal{B}$ with the s-topology. Then a subset $H \subset \widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ is relatively compact if and only if the following conditions are satisfied.*

(i) $\delta(H) = \{\delta^*(y|M(\cdot));\, M \in H,\, y \in E',\, |y| \leq 1\}$ *is relatively compact in the space $M_+(T,\mathcal{K})$ endowed with the s-topology.*

(ii) $\{M(T);\, M \in H\}$ *and* $\{M(G);\, M \in H\}$ *for any $G \in \mathcal{G}$ are relatively compact in the space* $\mathrm{ck}(E)$.

Proof. Assume that H is relatively compact. Since the maps $M \mapsto M(A)$, $A \in \mathcal{B}$, are continuous on $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$, (ii) is satisfied. Consider the closed unit ball $B'(0,1) = \{y;\, y \in E',\, |y| \leq 1\}$ of the dual space

E' of E with the relative topology defined by the weak topology $\sigma(E', E)$ of E', and consider $B'(0,1) \times \widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ with its product topology. For any $K \in \mathrm{ck}(E)$ the map $\delta^*(\cdot|K) : B'(0,1) \to \mathbb{R}$ is continuous. For all $A \in \mathcal{B}$, $M, M' \in \widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ and $y, y' \in E'$ we have

$$\begin{aligned}|\delta^*(y|M(A)) - \delta^*(y'|M'(A))| \le \sup_{|y|\le 1} &|\delta^*(y|M(A)) - \delta^*(y|M'(A))| \\ &+ |\delta^*(y|M'(A)) - \delta^*(y'|M'(A))|.\end{aligned}$$

It follows that the map $\theta : B'(0,1) \times \widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E)) \to M_+(T, \mathcal{K})$, $(y, M) \mapsto \theta(y, M) = \delta^*(y|M(\cdot))$, is continuous. Denote by $\overline{H}$ the closure of H. Then $B'(0,1) \times \overline{H}$ is a compact subset of the product space $B'(0,1) \times \widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$. We have $\delta(H) \subset \theta(B'(0,1) \times \overline{H})$, hence $\delta(H)$ is a relatively compact subset of $M_+(T, \mathcal{K})$.

Let us now prove the sufficiency. Assume that (i) and (ii) hold. To show that H is a relatively compact subset of $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ it suffices to prove that every universal net on H is convergent. Let $(M_i)_{i\in I}$ be a universal net on H. According to (ii) the universal nets $(M_i(T))$ and $(M_i(G))$ are convergent in $\mathrm{ck}(E)$. Put $N(G) = \lim_i M_i(G)$ for all $G \in \mathcal{G}$, and $\widetilde{M}(A) = \overline{\mathrm{co}} \bigcup_{K\subseteq A} \bigcap_{G\supseteq K} N(G)$ for all $A \in \mathcal{B}$. It is clear that $N(G) \subseteq \lim_i M_i(T)$ for all $G \in \mathcal{G}$, and $\widetilde{M}(A) \in \mathrm{ck}(E)$.

By [11, Theorem 2], the map $\widetilde{M} : \mathcal{B} \to \mathrm{ck}(E)$ is in $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$. For all $y \in E'$ we have

$$\delta^*(y|\widetilde{M}(A)) = \sup_{K\subseteq A} \inf_{G\supseteq K} \delta^*(y|N(G))$$

([11, Lemma 1–3]) and $\delta^*(y|N(G)) = \lim_i \delta^*(y|M_i(G))$. By (i) the universal net $(\delta^*(y|M_i(\cdot)))_{i\in I}$ on $\delta(H)$ is convergent in the space $M_+(T, \mathcal{K})$. Denote by m_y its limit. For all $A \in \mathcal{B}$, $\lim_i \delta^*(y|M_i(A)) = m_y(A)$, and for all $K \in \mathcal{K}$,

$$\begin{aligned}\delta^*(y|\widetilde{M}(K)) &= \inf_{G\supseteq K} \lim_i \delta^*(y|M_i(G)) \\ &= \inf_{G\supseteq K} \{\lim_i \delta^*(y|M_i(K)) + \lim_i \delta^*(y|M_i(G \setminus K))\} \\ &= \lim_i \delta^*(y|M_i(K)) \qquad \text{(Prop. 1(2)(i))}.\end{aligned}$$

For any $A \in \mathcal{B}$ we have

$$\begin{aligned}\delta^*(y|\widetilde{M}(A)) &= \sup_{K\subseteq A} \delta^*(y|\widetilde{M}(K)) = \sup_{K\subseteq A} \lim_i \delta^*(y|M_i(K)) = \sup_{K\subseteq A} m_y(K) \\ &= m_y(A) = \lim_i \delta^*(y|M_i(A)).\end{aligned}$$

We have just proved that $\lim_i \delta^*(y|M_i(A)) = \delta^*(y|\widetilde{M}(A))$ for all $A \in \mathcal{B}$ and $y \in B'(0,1)$. To finish the proof it suffices to show that for every $A \in \mathcal{B}$, $(M_i(A))$ is convergent in $\mathrm{ck}(E)$. Since $\mathrm{ck}(E)$ is a complete metric space it suffices to prove that $(M_i(A))$ is a Cauchy net. If this were not so, there would exist $B_0 \in \mathcal{B}$ and $\varepsilon_0 > 0$ such that for every $i \in I$ we would be able to find $k_i, j_i \in I$ with $k_i, j_i \geq i$ and $y_i \in B'(0,1)$ such that $|\delta^*(y_i|M_{k_i}(B_0)) - \delta^*(y_i|M_{j_i}(B_0))| > \varepsilon_0$. Consider now the nets $(\delta^*(y_i|M_{k_i}(\cdot))_{i\in I}$ and $(\delta^*(y_i|M_{j_i}(\cdot))_{i\in I}$. According to (i) each of them admits a convergent subnet. We may choose the subnets with the same indices. Assume for simplicity that the nets $(\delta^*(y_i|M_{k_i}(\cdot)))$ and $(\delta^*(y_i|M_{j_i}(\cdot)))$ converge to μ and μ' respectively. Since for every $G \in \mathcal{G}$ the universal net $(M_i(G))$ is convergent in $\mathrm{ck}(E)$, one has

$$\lim_i \sup_{|y|\leq 1} |\delta^*(y|M_{k_i}(G)) - \delta^*(y|M_{j_i}(G))| = 0.$$

On the other hand,

$$|\delta^*(y_i|M_{k_i}(G)) - \delta^*(y_i|M_{j_i}(G))| \leq \sup_{|y|\leq 1} |\delta^*(y|M_{k_i}(G)) - \delta^*(y|M_{j_i}(G))|.$$

Then

$$\mu(G) = \lim_i \delta^*(y_i|M_{k_i}(G)) = \lim_i \delta^*(y_i|M_{j_i}(G)) = \mu'(G) \quad \text{for all } G \in \mathcal{G}.$$

Hence $\mu(K) = \mu'(K)$ for all $K \in \mathcal{K}$ (Lemma 1). Since μ and μ' are $\mathcal{K}$-inner regular we conclude that $\mu = \mu'$ and that there exists $i_0 \in I$ such that $|\delta^*(y_i|M_{k_i}(B_0)) - \delta^*(y_i|M_{j_i}(B_0))| \leq \varepsilon_0/2$ for all $i \in I$ with $i \geq i_0$. That is a contradiction.

THEOREM 2 ([10]). *Let E be a Banach space, let T be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be sets of subsets of T. Assume that $\mathcal{G}$ and $\mathcal{K}$ satisfy axioms* (I)–(V) *and consider the space $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ with the s-topology. Then a subset H of $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ is relatively compact if and only if the following three conditions hold*:

(i) *H is net-compact in the narrow topology*,
(ii) *$\{M(G);\ M \in H\}$ for all $G \in \mathcal{G}$ and $\{M(T);\ M \in H\}$ are relatively compact in* $\mathrm{ck}(E)$.
(iii) *For any sequence $(G_n)_{n\geq 1}$ of pairwise disjoint sets in $\mathcal{G}$ we have $\lim_{n\to\infty} M(G_n) = \{0\}$ uniformly with respect to M in H.*

Proof. Assume that H is relatively compact. Then (i) is obvious. Since for each $A \in \mathcal{B}$ the map $M \mapsto M(A)$ from $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ to $\mathrm{ck}(E)$ is continuous, (ii) holds. If (iii) did not hold we would be able to find a sequence $(G_n)_{n\geq 1}$ of pairwise disjoint sets, an $\varepsilon > 0$, a sequence (M_n) of set-valued measures in H, and a sequence (y_n) in $B'(0,1)$ such that $\delta^*(y_n|M_n(G_n)) \geq \varepsilon$ for all $n \geq 1$. Put $\mu_n = \delta^*(y_n|M_n(\cdot))$ for all $n \geq 1$. By Theorem 1, $\delta(H) = \{\delta^*(y|M(\cdot));\ M \in H,\ y \in E',\ |y| \leq 1\}$ is a relatively compact subset of $M_+(T,\mathcal{K})$ in the s-topology. Then the sequence (μ_n) has a cluster point μ. Hence there exists a subnet (μ_{n_i}) of (μ_n) which converges to μ. In particular,

$(\mu_{n_i}(G_k))$ converges to $\mu(G_k)$ for all $k \geq 1$, and $(\mu_{n_i}(\bigcup_j G_j))$ converges to $\mu(\bigcup_j G_j)$. By Lemma 3, $\sum_{k=1}^{\infty} |\mu_{n_i}(G_k) - \mu(G_k)| \to 0$. Choose k_0 such that $\mu(G_k) < \varepsilon/2$ for all $k \geq k_0$ and choose i such that $n_i \geq k_0$ and $\sum_{k=1}^{\infty} |\mu_{n_i}(G_k) - \mu(G_k)| < \varepsilon/2$. We have

$$\mu_{n_i}(G_{n_i}) \leq |\mu_{n_i}(G_{n_i}) - \mu(G_{n_i})| + \mu(G_{n_i}) \leq \sum_{k=1}^{\infty} |\mu_{n_i}(G_k) - \mu(G_k)| + \mu(G_{n_i}) < \varepsilon.$$

That is a contradiction.

Let us now prove that H is relatively compact if (i)–(iii) hold. According to Theorem 1 it suffices to prove that $\delta(H) = \{\delta^*(y|M(\cdot));\ M \in H,\ y \in E',\ |y| \leq 1\}$ is relatively compact in $M_+(T, \mathcal{K})$ endowed with the s-topology. Since $\{M(T);\ M \in H\}$ is relatively compact, $\bigcup\{M(T);\ M \in H\}$ is a bounded subset in E. Hence $\sup\{\delta^*(y|M(T));\ M \in H,\ y \in E',\ |y| \leq 1\} < \infty$. By Proposition 1 and Lemmas 4–5, $\delta(H)$ is relatively compact.

This proof is an adaptation of that of Topsøe ([12, Lemma 4, pp. 208–209]).

Theorem 3. *Let E be a Banach space, let T be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be sets of subsets of T. Assume that $\mathcal{G}$ and $\mathcal{K}$ satisfy axioms* (I)–(V) *and the condition*

(C) *for all $K \in \mathcal{K}$ and $G \in \mathcal{G}$,*

$$K \subseteq G \ \Rightarrow\ \exists G', G'' \text{ such that } K \subseteq G' \subseteq T \setminus G'' \subseteq G.$$

Consider the space $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ with the s-topology. Then a subset H of $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ is relatively compact if and only if the following two conditions are satisfied:

(i) *For all $G \in \mathcal{G}$, $\{M(G);\ M \in H\}$ is relatively compact in* $\mathrm{ck}(E)$.
(ii) *For any sequence $(G_n)_{n \geq 1}$ of pairwise disjoint sets in $\mathcal{G}$ we have* $\lim_{n\to\infty} M(G_n) = \{0\}$ *uniformly with respect to M in H.*

Proof. Condition (C) implies that $T \in \mathcal{G}$. We only have to prove that H is net-compact in the narrow topology, i.e. every universal net $(M_i)_{i \in I}$ on H converges narrowly in $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ if (i) and (ii) hold. Assume that (i) and (ii) hold. Then for each $G \in \mathcal{G}$ the universal net $(M_i(G))$ converges in $\mathrm{ck}(E)$. Consider the map N from $\mathcal{G}$ to $\mathrm{ck}(E)$ defined by $N(G) = \lim_i M_i(G)$, and the map $\widetilde{M} : \mathcal{B} \to \mathrm{ck}(E)$ defined by

$$\widetilde{M}(A) = \overline{\mathrm{co}} \bigcup_{K \subseteq A} \bigcap_{G \supseteq K} N(G), \qquad A \in \mathcal{B}.$$

The map $\widetilde{M}$ is an element of $\widetilde{M}_+(T, \mathcal{K}, \mathrm{ck}(E))$ ([11, Theorem 2]). Let us prove that (M_i) converges narrowly to $\widetilde{M}$. It is obvious that $\liminf_i \delta^*(y|M_i(G)) \geq \delta^*(y|\widetilde{M}(G))$ for all $y \in E'$ and all $G \in \mathcal{G}$. To finish the proof we have to show that $\lim_i M_i(T) = \widetilde{M}(T)$. Since $(M_i(T))$ is convergent in $\mathrm{ck}(E)$ we need only prove that $\lim_i \delta^*(y|M_i(T)) = \delta^*(y|\widetilde{M}(T))$ for all $y \in B'(0,1)$, i.e.

$$\forall y \in B'(0,1) \qquad \inf_{K\in\mathcal{K}} \sup_{G\supseteq K} \lim_i \delta^*(y|M_i(T \setminus G)) = 0$$

([11, Lemmas 1–3]). If this were not so, we would be able to find $\varepsilon > 0$ and $y \in B'(0,1)$ such that for every $K \in \mathcal{K}$ there exist $G_K \in \mathcal{G}$ with $G_K \supseteq K$ and $M_K \in H$ such that $\delta^*(y|M_K(T \setminus G_K)) > \varepsilon$. We can then construct by induction a sequence $(K_n)_{n\geq 0}$ of sets in $\mathcal{K}$, two sequences $(G'_n)_{n\geq 1}$ and $(G''_n)_{n\geq 1}$ of sets in $\mathcal{G}$, and a sequence $(M_n)_{n\geq 1}$ of elements in H such that the following conditions hold:

(a) The sets K_n are parwise disjoint.
(b) $\delta^*(y|M_n(K_n)) \geq \varepsilon$ for all $n \geq 1$.
(c) $G'_n \supseteq K_n$ for all $n \geq 1$.
(d) $G''_n \supseteq \bigcup_{i=0}^{n-1} K_i$ for all $n \geq 1$.
(e) $G'_n \cap G''_n = \emptyset$ for all $n \geq 1$.
(f) $\sup_{M\in H} \delta^*(y|M(T \setminus (G''_n \cup K_n)) \leq \varepsilon/2^{n+1}$ for all $n \geq 1$.

The construction is identical with that of Topsøe ([12, Theorem 8, p. 209]). Lemma 4 in [12] should be replaced by our Lemma 5. Let (O_n) be the sequence defined by $O_1 = G'_1$ and $O_n = G'_n \cap \bigcap_{i=1}^{n-1} G''_i$ for $n \geq 2$. Then $O_n \in \mathcal{G}$ for all n and $O_n \cap O_m = \emptyset$ for $n < m$, because $O_n \cap O_m \subset G'_n \cap G''_n = \emptyset$. Put $\mu_n = \delta^*(y|M_n(\cdot))$. For all $n \geq 1$, we have

$$\begin{aligned}\mu_n(O_n) &= \mu_n(G'_n) - \mu_n\Big(G'_n \setminus \bigcap_{i=1}^{n-1} G''_i\Big) \geq \mu_n(K_n) - \mu_n\Big(\bigcup_{i=1}^{n-1}((T \setminus G''_i) \cap G'_n)\Big)\\ &\geq \varepsilon - \mu_n\Big(\bigcup_{i=1}^{n-1} T \setminus (G''_i \cup K_i)\Big) \geq \varepsilon - \sum_{i=1}^{\infty} \mu_n(T \setminus (G''_i \cup K_i))\\ &\geq \varepsilon - \sum_{i=1}^{\infty} \varepsilon/2^{i+1} = \varepsilon/2.\end{aligned}$$

This contradicts condition (ii) of the theorem.

Note that if T is a normal space (resp. locally compact space), if $\mathcal{G}$ is the family of all open subsets of T, and if $\mathcal{K}$ is the family of all closed (resp. compact) subsets of T then axioms (I)–(V) and condition (C) are simultaneously satisfied.

The following result is proved for real non-negative measures by Grothendieck ([6, p. 150]), Dieudonné ([4, Proposition 8]), Topsøe ([12, Corollary 3]) and by Brooks for vector measures ([1]). The result of Topsøe generalizes those of [6] and [4]. The following result is a generalization of that of Topsøe to set-valued measures.

THEOREM 4. *Let E be a separable Banach space, let T be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be sets of subsets of T. Assume that axioms* (I)–(V) *and condition* (C) *from Theorem* 3 *are satisfied, and that $\mathcal{G}$ is closed under countable unions. Consider $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ with the s-topology. Then a sequence (M_n) in $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$ is convergent if and only if for every $G\in\mathcal{G}$ the sequence $(M_n(G))$ is convergent.*

Proof. Assume that $(M_n(G))$ is convergent for all $G\in\mathcal{G}$. Put $H=\{M_n;\, n\in\mathbb{N}\}$. Let us prove that H is relatively compact in $\widetilde{M}_+(T,\mathcal{K},\mathrm{ck}(E))$, i.e. H satisfies conditions (i) and (ii) of Theorem 3. It is clear that (i) holds. If (ii) failed we would be able to find an $\varepsilon>0$, a sequence $(M_i)_{i\ge1}$ of set-valued measures in H, a sequence $(G_i)_{i\ge1}$ of pairwise disjoint sets, and a sequence $(y_i)_{i\ge1}$ in $B'(0,1)$ such that $\delta^*(y_i|M_i(G_i))>\varepsilon$ for all $i\ge1$. The closed unit ball $B'(0,1)$ of E' is a metrizable compact subset of E' in the weak topology $\sigma(E',E)$ ([5, 4.2 and 5.1]). Hence the sequence (y_i) admits a convergent subsequence (y_{i_k}). Put $y=\lim_{k\to\infty}y_{i_k}$ and $N(G)=\lim_{k\to\infty}M_{i_k}(G)$. We have $\lim_{k\to\infty}\delta^*(y_{i_k}|M_{i_k}(G))=\delta^*(y|N(G))$ for all $G\in\mathcal{G}$. Put $\mu_k=\delta^*(y_{i_k}|M_{i_k}(\cdot))$ for $k\in\mathbb{N}$. Let l^1 be the Banach space of summable scalar sequences under its natural norm. For all $\mu\in M_+(T,\mathcal{K})$ let $\widetilde{\mu}$ be the element of l^1 defined by $\widetilde{\mu}(j)=\mu(G_j)$. For all subsets I of $\mathbb{N}$, $\lim_{k\to\infty}\sum_{p\in I}\widetilde{\mu}_k(p)=\lim_{k\to\infty}\mu_k(\bigcup_{p\in I}G_p)$ exists because $\mathcal{G}$ is closed under countable unions. We deduce from this that the sequence $(\widetilde{\mu}_k)_{k\ge1}$ is convergent in the weak topology $\sigma(l^1,l^\infty)$. Hence the subset $\{\widetilde{\mu}_k;\, k\ge1\}$ is weakly relatively compact. By [8, pp. 281–282],

$$\lim_{p\to\infty}\sup_{k\in\mathbb{N}}\widetilde{\mu}_k(p)=\lim_{p\to\infty}\sup_{k\in\mathbb{N}}\delta^*(y_{i_k}|M_{i_k}(G_p))=0.$$

This contradicts the condition $\delta^*(y_i|M_i(G_i))\ge\varepsilon$ for all $i\ge1$. Hence H is relatively compact in the s-topology. Now let M and M' be two cluster points of the sequence $(M_n)_{n\in\mathbb{N}}$. We have $M(G)=M'(G)$ for all $G\in\mathcal{G}$. By Lemma 2 we have $M(K)=M'(K)$ for all $K\in\mathcal{K}$. Since M and M' are $\mathcal{K}$-inner regular we conclude that $M(A)=M'(A)$ for all $A\in\mathcal{B}$. The sequence $(M_n)_{n\in\mathbb{N}}$ has only one cluster point. Since H is relatively compact, the sequence (M_n) is convergent.

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