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ASSOCIATED PRIMES AND PRIMAL DECOMPOSITION OF MODULES OVER COMMUTATIVE RINGS

BY

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Abstract. Let R be a commutative ring and let M be an R-module. The aim of this paper is to establish an efficient decomposition of a proper submodule N of M as an intersection of primal submodules. We prove the existence of a canonical primal decomposition, $N = \bigcap_{\mathfrak{p}} N_{(\mathfrak{p})}$, where the intersection is taken over the isolated components $N_{(\mathfrak{p})}$ of N that are primal submodules having distinct and incomparable adjoint prime ideals $\mathfrak{p}$. Using this decomposition, we prove that for $\mathfrak{p} \in \operatorname{Supp}(M/N)$, the submodule N is an intersection of $\mathfrak{p}$-primal submodules if and only if the elements of $R \setminus \mathfrak{p}$ are prime to N. Also, it is shown that M is an arithmetical R-module if and only if every primal submodule of M is irreducible. Finally, we determine conditions for the canonical primal decomposition to be irredundant or residually maximal, and for the unique decomposition of N as an irredundant intersection of isolated components.

1. Introduction. Throughout this paper, all rings considered will be commutative and will have non-zero identity elements and all modules will be unitary. Such a ring will be denoted by R, and the terminology is, in general, the same as that in [1] and [6]. Associated primes and primary decompositions are the most basic notions in the study of modules over commutative Noetherian rings. However, for modules over non-Noetherian rings the classical associated primes and primary decompositions are also interesting (see [7] and [8]). Let M be an R-module and N a proper submodule of M. In [13] (resp. [16]) Krull (resp. Noether) has introduced the most useful concept of associated primes (resp. primary decomposition) of N.

A prime ideal $\mathfrak{p}$ of R is said to be a *Krull associated prime* of the submodule N if for every element $x \in \mathfrak{p}$, there exists $m \in M$ such that $x \in N :_R m \subseteq \mathfrak{p}$. A prime ideal $\mathfrak{p}$ of R is called a *weakly* (resp. *Zariski–Samuel*) *associated prime* to N if there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal over the annihilator $N :_R m$ (resp. $\mathfrak{p} = \operatorname{Rad}(N :_R m)$). We will denote the set of Krull (resp. weakly) associated primes to N by $\operatorname{Ass}_R M/N$

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(resp. $\mathrm{Ass}_f\, M/N$). Many of the basic properties of these primes are found in [1], [10], [12], [9], [15] and have led to some interesting results.

Let M be an R-module. Recall that a submodule N of M is said to be *decomposable* if N is a finite intersection of primary submodules of M, and M is said to be *Laskerian* if each proper submodule of M is decomposable. We say that M is *uniserial* if the submodules of M are linearly ordered with respect to inclusion. Finally, M is said to be an *arithmetical module* if $M_{\mathfrak{m}}$ is a uniserial $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.

The purpose of the present paper is to establish an efficient decomposition of a proper submodule N of M as an intersection of primal submodules. We prove the existence of a canonical primal decomposition, $N = \bigcap_{\mathfrak{p}} N_{(\mathfrak{p})}$, where the intersection is taken over the isolated components $N_{(\mathfrak{p})}$ of N that are primal submodules having distinct and incomparable adjoint prime ideals $\mathfrak{p}$. (Recall that a submodule N of M is said to be *primal* if the set of all zero-divisors of the factor module M/N is an ideal of R.) Using the above canonical primal decomposition, we prove that for $\mathfrak{p} \in \mathrm{Supp}(M/N)$, the submodule N is an intersection of $\mathfrak{p}$-primal submodules if and only if the elements of $R \setminus \mathfrak{p}$ are prime to N. Also, it is shown that M is an arithmetical R-module if and only if every primal submodule of M is irreducible.

Let $\mathfrak{p}$ be a prime ideal of R. For a submodule N of M, we use $N_{(\mathfrak{p})}$ to denote the submodule $\bigcup_{s \in R\setminus\mathfrak{p}} (N :_M s)$ of M. For any ideal $\mathfrak{a}$ of R, the *radical* of $\mathfrak{a}$, denoted by $\mathrm{Rad}(\mathfrak{a})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$. Finally, for any R-module L, the set of all zero-divisors of L is denoted by $Z_R(L)$.

In the second section, we prove some basic results about primal submodules. Among other things, we prove the following theorem:

THEOREM 1.1. *Let M be an R-module. Then the following conditions are equivalent*:

(i) *M is an arithmetical module.*
(ii) *Every primal submodule of M is irreducible.*
(iii) *For each maximal ideal $\mathfrak{m}$ of R and each finitely generated submodule N of M with $N_{\mathfrak{m}} \neq 0$, the submodule $\mathfrak{m}N_{(\mathfrak{m})}$ is irreducible.*

This result of 1.1 is proved as Theorem 2.7. Let M be an R-module and N a proper submodule of M. An intersection $N = \bigcap_{i\in I} N_i$ is said to be *residually maximal at N_i* if replacing N_i by a residue $N_i :_M x$ that properly contains N_i leads to a submodule larger than N. If the intersection is residually maximal at each N_i then the intersection is said to be *residually maximal.* Set

$$\Omega_N = \{\mathfrak{p} : \mathfrak{p} \text{ is a maximal member of } \mathrm{Ass}_R\, M/N\}.$$

As the main result of Section 3, we characterize the primal decomposition property of a proper submodule of a non-zero module M over a commutative ring R. More precisely, we prove:

THEOREM 1.2. *Let M be an R-module and N a proper submodule of M. Let $\mathfrak{p}$ be a prime ideal of R. Then $N_{(\mathfrak{p})}$ is a $\mathfrak{p}$-primal submodule of M if and only if $\mathfrak{p}$ is a Krull associated prime to N. Consequently, $N = \bigcap_{\mathfrak{p}\in\Omega_N} N_{(\mathfrak{p})}$ is a primal decomposition of N.*

Moreover, if $\mathfrak{p} \in \Omega_N$ is a Zariski–Samuel associated prime to N, then the above primal decomposition of N is residually maximal at $N_{(\mathfrak{p})}$, and so if each $\mathfrak{p} \in \Omega_N$ is a Zariski–Samuel associated prime to N, then this primal decomposition of N is residually maximal.

This result is given in Theorems 3.3 and 3.9. Finally, we determine conditions for the canonical primal decomposition to be irredundant or residually maximal, and for the unique decomposition of N as an irredundant intersection of isolated components of N.

The results of this paper are generalizations of the corresponding results of [4], where the focus was on ideals of the given ring R, rather than on R-modules.

2. Some basic results on primal submodules. In this section we prove some basic results about primal submodules and we characterize the arithmetical modules over a commutative ring in terms of primal submodules. The main result of this section is Theorem 2.7, which is a generalization of [4, Theorem 1.8]. We begin with:

REMARK 2.1. Let M be an R-module and N a proper submodule of M. For any prime ideal $\mathfrak{p}$ of R, denote by $N_{(\mathfrak{p})}$ the submodule $\bigcup_{s\in R\setminus\mathfrak{p}}(N :_M s)$, the *isolated* $\mathfrak{p}$*-component* of N in the sense of Krull [14]. It is easy to check that $m \in N_{(\mathfrak{p})}$ if and only if $N :_R m \nsubseteq \mathfrak{p}$, and $(N :_R m)_{(\mathfrak{p})} = N_{(\mathfrak{p})} :_R m$ for each $m \in M$.

The important notion of primal ideals in a commutative ring (with identity) was introduced by L. Fuchs in [3] and developed further in [4] and [5]. They have proved useful in many situations in commutative algebra. J. Dauns [2] extended this concept to modules over an arbitrary ring.

An element $x \in R$ is called *non-prime* to N if $N \subsetneq N :_M x$, i.e., there exists an element $m \in M \setminus N$ such that $xm \in N$. Evidently, the set of all non-prime elements to N corresponds to the set of zero-divisors of the factor R-module M/N; which is a union of prime ideals.

Recall that a proper submodule N of an R-module M is said to be *primal* if $Z_R(M/N)$, the set of zero-divisors of the R-module M/N, is an ideal of R. Then the ideal $Z_R(M/N)$ is a prime ideal, called the *adjoint prime* of N. In this case we say that N is a $\mathfrak{p}$-*primal submodule* of M.

The following lemma was proved by Fuchs [3] when $R = M$. It is easy to carry it over to modules, so we omit the proof.

LEMMA 2.2. *Let M be an R-module. Then*:

(i) *Every irreducible submodule of M is primal.*
(ii) *Every submodule of M is an intersection of primal submodules.*

The following two lemmas will be needed in the proof of the main result of this section.

LEMMA 2.3. *Let M be an R-module and $\mathfrak{p} \in \operatorname{Spec} R$. Let N be a $\mathfrak{p}$-primal submodule of M.*

(i) *If $\mathfrak{q}$ is a prime ideal containing $\mathfrak{p}$, then $N_{(\mathfrak{q})} = N$.*
(ii) *If $\mathfrak{q}$ is a prime ideal not containing $\mathfrak{p}$, then $N_{(\mathfrak{q})} \supsetneq N$.*
(iii) *If $N_{(\mathfrak{q})}$ is $\mathfrak{q}$-primal submodule for some $\mathfrak{q} \in \operatorname{Supp}(M/N)$, then $\mathfrak{q} \subseteq \mathfrak{p}$.*

Proof. (i) Let $m \in N_{(\mathfrak{q})}$. Then there exists $s \in R \setminus \mathfrak{q}$ such that $sm \in N$. Since N is $\mathfrak{p}$-primal it follows that $m \in N$, as required.

(ii) Let $\mathfrak{q}$ be a prime ideal not containing $\mathfrak{p}$. Hence there exists $c \in \mathfrak{p} \setminus \mathfrak{q}$. Then $cm \in N$ for some $m \in M \setminus N$, and so $m \in N_{(\mathfrak{q})} \setminus N$.

(iii) Let $x \in \mathfrak{q}$. By our assumption $\mathfrak{q} = \bigcup_{m \in M \setminus N_{(\mathfrak{q})}} (N_{(\mathfrak{q})} :_R m)$. Then there exists $m \in M \setminus N_{(\mathfrak{q})}$ such that $xm \in N_{(\mathfrak{q})}$. Hence there exists $y \in R \setminus \mathfrak{q}$ such that $yxm \in N$. Therefore it is enough to show that $N :_R ym \subseteq \mathfrak{p}$. Suppose the contrary and that $ysm \in N$ for some $s \in R \setminus \mathfrak{p}$. Then in view of part (i) we have $ym \in N$, which implies that $y \in \mathfrak{q}$. This is a contradiction. ∎

REMARK 2.4. Although, by Lemma 2.3, primal submodules behave like primary submodules, there are notable differences. For example, the intersection of two primal submodules with the same adjoint prime need not be primal. As an example consider the polynomial ring $R = k[x, y, z]$, where x, y and z are indeterminates over a field k. Then the ideal (xy, z) is not primal, but

$$(xy, z) = (x^2, xy, z) \cap (y^2, xy, z),$$

and (x^2, xy, z) and (y^2, xy, z) are (x, y, z)-primal. Launs [2] proved that a reduced intersection $N = \bigcap_{i=1}^n N_i$ of $\mathfrak{p}_i$-primal submodules of an R-module M is primal if and only if there exists a unique maximal member in the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. This maximal element is then the adjoint prime of N.

DEFINITION 2.5 (see [11]). Let M be an R-module. We say that M is an *arithmetical module* if $M_{\mathfrak{m}}$ is a uniserial $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of R, i.e., the submodules of $M_{\mathfrak{m}}$ are linearly ordered with respect to inclusion.

It is easy to see that an R-module M is arithmetical if and only if for each proper submodule N of M, $N_{\mathfrak{m}}$ is an irreducible submodule of $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.

LEMMA 2.6. *Let N be a finitely generated submodule of an R-module M and let $\mathfrak{p} \in \operatorname{Supp} N$. Then $(\mathfrak{p}N)_{(\mathfrak{p})}$ is a $\mathfrak{p}$-primal submodule of M.*

Proof. Obviously those elements of R that are non-prime to $(\mathfrak{p}N)_{(\mathfrak{p})}$ are in $\mathfrak{p}$. Now since $N_{\mathfrak{p}} \neq 0$ by Nakayama's lemma we have $(\mathfrak{p}N)_{(\mathfrak{p})} \subsetneq N_{(\mathfrak{p})} \subseteq (\mathfrak{p}N)_{(\mathfrak{p})} :_M \mathfrak{p}$. Hence there exists $y \in ((\mathfrak{p}N)_{(\mathfrak{p})} :_M \mathfrak{p}) \setminus (\mathfrak{p}N)_{(\mathfrak{p})}$. It follows that the elements of $\mathfrak{p}$ are not prime to $(\mathfrak{p}N)_{(\mathfrak{p})}$, and this completes the proof of the lemma. ∎

We are now ready to state and prove the main theorem of this section, generalizing [4, Theorem 1.8] which concerned the case $M = R$.

THEOREM 2.7. *Let M be an R-module. Then the following conditions are equivalent*:

(i) *M is an arithmetical module.*
(ii) *Every primal submodule of M is irreducible.*
(iii) *For each maximal ideal $\mathfrak{m}$ of R and each finitely generated submodule N of M with $N_{\mathfrak{m}} \neq 0$, the submodule $\mathfrak{m}N_{(\mathfrak{m})}$ is irreducible.*

Proof. (i)⇒(ii). Let N be a primal submodule of M. Then by Lemma 2.3, $N = N_{(\mathfrak{p})}$ for some prime ideal $\mathfrak{p}$. Since M is an arithmetical module, $N_{\mathfrak{p}}$ and hence N are irreducible, and this proves (ii).

(ii)⇒(iii). Let N be a finitely generated submodule of M and let $\mathfrak{m}$ be a maximal ideal in $\operatorname{Supp} N$. First we show that $\mathfrak{m}N_{(\mathfrak{m})} = (\mathfrak{m}N)_{(\mathfrak{m})}$. To this end, let $\mathfrak{n}$ be an arbitrary maximal ideal of R such that $\mathfrak{n} \neq \mathfrak{m}$. Since $\mathfrak{m}N_{(\mathfrak{m})} \subseteq (\mathfrak{m}N)_{(\mathfrak{m})}$ it follows that

$$(N_{(\mathfrak{m})})_{\mathfrak{n}} = (\mathfrak{m}N_{(\mathfrak{m})})_{\mathfrak{n}} \subseteq ((\mathfrak{m}N)_{(\mathfrak{m})})_{\mathfrak{n}} \subseteq (N_{(\mathfrak{m})})_{\mathfrak{n}}.$$

Therefore

$$(\mathfrak{m}N_{(\mathfrak{m})})_{\mathfrak{n}} = ((\mathfrak{m}N)_{(\mathfrak{m})})_{\mathfrak{n}},$$

and this implies the desired equality.

Now, by Lemma 2.6, $(\mathfrak{m}N)_{(\mathfrak{m})}$ is primal, and so irreducible by hypothesis.

(iii)⇒(i). Let $x, y \in M$, and let $\mathfrak{m}$ be a maximal ideal of R. Let $N = xR + yR$. By assumption $\mathfrak{m}N_{(\mathfrak{m})}$, and so $\mathfrak{m}N_{\mathfrak{m}}$, is irreducible. Therefore, as $N_{\mathfrak{m}}/\mathfrak{m}N_{\mathfrak{m}}$ is a finite-dimensional vector space over $R/\mathfrak{m}$, it follows that $N_{\mathfrak{m}} = (Rx)_{\mathfrak{m}} + \mathfrak{m}N_{\mathfrak{m}}$ or $N_{\mathfrak{m}} = (Ry)_{\mathfrak{m}} + \mathfrak{m}N_{\mathfrak{m}}$. Hence by Nakayama's lemma $N_{\mathfrak{m}} = (Rx)_{\mathfrak{m}}$ or $N_{\mathfrak{m}} = (Ry)_{\mathfrak{m}}$. Consequently,

$$(Rx)_{\mathfrak{m}} \subseteq (Ry)_{\mathfrak{m}} \quad \text{or} \quad (Ry)_{\mathfrak{m}} \subseteq (Rx)_{\mathfrak{m}}. \ ∎$$

3. Associated primes and primal decomposition. In this section we characterize the primal decomposition property of a proper submodule N of a non-zero module M over a commutative ring R. The main results of this section are Theorems 3.3 and 3.9. Recall that saying that $(R, \mathfrak{m})$ is a *quasilocal ring* means that R is a commutative ring with unique maximal ideal $\mathfrak{m}$.

LEMMA 3.1. *Let M be an R-module and N a proper submodule of M. A prime ideal $\mathfrak{q}$ is a Krull associated prime ideal of N if and only if $\mathfrak{q}$ is a union of weakly associated prime ideals of N.*

Proof. Let $\mathfrak{p}$ be a weakly associated prime to N. Then there exists $m \in M \setminus N$ such that $\mathfrak{p}$ is a minimal prime ideal of $(N :_R m)$. Since

$$N_{(\mathfrak{p})} :_R m = (N :_R m)_{(\mathfrak{p})}$$

is a $\mathfrak{p}$-primary ideal of R, for every $u \in \mathfrak{p}$ there exists a smallest integer $k \geq 1$ such that $u^k sm \in N$ for some $s \in R \setminus \mathfrak{p}$. Now $u \in N :_R u^{k-1}sm$, and the minimality of k shows that $N :_R u^{k-1}sm \subseteq \mathfrak{p}$. This means that every weakly associated prime ideal of N is a Krull associated prime ideal of N. Therefore if the prime ideal $\mathfrak{q}$ is a union of weakly associated primes of N, then for every $u \in \mathfrak{q}$ there exists a weakly associated prime ideal $\mathfrak{p}$ of N such that $u \in \mathfrak{p}$. This implies that $x \in N :_R m \subseteq \mathfrak{q}$ for some $m \in M \setminus N$. So $\mathfrak{q}$ is a Krull associated prime ideal of N. The converse is obvious. ∎

The next result provides a slight generalization of [4, Theorems 3.4 and 3.5].

THEOREM 3.2. *Let M be an R-module, N a proper submodule of M, and $\mathfrak{p}$ a prime ideal of R. Then $N_{(\mathfrak{p})}$ is a $\mathfrak{p}$-primal submodule of M if and only if $\mathfrak{p}$ is a Krull associated prime to N. Consequently, $N = \bigcap_{\mathfrak{p} \in \Omega_N} N_{(\mathfrak{p})}$, which we call the canonical primal decomposition of N, is a decomposition of N into primal submodules, where the isolated components $N_{(\mathfrak{p})}$ are primal submodules with distinct adjoint primes.*

Proof. If $N_{(\mathfrak{p})}$ is a $\mathfrak{p}$-primal submodule of M then it is easy to see that

$$\mathfrak{p} = \bigcup_{\mathfrak{q} \in \operatorname{Ass}_f M/N_{(\mathfrak{p})}} \mathfrak{q}.$$

Now Lemma 3.1 implies that $\mathfrak{p}$ is a Krull associated prime ideal of $N_{(\mathfrak{p})}$. Conversely, let $\mathfrak{p}$ be a Krull associated prime of $N_{(\mathfrak{p})}$. We have to show that the elements of $\mathfrak{p}$ are non-prime to $N_{(\mathfrak{p})}$. Let $x \in \mathfrak{p}$. Then there exists $y \in M \setminus N_{(\mathfrak{p})}$ such that $x \in N_{(\mathfrak{p})} :_R y \subseteq \mathfrak{p}$. This yields $\mathfrak{p} = \bigcup_{y \in M \setminus N_{(\mathfrak{p})}} (N_{(\mathfrak{p})} :_R y)$. Therefore $N_{(\mathfrak{p})}$ is a $\mathfrak{p}$-primal submodule of M. This proves the first assertion. Now, let $m \in \bigcap_{\mathfrak{p} \in \Omega_N} N_{(\mathfrak{p})}$. Then for all $\mathfrak{p} \in \Omega_N$ there exists an element $s_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ such that $s_{\mathfrak{p}} \in N :_R m$. Therefore $N :_R m$ contains no minimal prime ideal, which implies that $m \in N$. The final assertion is obvious by Lemma 3.1. ■

Now we are prepared to prove the first main theorem of this section, which is an extension of [4, Proposition 3.7].

THEOREM 3.3. *Let M be an R-module, N a proper submodule of M, and $\mathfrak{p}$ a prime ideal in* $\mathrm{Supp}(M/N)$. *Then N is an intersection of $\mathfrak{p}$-primal submodules if and only if all the elements of $R \setminus \mathfrak{p}$ are prime to N. In particular, $N_{(\mathfrak{p})}$ is an intersection of $\mathfrak{p}$-primal submodules.*

Proof. Suppose that $N = \bigcap N_i$ is an intersection of $\mathfrak{p}$-primal submodules N_i. Then by Lemma 2.3, $(N_i)_{(\mathfrak{p})} = N_i$. Hence $N_{(\mathfrak{p})} = N$, and so the elements of $R \setminus \mathfrak{p}$ are prime to N. Conversely, suppose that the elements of $R \setminus \mathfrak{p}$ are prime to N. We may assume that $(R, \mathfrak{p})$ is a quasilocal ring. Let $m \in M \setminus N$. Then the ideal $N :_R m$ is contained in $\mathfrak{p}$, and so $m \notin N + \mathfrak{p}m$. Enlarge $N + \mathfrak{p}m$ to a submodule, say $N(m)$, which is maximal not containing m. Obviously $N(m)$ is a $\mathfrak{p}$-primal submodule of M and we have $N = \bigcap_{m \notin N} N(m)$. ■

LEMMA 3.4. *Let M be an R-module and N a proper submodule of M. Suppose that $\mathfrak{p} \in \Omega_N$ satisfies one of the following conditions*:

(i) $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}} \mathfrak{q}$.
(ii) $\mathfrak{p}$ *is a Zariski–Samuel associated prime of N.*

Then in the canonical primal decomposition of N (*see Theorem* 3.2), *the isolated $\mathfrak{p}$-component $N_{(\mathfrak{p})}$ is relevant.*

Proof. First let $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}} \mathfrak{q}$. Then there exists $x \in \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}} \mathfrak{q}$. Since $N \subsetneq N :_M x$ and $N_{(\mathfrak{q})} :_M x = N_{(\mathfrak{q})}$ for all $\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}$, it follows that

$$N \subsetneq N :_M x = (N :_M x) \cap \bigcap_{\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}} N_{(\mathfrak{q})} \subseteq \bigcap_{\mathfrak{q} \in \Omega_N,\, \mathfrak{q} \neq \mathfrak{p}} N_{(\mathfrak{q})}.$$

Therefore $N_{(\mathfrak{p})}$ is relevant.

Now suppose that $\mathfrak{p}$ is a Zariski–Samuel associated prime of N. Then $\mathfrak{p} = \mathrm{Rad}(N :_R m)$ for some $m \in M \setminus N$. Let $\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}$. Maximality of $\mathfrak{p}$ in Ω_N implies that $m \in N_{(\mathfrak{q})}$ for all $\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}$. Consequently, $m \in \bigcap_{\mathfrak{q} \in \Omega_N,\, \mathfrak{q} \neq \mathfrak{p}} N_{\mathfrak{q}}$, which implies the relevance of $N_{(\mathfrak{p})}$. ■

COROLLARY 3.5. *Let M be an R-module and N a proper submodule of M. Suppose that each $\mathfrak{p} \in \Omega_N$ satisfies one of the following conditions*:

(i) $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q} \in \Omega_N \setminus \{\mathfrak{p}\}} \mathfrak{q}$.
(ii) $\mathfrak{p}$ *is a Zariski–Samuel associated prime of N.*

Then the canonical primal decomposition of N is irredundant.

Proposition 3.6. *Let R be a ring such that every radical ideal $\mathfrak{a}$ of R has a minimal prime divisor $\mathfrak{p}$ such that $\mathfrak{p}/\mathfrak{a}$ is the radical of a finitely generated ideal in the quotient ring $R/\mathfrak{a}$. Let M be an R-module and let N be a proper submodule of M. Then $N = \bigcap_{\mathfrak{p}\in\mathcal{Z}(N)} N_{(\mathfrak{p})}$, where $\mathcal{Z}(N)$ denotes the set of Zariski–Samuel associated primes of N. Furthermore, if R satisfies the ascending chain condition on prime ideals and $\mathcal{Z}^*(N)$ denotes the maximal elements of $\mathcal{Z}(N)$, then $N = \bigcap_{\mathfrak{p}\in\mathcal{Z}^*(N)} N_{(\mathfrak{p})}$, and this intersection is irredundant.*

Proof. First of all we have to show that $\mathcal{Z}(N) \neq \emptyset$. Let $m \in M\backslash N$ and set $\mathfrak{a} = \mathrm{Rad}\,(N :_R m)$. Our hypothesis yields the existence of a finitely generated ideal $\mathfrak{c}$ and a minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$ such that $\mathfrak{p} = \mathrm{Rad}(\mathfrak{a}+\mathfrak{c})$. Since $\mathfrak{p} = \mathfrak{a}_{(\mathfrak{p})}$ it follows that $\mathfrak{c} \subseteq \mathfrak{a}_{(\mathfrak{p})}$. As $\mathfrak{c}$ is finitely generated there exist $n \in \mathbb{N}$ and $y \in R \setminus \mathfrak{p}$ such that $\mathfrak{c}^n \subseteq N :_R ym$. Therefore

$$\mathfrak{c}^n + N :_R m \subseteq N :_R ym \subseteq \mathfrak{p}.$$

Hence $\mathrm{Rad}(N :_R ym) = \mathfrak{p}$, and so $\mathcal{Z}(N) \neq \emptyset$. It is easy to see that this implies $N = \bigcap_{\mathfrak{p}\in\mathcal{Z}(N)} N_{(\mathfrak{p})}$.

Now suppose that R satisfies the ascending chain condition on prime ideals. Then for each $\mathfrak{p} \in \mathcal{Z}(N)$ there exists $\mathfrak{q} \in \mathcal{Z}^*(N)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. This implies that $N = \bigcap_{\mathfrak{p}\in\mathcal{Z}^*(N)} N_{(\mathfrak{p})}$. To prove that this intersection is irredundant, let $\mathfrak{p} \in \mathcal{Z}^*(N)$. Then $\mathfrak{p} = \mathrm{Rad}(N :_R m)$ for some $m \in M \setminus N$. Hence $N :_R m \nsubseteq \mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{Z}^*(N)$ and so $m \in \bigcap_{\mathfrak{q}\in\mathcal{Z}^*(N),\, \mathfrak{q}\neq\mathfrak{p}} N_{(\mathfrak{q})} \setminus N_{(\mathfrak{p})}$, as required. ∎

Definition 3.7. Let M be an R-module and N a proper submodule of M. An intersection $N = \bigcap_{i\in I} N_i$ is said to be *residually maximal at N_i* if replacing N_i by a residue $N_i :_M x$, $x \in R$, that properly contains N_i leads to a submodule larger than N. If the intersection is residually maximal at each N_i then the intersection is said to be *residually maximal.*

Proposition 3.8. *Let M be an R-module and N a proper submodule of M. If $\mathfrak{p} \in \Omega_N$ is a Zariski–Samuel associated prime of N then the canonical primal decomposition of N (see Theorem 3.2) is residually maximal at $N_{(\mathfrak{p})}$. Consequently if each $\mathfrak{p} \in \Omega_N$ is a Zariski–Samuel associated prime to N, then the canonical primal decomposition of N is residually maximal.*

Proof. Suppose that $N_{(\mathfrak{p})}$ can be replaced by $N_{(\mathfrak{p})} :_M x$ for some $x \in R$. We have to show that $N_{(\mathfrak{p})} = N_{(\mathfrak{p})} :_M x$. By hypothesis, $\mathfrak{p} = \mathrm{Rad}(N :_R m)$ for some $m \in M$. Then $m \in \bigcap_{\mathfrak{p}\in\Omega_N,\, \mathfrak{q}\neq\mathfrak{p}} N_{(\mathfrak{q})}$. Since $N :_R m$ is a $\mathfrak{p}$-primary ideal of R and

$$N :_R m = N_{(\mathfrak{p})} :_R m = (N_{(\mathfrak{p})} :_M x) :_R m = (N_{(\mathfrak{p})} :_R m) :_R x = (N :_R m) :_R x,$$

it follows that $x \notin \mathfrak{p}$. As $N_{(\mathfrak{p})}$ is primal, we have $N_{(\mathfrak{p})} :_M x = N_{(\mathfrak{p})}$. ∎

By a *primal isolated component* of a submodule N we mean a primal submodule B such that $B = N_{(\mathfrak{q})}$ for some prime ideal $\mathfrak{q}$. If $\mathfrak{p}$ is the adjoint prime of $N_{(\mathfrak{q})}$ then necessarily $\mathfrak{p} \subseteq \mathfrak{q}$, which yields $N_{(\mathfrak{q})} \subseteq N_{(\mathfrak{p})}$. Since $(N_{(\mathfrak{q})})_{(\mathfrak{p})} = N_{(\mathfrak{q})}$ and $N \subseteq N_{(\mathfrak{q})}$, it follows that $N_{(\mathfrak{q})} = N_{(\mathfrak{p})}$. This shows that if B is a primal isolated component of N then $B = N_{(\mathfrak{p})}$ for some $\mathfrak{p} \in \operatorname{Ass}_R M/N$.

We are now ready to state and prove the second main theorem of this section, which is a generalization of [4, Theorem 4.6].

THEOREM 3.9. *Let M be an R-module and N a proper submodule of M. For a prime ideal $\mathfrak{p} \in \Omega_N$, the following conditions are equivalent*:

(i) *$\mathfrak{p}$ is a Zariski–Samuel associated prime ideal to N.*
(ii) *The isolated component $N_{(\mathfrak{p})}$ must appear in every representation of N as an intersection of primal isolated components.*

Proof. The implication (i)⇒(ii) is proved in Lemma 3.4.

(ii)⇒(i). Let $m \in \bigcap_{\mathfrak{q} \in \operatorname{Ass}_R M/N,\, \mathfrak{q} \neq \mathfrak{p}} N_{(\mathfrak{q})} \setminus N_{(\mathfrak{p})}$. Then, as $m \notin N$, it is easy to see that $N :_R m \subseteq \mathfrak{p}$, and if $\mathfrak{p}'$ is a minimal prime ideal of $N :_R m$ such that $\mathfrak{p}' \neq \mathfrak{p}$, then $\mathfrak{p}' \in \operatorname{Ass}_R M/N$, and so $m \in N_{(\mathfrak{p}')}$. Hence there exists $s \in R \setminus \mathfrak{p}'$ such that $s \in N :_R m$, which is a contradiction. Therefore $\mathfrak{p}$ is the unique minimal prime ideal of $N :_R m$. Hence $\mathfrak{p} = \operatorname{Rad}(N :_R m)$, and so $\mathfrak{p}$ is a Zariski–Samuel associated prime to N. ∎

COROLLARY 3.10. *Let M be an R-module and N a proper submodule of M. Then the following statements are equivalent*:

(i) *$\Omega_N \subseteq \mathcal{C}$ for each $\mathcal{C} \subseteq \operatorname{Ass}_R M/N$ with $N = \bigcap_{\mathfrak{p} \in \mathcal{C}} N_{(\mathfrak{p})}$.*
(ii) *Each $\mathfrak{p} \in \Omega_N$ is a Zariski–Samuel associated prime to N.*

In particular, if each $\mathfrak{p} \in \Omega_N$ is a Zariski–Samuel associated prime to N, then the canonical primal decomposition of N is irredundant.

Proof. (i)⇒(ii). By Theorem 3.2, we have $N = \bigcap_{\mathfrak{p} \in \Omega_N} N_{(\mathfrak{p})}$ and $N_{(\mathfrak{p})}$ is $\mathfrak{p}$-primal. Now the result follows easily from Proposition 3.6.

(ii)⇒(i). Suppose $N = \bigcap_{\mathfrak{p} \in \mathcal{C}} N_{(\mathfrak{p})}$ for some $\mathcal{C} \subseteq \Omega_N$. By Proposition 3.6 for each $\mathfrak{p} \in \Omega_N$ there exists $\mathfrak{q} \in \Omega_N$ such that $N_{(\mathfrak{p})} = N_{(\mathfrak{q})}$. But $\mathfrak{p}$ and $\mathfrak{q}$ belong to $\operatorname{Ass}_R M/N$. Consequently, $N_{(\mathfrak{p})}$ is $\mathfrak{p}$-primal and $N_{(\mathfrak{q})}$ is $\mathfrak{q}$-primal. Therefore $\mathfrak{q} = \mathfrak{p}$ and so $\Omega_N \subseteq \mathcal{C}$. ∎

The following theorem extends the main result of [4, Theorem 5.1].

THEOREM 3.11. *Let M be an R-module and N a proper submodule of M. Let $\mathcal{C}$ be a collection of prime ideals of R such that $N = \bigcap_{\mathfrak{p} \in \mathcal{C}} N_{(\mathfrak{p})}$ is an irredundant intersection of irreducible isolated components of N. Then this intersection is residually maximal and is the unique irredundant decomposition of N into irreducible isolated components of N.*

Proof. Let $\mathfrak{q}$ be the adjoint prime of $N_{(\mathfrak{p})}$. Without loss of generality we may assume that $\mathfrak{q} = \mathfrak{p}$, and therefore $\mathcal{C} \subseteq \operatorname{Ass}_R M/N$. Now suppose that for some $\mathfrak{p} \in \mathcal{C}$ there exists $x \in R$ such that

$$N = (N_{(\mathfrak{p})} :_M x) \cap \bigcap \{N_{(\mathfrak{q})} : \mathfrak{q} \in \mathcal{C},\ \mathfrak{q} \neq \mathfrak{p}\}.$$

Since $N_{(\mathfrak{p})}$ is irreducible, it follows that $N_{(\mathfrak{p})} = N_{(\mathfrak{p})} :_M x$, and so the representation is residually maximal. It only remains to show that it is unique among irredundant intersections of irreducible isolated components of N. To do this, suppose that $\mathcal{C}' = \{\mathfrak{q}_i\}_{i \in I} \subseteq \operatorname{Ass}_R M/N$ is such that $N = \bigcap_{i \in I} N_{(\mathfrak{q}_i)}$ is also an irredundant intersection of irreducible components $B_i = N_{(\mathfrak{q}_i)}$ of N with adjoint primes $\mathfrak{q}_i$. It is enough to show that for each $i' \in I$, there exists $\mathfrak{p}_{i'}$ such that $\mathfrak{p}_{i'} \subseteq \mathfrak{q}_i$. Fix $j \in I$, and define $C_j = \bigcap_{i \neq j} B_i$. Since $N_{(\mathfrak{p})}$ is irreducible and for each $\mathfrak{p} \in \mathcal{C}$, we have

$$N_{(\mathfrak{p})} = (B_j)_{(\mathfrak{p})} \cap (C_j)_{(\mathfrak{p})},$$

it follows that $N_{(\mathfrak{p})} = (B_j)_{(\mathfrak{p})}$ or $N_{(\mathfrak{p})} = (C_j)_{(\mathfrak{p})}$. As the decomposition $N = \bigcap_{i \in I} B_i$ is irredundant, it follows that $N_{(\mathfrak{p})} = (B_j)_{(\mathfrak{p})}$ and so $(B_j)_{(\mathfrak{p})}$ is $\mathfrak{p}$-primal. As B_j is $\mathfrak{q}_j$-primal, we find that $\mathfrak{p} \subseteq \mathfrak{q}_j$. By symmetry, $\mathfrak{q}_j \subseteq \mathfrak{p}_j$ for some $\mathfrak{p}_j \in \mathcal{C}$, and so $N_{(\mathfrak{p}_j)} \subseteq N_{(\mathfrak{q}_j)} \subseteq N_{(\mathfrak{p})}$. Again by irredundancy, $N_{(\mathfrak{p}_j)} = N_{(\mathfrak{q}_j)} = N_{(\mathfrak{p})}$, and so $\mathfrak{p}_j = \mathfrak{q}_j = \mathfrak{p}$. Therefore

$$\{N_{(\mathfrak{p})} : \mathfrak{p} \in \mathcal{C}\} = \{N_{(\mathfrak{q})} : i \in I\}. \blacksquare$$

REMARK 3.12. Let M be an R-module and N a proper submodule of M. Assume that $\mathcal{C} \subseteq \operatorname{Ass}_R M/N$ contains at least two elements. If $N = \bigcap_{\mathfrak{p} \in \mathcal{C}} N_{(\mathfrak{p})}$ and if the intersection is irredundant, then $\mathcal{C}$ consists of incomparable primes. Also, for each $\mathfrak{p} \in \mathcal{C}$ there exists $\mathfrak{q} \in \Omega_N$ such that $\mathfrak{p} \subseteq \mathfrak{q}$, and so $N_{(\mathfrak{q})} \subseteq N_{(\mathfrak{p})}$. Thus if $\mathcal{C}' = \mathcal{C} \setminus \{\mathfrak{p}\}$ and $B = \bigcap_{\mathfrak{p}' \in \mathcal{C}'}$, then $N_{(\mathfrak{q})}$ is relevant in the intersection $N = N_{(\mathfrak{q})} \cap B$. Note that

$$N \subseteq N_{(\mathfrak{q})} \cap B \subseteq N_{(\mathfrak{p})} \cap B = \bigcap_{\mathfrak{p} \, \in \mathcal{C}} N_{(\mathfrak{p})} = N.$$

Since $N_{(\mathfrak{p})}$ is $\mathfrak{p}$-primal and $N_{(\mathfrak{q})}$ is $\mathfrak{q}$-primal, we have $N_{(\mathfrak{p})} = N_{(\mathfrak{q})}$ if and only if $\mathfrak{p} = \mathfrak{q}$. If $N_{(\mathfrak{q})}$ is irreducible, from $N = N_{(\mathfrak{p})} \cap B$ we obtain

$$N_{(\mathfrak{q})} = (N_{(\mathfrak{p})})_{(\mathfrak{q})} \cap B_{(\mathfrak{q})} = N_{(\mathfrak{p})} \cap B_{(\mathfrak{q})},$$

and this yields $N_{(\mathfrak{q})} = N_{(\mathfrak{p})}$. Thus $\mathfrak{q} = \mathfrak{p}$. Therefore if $N_{(\mathfrak{q})}$ is irreducible for each $\mathfrak{q} \in \Omega_N$, then $\mathcal{C} \subseteq \Omega_N$.

COROLLARY 3.13. *Let M be an R-module and N a proper submodule of M. Suppose that $\mathcal{C} \subseteq \operatorname{Ass}_R M/N$. Let $N_{(\mathfrak{q})}$ be irreducible for each $\mathfrak{q} \in \Omega_N$ and let $N = \bigcap_{\mathfrak{q} \in \Omega_N} N_{(\mathfrak{q})}$ be an irredundant intersection. Then $\mathcal{C} \subseteq \Omega_N$ and $N = \bigcap_{\mathfrak{q} \in \Omega_N} N_{(\mathfrak{q})}$ is the unique decomposition of N into an intersection of irreducible isolated components.*

REMARK 3.14. Let M be an arithmetical R-module and N a proper submodule of M. Suppose $N = \bigcap_{i\in I} N_i$. If for each $i \in I, N_i$ is a primal submodule of M that is relevant to this decomposition, then $N_i = (N_i)_{(\mathfrak{p}_i)}$ for some prime ideal $\mathfrak{p}_i$. Let $i \in I$, and suppose N_i is relevant to the given decomposition. Since $N_{(\mathfrak{p}_i)}$ is irreducible, it follows that $N_i = N_{(\mathfrak{p}_i)}$.

We close this section with a characterization of arithmetical modules in terms of irredundant irreducible decompositions, which is an extension of Fuchs–Heinzer–Olberding's result in [4].

THEOREM 3.15 (cf. [4, Theorem 5.8]). *Let M be an R-module and N a proper submodule of M. The following statements are equivalent.*

(i) *M is an arithmetical R-module.*
(ii) *For each proper submodule N of M, $N = \bigcap_{\mathfrak{p}\in\Omega_N} N_{(\mathfrak{p})}$ is an intersection of irreducible submodules.*
(iii) *Each proper submodule N can be represented as an intersection of irreducible isolated components.*

Proof. By Theorems 2.7 and 3.2 the implications (i)⇒(ii) and (ii)⇒(iii) are obvious. To show (iii)⇒(i), in view of Theorem 2.7, we establish that, for each maximal ideal $\mathfrak{m}$ of R and each finitely generated submodule N of M with $N_{\mathfrak{m}} \neq 0$, the submodule $\mathfrak{m}N_{(\mathfrak{m})}$ is irreducible. To do this, by hypothesis there exists a collection $\mathcal{C}$ of prime ideals of R such that

$$\mathfrak{m}N = \bigcap_{\mathfrak{p}\in\mathcal{C}} (\mathfrak{m}N)_{(\mathfrak{p})},$$

and each $(\mathfrak{m}N)_{(\mathfrak{p})}$ is irreducible. Now, if $\mathfrak{m} \notin \mathcal{C}$, then $(\mathfrak{m}N)_{(\mathfrak{p})} = N_{(\mathfrak{p})}$ for all $\mathfrak{p} \in \mathcal{C}$. Hence

$$N \subseteq \bigcap_{\mathfrak{p}\in\mathcal{C}} N_{(\mathfrak{p})} = \mathfrak{m}N.$$

As N is a finitely generated submodule, Nakayama's lemma yields $N_{\mathfrak{m}} = 0$, which is a contradiction. Therefore $\mathfrak{m} \in \mathcal{C}$ and $(\mathfrak{m}N)_{(\mathfrak{m})}$ is an irreducible submodule of M. Since $\mathfrak{m}N_{(\mathfrak{m})} = (\mathfrak{m}N)_{(\mathfrak{m})}$, by the proof of Theorem 2.7 ((ii)⇒(iii)), the result follows. ■

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