COLLOQUIUM MATHEMATICUM

VOL. 114 2009 NO. 2

GLOBAL WELL-POSEDNESS, SCATTERING AND BLOW-UP FOR THE ENERGY-CRITICAL, FOCUSING HARTREE EQUATION IN THE RADIAL CASE

BY

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Abstract. We establish global existence and scattering for radial solutions to the energy-critical focusing Hartree equation with energy and $\dot{H}^1$ norm less than those of the ground state in $\mathbb{R} \times \mathbb{R}^d$, $d \geq 5$.

1. Introduction. We consider the following initial value problem

$$\begin{cases} iu_t + \Delta u = f(u) & \text{in } \mathbb{R}^d \times \mathbb{R},\ d \geq 5, \\ u(0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function in space-time $\mathbb{R} \times \mathbb{R}^d$ and Δ is the Laplacian in $\mathbb{R}^d$, $f(u) = -(|x|^{-4} * |u|^2)u$. It was introduced as a classical model in [32]. In practice, we use the integral formulation of (1.1),

$$u(t) = U(t)u_0(x) - i\int_0^t U(t-s)f(u(s))\,ds, \tag{1.2}$$

where $U(t) = e^{it\Delta}$.

We are primarily interested in (1.1) since it is critical with respect to the energy norm. That is, the scaling $u \mapsto u_\lambda$, where

$$u_\lambda(t,x) = \lambda^{(d-2)/2}u(\lambda^2 t, \lambda x), \qquad \lambda > 0, \tag{1.3}$$

maps a solution to (1.1) to another solution to (1.1), and u and u_λ have the same energy (2.2).

It is known that if the initial data $u_0(x)$ has finite energy, then (1.1) is locally well-posed (see, for instance, [24]). That is, there exists a unique local-in-time solution that lies in $C_t^0\dot{H}_x^1 \cap L_t^6 L_x^{6d/(3d-8)}$ and the map from the initial data to the solution is locally Lipschitz in these norms. If the energy is small, it is known that the solution exists globally in time and scattering occurs, that is, there exist solutions $u_\pm$ of the free Schrödinger

2000 *Mathematics Subject Classification*: 35Q40, 35Q55, 47J35.
Key words and phrases: Hartree equation, global well-posedness, scattering.

DOI: 10.4064/cm114-2-5

equation $(i\partial_t + \Delta)u_\pm = 0$ such that

$$\|u(t) - u_\pm(t)\|_{\dot{H}^1_x} \to 0 \quad \text{as } t \to \pm\infty.$$

However, for initial data with large energy, the local well-posedness argument does not extend to give global well-posedness, only under the conservation of the energy (2.2), because the time of existence given by the local theory depends on the profile of the data as well as on $\|u_0\|_{\dot{H}^1_x}$.

A large amount of work has been devoted to the theory of scattering for the Hartree equation: see [4]–[9], [17], [23]–[26], [28] and [29]. In particular, we have recently obtained global well-posedness in $\dot{H}^1_x$ for the energy-critical, defocusing Hartree equation in the case of large finite-energy initial data [25], [26]. In this paper, we continue this investigation and establish a scattering result for radial solutions to the energy-critical, focusing Hartree equation for data with energy and $\dot{H}^1$ norm less than those of the ground state $W(x)$.

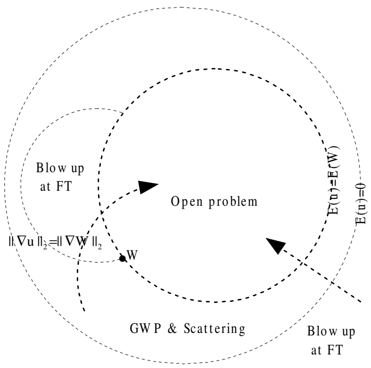

Fig. 1. A description of the solutions with radial data in the energy space, where "FT" means finite time

The main result of this paper is the following global well-posedness and blow up result for (1.1) in the energy space.

THEOREM 1.1. *Let $d \geq 5$, let $u_0 \in \dot{H}^1(\mathbb{R}^d)$ be radial and let u be the corresponding solution to* (1.1) *in $\dot{H}^1(\mathbb{R}^d)$ with maximal forward time interval of existence $[0, T)$. Suppose $E(u_0) < E(W)$.*

(1) *If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then $T = \infty$ and u scatters in $\dot{H}^1$.*
(2) *If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, then $T < \infty$, and thus, the solution blows up in finite time.*

Concerning the blow up result, we also have

THEOREM 1.2. *Let $d \geq 5$, $u_0 \in \dot{H}^1(\mathbb{R}^d)$ and let u be the corresponding solution to* (1.1) *in $\dot{H}^1(\mathbb{R}^d)$ with maximal forward time interval of existence*

$[0,T)$. *Suppose* $E(u_0) < E(W)$, $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ *and* $|x|u_0 \in L^2$. *Then* $T < \infty$, *i.e., the solution blows up in finite time.*

Next, we introduce some notations. If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \le CY$ for some C which may depend on the critical energy E_{crit} (see Section 4) but not on any parameter such as η, and $X \approx Y$ to denote the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \le cY$ for some small constant c which is again allowed to depend on E_{crit}.

We use $C \gg 1$ to denote various large finite constants and $0 < c \ll 1$ to denote various small constants.

The Fourier transform on $\mathbb{R}^d$ is defined by

$$\widehat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx,$$

giving rise to the fractional differentiation operators $|\nabla|^s$, defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \widehat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s_x} := \|\,|\nabla|^s f\|_{L^2_x(\mathbb{R}^d)}.$$

Let $e^{it\Delta}$ be the free Schrödinger propagator. In the physical space this is given by the formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y)\, dy,$$

while in the frequency space one can write this as

$$\widehat{e^{it\Delta} f}(\xi) = e^{-it|\xi|^2} \widehat{f}(\xi).$$

In particular, the propagator preserves the above Sobolev norms and obeys the dispersive estimate

$$\|e^{it\Delta} f\|_{L^\infty_x(\mathbb{R}^d)} \lesssim |t|^{-d/2} \|f\|_{L^1_x(\mathbb{R}^d)}, \quad \forall t \neq 0. \tag{1.4}$$

Let $d \ge 5$. A pair (q, r) is L^2-*admissible* if

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) \quad \text{whenever} \quad 2 \le r \le \frac{2d}{d-2}.$$

For a space-time slab $I \times \mathbb{R}^d$, we define the *Strichartz norm* $\dot{S}^0(I)$ by

$$\|u\|_{\dot{S}^0(I)} := \sup_{(q,r)\ L^2\text{-admissible}} \|u\|_{L^q_t L^r_x(I\times\mathbb{R}^d)}.$$

and for some fixed number $0 < \varepsilon_0 \ll 1$, define $\mathcal{Z}^1(I)$ by

$$\|u\|_{\mathcal{Z}^1(I)} := \sup_{(q,r)\in\Lambda} \|u\|_{L^q_t L^r_x},$$

where

$$\Lambda = \left\{ (q, r) : \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) - 1, \, \frac{2d}{d-2} \le r \le \frac{2d}{d-4} - \varepsilon_0 \right\}.$$

When $d \ge 5$, the spaces $(\dot{S}^0(I), \|\cdot\|_{\dot{S}^0(I)})$ and $(\mathcal{Z}^1(I), \|\cdot\|_{\mathcal{Z}^1(I)})$ are Banach spaces.

We will occasionally use subscripts to denote spatial derivatives and will use the summation convention over repeated indices.

We work in the framework of [12], [13] and [16]. In Section 2, we recall some useful facts. In Section 3, we obtain some variational estimates and blow-up results (part (2) of Theorem 1.1 and Theorem 1.2). Finally, using a concentration-compactness argument, we obtain the scattering result (part (1) of Theorem 1.1) in Sections 4 and 5.

2. A review of the Cauchy problem. In this section, we will recall some basic facts about the Cauchy problem

$$\begin{cases} iu_t + \Delta u = f(u), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \, d \ge 5, \\ u(t_0) \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{2.1}$$

where $f(u) = -(|x|^{-4} * |u|^2)u$. It is the $\dot{H}^1$ critical, focusing Hartree equation.

In the above notations, we have the following *Strichartz inequalities*:

LEMMA 2.1 (Strichartz estimate [11], [31]). *Let u be an $\dot{S}^0$ solution to the Schrödinger equation* (2.1). *Then*

$$\|u\|_{\dot{S}^0} \lesssim \|u(t_0)\|_{L_x^2} + \|f(u)\|_{L_t^{q'} L_x^{r'}(I \times \mathbb{R}^d)}$$

for any $t_0 \in I$ and any admissible pair (q, r). The implicit constant is independent of the choice of the interval I.

From Sobolev embedding, we have

LEMMA 2.2. *For any function u on $I \times \mathbb{R}^d$,*

$$\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^6 L_x^{6d/(3d-2)}} + \|\nabla u\|_{L_t^3 L_x^{6d/(3d-4)}} + \|u\|_{L_t^\infty L_x^{2d/(d-2)}} + \|u\|_{L_t^6 L_x^{6d/(3d-8)}} \lesssim \|\nabla u\|_{\dot{S}^0},$$

where all space-time norms are on $I \times \mathbb{R}^d$.

For convenience, we introduce two abbreviated notations. For a time interval I, we set

$$\|u\|_{X(I)} := \|u\|_{L_t^6(I; L_x^{6d/(3d-8)})}, \qquad \|u\|_{Y(I)} := \|\nabla u\|_{L_t^6(I; L_x^{6d/(3d-2)})},$$
$$\|u\|_{W(I)} := \|\nabla u\|_{L_t^3(I; L_x^{6d/(3d-4)})}.$$

We develop a local well-posedness and blow-up criterion for the $\dot{H}^1$-critical Hartree equation. First, we have

Proposition 2.1 (Local well-posedness [25]). *Suppose* $\|u(t_0)\|_{\dot{H}^1} \le A$, *and let* I *be a compact time interval that contains* t_0 *such that*

$$\|U(t-t_0)u(t_0)\|_{X(I)} \le \delta$$

for a sufficiently small absolute constant $\delta = \delta(A) > 0$. *Then there exists a unique solution* $u \in C_t^0 \dot{H}_x^1$ *to* (2.1) *on* $I \times \mathbb{R}^d$ *such that*

$$\|u\|_{W(I)} < \infty, \quad \|u\|_{X(I)} \le 2\delta.$$

Moreover, if $u_{0,k} \to u_0$ *in* $\dot{H}^1(\mathbb{R}^d)$, *the corresponding solutions* u_k *tend to* u *in* $C(I; \dot{H}^1(\mathbb{R}^d))$.

Remark 2.1. There exists $\widetilde{\delta} > 0$ such that if $\|u(t_0)\|_{\dot{H}^1} \le \widetilde{\delta}$, the conclusion of Proposition 2.1 applies to any interval I. In fact, by the Strichartz estimates, we have

$$\|e^{i(t-t_0)\Delta}u(t_0)\|_{X(I)} \le C\|e^{i(t-t_0)\Delta}u(t_0)\|_{Y(I)} \le C\widetilde{\delta},$$

and the claim follows.

Remark 2.2. Given $u_0 \in \dot{H}^1$, there exists I such that $0 \in I$ and the hypothesis of Proposition 2.1 is satisfied on I. In fact, by the Strichartz estimates, we have

$$\|e^{it\Delta}u_0\|_{Y(I)} < \infty,$$

and the claim follows from the Sobolev inequality and absolute continuity theorem.

Remark 2.3 (Energy identity). By the standard limiting argument, if u is the solution constructed in Proposition 2.1, then

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{1}{4}\iint \frac{1}{|x-y|^4}\,|u(t,x)|^2|u(t,y)|^2\,dx\,dy \tag{2.2}$$

is constant for $t \in I$.

Now let $t_0 \in I$. We say that $u \in C(I; \dot{H}^1(\mathbb{R}^d)) \cap W(I)$ is a solution of (2.1) if

$$u(t) = e^{i(t-t_0)\Delta}u_0 - i\int_{t_0}^{t} e^{i(t-s)\Delta} f(u)\,ds$$

with $f(u) = -(|x|^{-4} * |u|^2)u$. Note that if $u^{(1)}$, $u^{(2)}$ are solutions of (2.1) on I, and $u^{(1)}(t_0) = u^{(2)}(t_0)$, then $u^{(1)} \equiv u^{(2)}$ on $I \times \mathbb{R}^d$. In fact, let

$$A = \sup_{t\in I}\max_{i=1,2}\|u^{(i)}(t)\|_{\dot{H}^1}$$

and partition I into a finite collection of subintervals I_j. If j_0 is such that $t_0 \in I_{j_0}$, then the uniqueness of the fixed point in the proof of Proposition 2.1, combined with Remark 2.2, gives an interval $\widetilde{I} \ni t_0$ such that $u^{(1)}(t) =$

$u^{(2)}(t)$ for $t \in \widetilde{I}$. A continuation argument now easily gives $u^{(1)}(t) = u^{(2)}(t)$ for $t \in I$.

DEFINITION 2.1. The above analysis allows us to define the *maximal interval* $(t_0 - T_-(u_0), t_0 + T_+(u_0))$, with $T_\pm(u_0) > 0$, where the solution is defined. If $T_1 < t_0 + T_+(u_0)$, $T_2 > t_0 - T_-(u_0)$, $T_2 < t_0 < T_1$, then u solves (2.1) in $[T_2, T_1] \times \mathbb{R}^d$, so that $u \in C([T_2, T_1], \dot{H}^1(\mathbb{R}^d)) \cap X([T_2, T_1]) \cap W([T_2, T_1])$.

PROPOSITION 2.2 (Blow-up criterion [25]). *If $T_+(u_0) < \infty$, then*

$$\|u\|_{X(t_0, t_0+T_+(u_0))} = \infty.$$

A corresponding result holds for $T_-(u_0)$.

DEFINITION 2.2. Let $v_0 \in \dot{H}^1$, $v(t) = e^{it\Delta}v_0$ and let t_n be a sequence with $\lim_{n\to\infty} t_n = \bar{t} \in [-\infty, \infty]$. We say that $u(t,x)$ is a *nonlinear profile* associated with $(v_0, \{t_n\})$ if there exists an interval I with $\bar{t} \in I$ (if $\bar{t} = \pm\infty$ then $I = [a, \infty)$ or $(-\infty, a]$) such that u is a solution of (2.1) in I and

$$\lim_{n\to\infty} \|u(t_n, \cdot) - v(t_n, \cdot)\|_{\dot{H}^1} = 0.$$

REMARK 2.4. As in [12], there always exists a unique nonlinear profile $u(t)$ associated to $(v_0, \{t_n\})$, with maximal interval I.

Lastly, in order to meet our needs in Lemma 4.2, we give a stability theory, which is somewhat different from that in [26], but the proofs are similar in essence.

PROPOSITION 2.3 (Long-time perturbations). *Let I be a compact interval, and let $\widetilde{u}$ be a function on $I \times \mathbb{R}^d$ which obeys the bounds*

$$\|\widetilde{u}\|_{X(I)} \le M \tag{2.3}$$

and

$$\|\widetilde{u}\|_{L_t^\infty(I;\dot{H}_x^1)} \le E \tag{2.4}$$

for some $M, E > 0$. Suppose also that $\widetilde{u}$ is a near-solution to (2.1) in the sense that it solves

$$(i\partial_t + \Delta)\widetilde{u} = -(|x|^{-4} * |\widetilde{u}|^2)\widetilde{u} + e \tag{2.5}$$

for some function e. Let $t_0 \in I$, and let $u(t_0)$ be close to $\widetilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \widetilde{u}(t_0)\|_{\dot{H}_x^1} \le E'$$

for some $E' > 0$. Assume also that we have the smallness conditions

$$\|e^{i(t-t_0)\Delta}(u(t_0) - \widetilde{u}(t_0))\|_{\mathcal{Z}^1(I)} \le \varepsilon, \tag{2.6}$$

$$\|e\|_{L_t^{3/2}(I;\dot{H}_x^{1,6d/(3d+4)})} \le \varepsilon \tag{2.7}$$

for some $0 < \varepsilon < \varepsilon_1$, where ε_1 is some constant $\varepsilon_1 = \varepsilon_1(E, E', M) > 0$.

Then there exists a solution u to (2.1) *on $I \times \mathbb{R}^d$ with the specified initial data $u(t_0)$ at t_0, and*

$$\|u\|_{\mathcal{Z}^1(I)} \le C(M,E,E').$$

Moreover,

$$\|\nabla u\|_{S^0(I)} \le C(M,E,E').$$

REMARK 2.5. Under the assumptions (2.3) and (2.7), the assumption (2.4) is equivalent to

$$\|\nabla \widetilde{u}(t_0)\|_{L^2} \le E.$$

REMARK 2.6. The long time perturbation theorem in [26] yields the following continuity fact, which will be used later: Let $\widetilde{u}_0 \in \dot{H}^1$ with $\|\widetilde{u}_0\|_{\dot{H}^1} \le A$, and let $\widetilde{u}$ be the solution of (2.1) with maximal interval of existence $(T_-(\widetilde{u}_0), T_+(\widetilde{u}_0))$. Let $u_{0,n} \to \widetilde{u}_0$ in $\dot{H}^1$, and let u_n be the corresponding solution of (2.1), with maximal interval of existence $(T_-(u_{0,n}), T_+(u_{0,n}))$. Then

$$T_-(\widetilde{u}_0) \ge \limsup_{n\to\infty} T_-(u_{0,n}), \qquad T_+(\widetilde{u}_0) \le \liminf_{n\to\infty} T_+(u_{0,n}),$$

and for each $t \in (T_-(\widetilde{u}_0), T_+(\widetilde{u}_0))$, $u_n(t) \to \widetilde{u}(t)$ in $\dot{H}^1$.

3. Some variational estimates and blow-up result. Let $W(x)$ be the *ground state*, i.e. the positive radial $\dot{H}^1$ solution to the elliptic equation

$$\Delta W + (|x|^{-4} * |W|^2)W = 0. \tag{3.1}$$

The existence and uniqueness of W have been established in [18] and [21]. By invariance of the equation, for $\theta_0 \in [-\pi,\pi]$, $\lambda_0 > 0$, $x_0 \in \mathbb{R}^d$,

$$W_{\theta_0,x_0,\lambda_0}(x) = \lambda_0^{-(d-2)/2} e^{i\theta_0} W\left(\frac{x-x_0}{\lambda_0}\right)$$

is still a solution. Now let C_d be the best constant in the Sobolev inequality in dimension d. That is,

$$\forall u \in \dot{H}^1, \quad \|(|x|^{-4} * |u|^2)|u|^2\|_{L^1}^{1/4} \le C_d \|\nabla u\|_{L^2}. \tag{3.2}$$

In addition, using the concentration-compactness argument [10], [19], [20], [27], we can obtain the following characterization of W:

If $\|(|x|^{-4} * |u|^2)|u|^2\|_{L^1}^{1/4} = C_d\|\nabla u\|_{L^2}$ and $u \ne 0$, then there exists $(\theta_0, \lambda_0, x_0)$ such that $u = W_{\theta_0,x_0,\lambda_0}$.

From the above, we have

$$\|(|x|^{-4} * |W|^2)|W|^2\|_{L^1} = C_d^4 \Big(\int |\nabla W|^2\,dx\Big)^2.$$

On the other hand, from (3.1), we obtain

$$\|(|x|^{-4} * |W|^2)|W|^2\|_{L^1} = \int |\nabla W|^2\,dx.$$

Hence,

$$\|\nabla W\|_{L^2}^2 = \frac{1}{C_d^4}, \qquad E(W) = \left(\frac{1}{2} - \frac{1}{4}\right)\|\nabla W\|_{L^2}^2 = \frac{1}{4C_d^4}.$$

LEMMA 3.1. *Assume that*

$$\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}.$$

Assume moreover that $E(u) \le (1-\delta_0)E(W)$ *for some* $\delta_0 > 0$. *Set* $\overline{\delta} = \delta_0^{1/2} > 0$. *Then*

$$\int |\nabla u|^2\,dx - \iint \frac{|u(x)|^2|u(y)|^2}{|x-y|^4}\,dx\,dy \ge \frac{\overline{\delta}}{2}\int |\nabla u|^2\,dx,$$
$$\int |\nabla u|^2\,dx \le (1-\overline{\delta})\int |\nabla W|^2\,dx, \qquad E(u) \ge 0.$$

Proof. Define

$$a = \int |\nabla u|^2\,dx \quad \text{and} \quad f(x) = \frac{1}{2}\,x - \frac{1}{4}\,C_d^4 x^2.$$

From (3.2), we have

$$(1-\delta_0)E(W) \ge E(u) \ge \frac{1}{2}\int |\nabla u|^2\,dx - \frac{1}{4}\,C_d^4\Big(\int |\nabla u|^2\,dx\Big)^2 = f(a). \tag{3.3}$$

Note that

$$f'(x) = \frac{1}{2} - \frac{1}{2}C_d^4 x.$$

This implies that

$$f'(x) = 0 \;\Leftrightarrow\; x = \frac{1}{C_d^4} = \int |\nabla W(x)|^2 dx.$$

On the other hand,

$$f'(x) > 0 \quad \text{for } x < 1/C_d^4,$$
$$f(0) = 0, \quad f\left(\frac{1}{C_d^4}\right) = \frac{1}{4C_d^4} = E(W).$$

Together with (3.3) and the fact that $a = \|\nabla u\|_{L^2}^2 \in [0, 1/C_d^4)$, these imply that

$$\|\nabla u\|_{L^2}^2 = a \le (1-\overline{\delta})\,\frac{1}{C_d^4} = (1-\overline{\delta})\int |\nabla W|^2\,dx, \qquad \overline{\delta} = \delta_0^{1/2},$$
$$E(u) \ge f(a) \ge 0.$$

Now define

$$g(x) = x - C_d^4 x^2.$$

From (3.2), we also have

$$(3.4)\quad \int |\nabla u|^2\,dx - \iint \frac{|u(x)|^2|u(y)|^2}{|x-y|^4}\,dx\,dy \geq \int |\nabla u|^2\,dx - C_d^4\Big(\int |\nabla u|^2\,dx\Big)^2 = g(a).$$

Note that

$$g(x) = 0 \iff x = 0 \text{ or } x = \frac{1}{C_d^4},$$
$$g'(0) = 1, \quad g'(1/C_d^4) = -1, \quad g''(x) = -2C_d^4 < 0.$$

Hence, we obtain

$$g(x) \geq \frac{1}{2}\min(x, 1/C_d^4 - x) \quad \text{for } 0 \leq x \leq 1/C_d^4.$$

Since $\|\nabla u\|_{L^2}^2 = a \in [0, (1-\overline{\delta})/C_d^4]$, the above inequality implies that

$$\text{(LHS) of (3.4)} \geq g(a) \geq \frac{1}{2}\min(a, 1/C_d^4 - a) \geq \frac{1}{2}\min(a, \overline{\delta}a) = \frac{\overline{\delta}}{2}\,a.$$

This completes the proof.

Corollary 3.1. *Assume that* $u \in \dot{H}^1(\mathbb{R}^d)$ *and* $\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}$. *Then* $E(u) \geq 0$.

Proof. If $E(u) < E(W)$, the conclusion follows from Lemma 3.1. If $E(u) \geq E(W) = 1/4C_d^4$, it is clear.

Proposition 3.1 (Lower bound on the convexity of the variance). *Let* u *be a solution of* (2.1) *with* $t_0 = 0$, $u(0) = u_0$ *such that for* $\delta_0 > 0$,

$$\int |\nabla u_0|^2\,dx < \int |\nabla W|^2\,dx, \quad E(u_0) < (1-\delta_0)E(W).$$

Let $I \ni 0$ *be the maximal interval of existence given by Definition* 2.1. *Let* $\overline{\delta} = \delta_0^{1/2}$ *be as in Lemma* 3.1. *Then for each* $t \in I$,

$$\int |\nabla u(t)|^2\,dx - \iint \frac{|u(t)|^2|u(t)|^2}{|x-y|^4}\,dx\,dy \geq \frac{\overline{\delta}}{2}\int |\nabla u(t)|^2\,dx,$$
$$\int |\nabla u(t)|^2\,dx \leq (1-\overline{\delta})\int |\nabla W|^2\,dx, \quad E(u(t)) \geq 0.$$

Proof. We use a continuity argument. Define

$$\Omega = \{t \in I : \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2},\ E(u(t)) < (1-\delta_0)E(W)\}.$$

It suffices to prove that Ω is both open and closed.

First, we see that $t_0 \in \Omega$. Second, Ω is open because of $u \in C_t^0(I, \dot{H}^1)$ and the conservation of energy. Lastly, to prove that Ω is also closed, pick any $t_n \in \Omega$ and $T \in I$ with $t_n \to T$. Then

$$\|\nabla u(t_n)\|_{L^2} < \|\nabla W\|_{L^2}, \quad E(u(t_n)) < (1-\delta_0)E(W).$$

From Lemma 3.1, we obtain

$$\|\nabla u(t_n)\|_{L^2}^2 < (1-\overline{\delta})\|\nabla W\|_{L^2}^2.$$

Using the fact that $u \in C_t^0(I, \dot{H}^1)$ and the conservation of energy again, we have

$$\|\nabla u(T)\|_{L^2}^2 \leq (1-\overline{\delta})\|\nabla W\|_{L^2}^2, \quad E(u(T)) = E(u(t_n)) < (1-\delta_0)E(W).$$

This implies that $T \in \Omega$ and completes the proof.

Corollary 3.2 (Comparability of gradient and energy). *Let u, u_0 be as in Proposition 3.1. Then for all $t \in I$,*

$$E(u(t)) \approx \int |\nabla u(t)|^2 \, dx \approx \int |\nabla u_0|^2 \, dx$$

with comparability constants which depend only on δ_0.

Proof. From Proposition 3.1, we have

$$\begin{aligned}\frac{1}{2}\int |\nabla u(t)|^2 \, dx &\geq E(u(t)) \\ &= \frac{1}{4}\int |\nabla u(t)|^2 \, dx + \frac{1}{4}\left(\int |\nabla u(t,x)|^2 \, dx - \iint \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} \, dx \, dy\right) \\ &\geq \frac{2+\overline{\delta}}{8}\int |\nabla u(t)|^2 \, dx \quad \forall t \in I.\end{aligned}$$

This together with the conservation of energy implies the claim.

In order to obtain blow up results, we first give the (local) virial identity, which we can verify by direct computations.

Lemma 3.2. *Let $\varphi \in C_0^\infty(\mathbb{R}^d)$, $V(x) = |x|^{-4}$, $t \in [0, T_+(u_0))$. Then*

(1) $\displaystyle \frac{d}{dt}\int |u|^2 \varphi \, dx = 2 \operatorname{Im} \int \overline{u} \nabla u \nabla \varphi \, dx,$

(2) $$\begin{aligned}\frac{d^2}{dt^2}\int |u|^2 \varphi \, dx = &-\int \Delta\Delta\varphi |u|^2 \, dx + 4 \operatorname{Re}\int \varphi_{jk} \overline{u}_j u_k \, dx \\ &- \operatorname{Re}\iint (\nabla\varphi(x) - \nabla\varphi(y)) \nabla V(x-y) |u(y)|^2 |u(x)|^2 \, dx \, dy.\end{aligned}$$

Proposition 3.2. *Assume that $u_0 \in \dot{H}^1(\mathbb{R}^d)$ and*

$$E(u_0) < E(W), \quad \int |\nabla u_0|^2 \, dx > \int |\nabla W|^2 \, dx.$$

If $|x|u_0 \in L^2$ or u_0 is radial, then the maximal interval I of existence must be finite.

Proof. Indeed, we can choose a suitable small number $\delta_0 > 0$ such that

$$E(u_0) < (1-\delta_0)E(W), \quad \int |\nabla u_0|^2 \, dx > \int |\nabla W|^2 \, dx.$$

Arguing as in Lemma 3.1, we find that there exists $\widetilde{\delta}$ such that

$$\int |\nabla u_0|^2\,dx > (1+\widetilde{\delta})\int |\nabla W|^2\,dx = \frac{1+\widetilde{\delta}}{C_d^4}.$$

This shows that

$$\int |\nabla u_0|^2\,dx - \iint \frac{|u_0(x)|^2|u_0(y)|^2}{|x-y|^4}\,dx\,dy = 4E(u_0) - \int |\nabla u_0|^2\,dx$$
$$< 4(1-\delta_0)E(W) - \frac{1+\widetilde{\delta}}{C_d^4} = \frac{1-\delta_0}{C_d^4} - \frac{1+\widetilde{\delta}}{C_d^4} = -\frac{\delta_0+\widetilde{\delta}}{C_d^4} < 0.$$

Now define

$$\Omega = \{t \in I : \|\nabla u(t)\|_{L^2} > \|\nabla W\|_{L^2},\ E(u(t)) < (1-\delta_0)E(W)\}.$$

Using the continuity argument and arguing as in Proposition 3.1, we obtain $\Omega = I$. Arguing as in Lemma 3.1 again, we have

$$\|\nabla u(t)\|_{L^2}^2 > (1+\widetilde{\delta})\|\nabla W\|_{L^2}^2.$$

Then

$$\int |\nabla u(t,x)|^2\,dx - \iint \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy = -\frac{\delta_0+\widetilde{\delta}}{C_d^4} < 0, \qquad \forall t \in I.$$

If $|x|u_0 \in L^2$, then from Lemma 3.2, we have

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2\,dx = 8\biggl(\int |\nabla u(t,x)|^2\,dx - \iint \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy\biggr) < 0.$$

This implies that I must be finite.

If u_0 is radial, then using the local virial identity [2], [3] and [30], we can also deduce the same result.

4. Existence and compactness of a critical element. Let us consider the statement

(SC) For all $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E(W)$, if u is the corresponding solution to (2.1), with maximal interval of existence I, then $I = (-\infty,\infty)$ and $\|u\|_{X(\mathbb{R})} < \infty$.

We say that (SC)(u_0) holds if whenever $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E(W)$, and u is the corresponding solution to (2.1) with maximal interval of existence I, then $I = (-\infty,\infty)$ and $\|u\|_{X(\mathbb{R})} < \infty$.

Note that, because of Remark 2.1, if $\|u_0\|_{\dot{H}^1} \le \widetilde{\delta}$, then (SC)($u_0$) holds. Thus, in light of Corollary 3.2, there exists $\eta_0 > 0$ such that if u_0 is as in (SC) and $E(u_0) < \eta_0$, then (SC)(u_0) holds. Moreover, $E(u_0) \ge 0$ in light of Proposition 3.1. Thus, there exists a number E_c with $\eta_0 \le E_c \le E(W)$ such that if u_0 is radial with $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E_c$, then (SC)(u_0) holds, and E_c is optimal with this property. If $E_c \ge E(W)$, then the first part of Theorem 1.1 is true. For the rest of this section, we will assume that $E_c < E(W)$ and ultimately deduce a contradiction in Section 5. By definition of E_c, we have

(C.1) If u_0 is radial and $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E_c$, then (SC)(u_0) holds.

(C.2) There exists a sequence of radial solutions u_n to (2.1) with corresponding initial data $u_{n,0}$ such that $\|\nabla u_{n,0}\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_{n,0}) \searrow E_c$ as $n \to \infty$, and (SC)$(u_{n,0})$ does not hold for any n.

The goal of this section is to use the above sequence $u_{n,0}$ to prove the existence of an $\dot{H}^1$ radial solution u_c to (2.1) with initial data $u_{c,0}$ such that $\|\nabla u_{c,0}\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_{c,0}) = E_c$ and (SC)$(u_{c,0})$ does not hold (see Proposition 4.1). Moreover, we will show that this critical solution has a compactness property up to symmetries of this equation (see Proposition 4.2).

Before stating and proving Proposition 4.1, we introduce some useful preliminaries in the spirit of the results of Keraani [14]. First we give the profile decomposition lemma.

LEMMA 4.1 (Profile decomposition). *Let $v_{n,0}$ be a radial uniformly bounded sequence in $\dot{H}^1$, i.e. $\|\nabla v_{n,0}\|_{L^2} \le A$. Assume that $\|e^{it\Delta} v_{n,0}\|_{X(\mathbb{R})} \ge \delta > 0$, where $\delta = \delta(d)$ is as in Proposition 2.1. Then for each J, there exists a subsequence of $v_{n,0}$, also denoted $v_{n,0}$, and*

- *for each $1 \le j \le J$, there exists a radial profile $V_{0,j}$ in $\dot{H}^1$,*
- *for each $1 \le j \le J$, there exists a sequence of $(\lambda_{j,n}, t_{j,n})$ with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} \to \infty \quad \text{as } n \to \infty \text{ for } j \neq j', \tag{4.1}$$

- *there exists a sequence of radial remainders w_n^J in $\dot{H}^1$,*

such that

$$v_{n,0}(x) = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\Big) + w_n^J(x) \tag{4.2}$$

with

$$V_j^l(t,x) = e^{it\Delta} V_{0,j}(x), \qquad \|V_{0,1}\|_{\dot{H}^1} \ge \alpha_0(A) > 0, \tag{4.3}$$

$$\|\nabla v_{n,0}\|_{L^2}^2 = \sum_{j=1}^{J} \|\nabla V_{0,j}\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \tag{4.4}$$

$$E(v_{n,0}) = \sum_{j=1}^{J} E\Big(V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}\Big)\Big) + E(w_n^J) + o_n(1), \tag{4.5}$$

$$\lim_{J\to\infty}\big[\lim_{n\to\infty}\|e^{it\Delta}w_n^J\|_{L^q(\mathbb{R},L^r)}\big]=0 \tag{4.6}$$
$$\textit{whenever}\quad \frac{2}{q}=d\Big(\frac{1}{2}-\frac{1}{r}\Big)-1,\ \frac{2d}{d-2}\le r<\frac{2d}{d-4}.$$

Proof. Here we only give the proof of the energy asymptotic Pythagorean expansion (4.5), the rest is standard (see [14]).

By the asymptotic Pythagorean expansion of kinetic energy, it suffices to show that

$$\begin{aligned}\iint\frac{1}{|x-y|^4}|v_{n,0}(x)|^2|v_{n,0}(y)|^2\,dx\,dy&\\ =\sum_{j=1}^{J}\iint\frac{1}{|x-y|^4}\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},x\Big)\Big|^2\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},y\Big)\Big|^2\,dx\,dy&\\ +\iint\frac{1}{|x-y|^4}\,|w_n^J(x)|^2|w_n^J(y)|^2\,dx\,dy+o_n(1),\qquad\forall J\ge 1.&\end{aligned}$$

We first claim that if $J\ge 1$ is fixed, the orthogonality condition (4.1) implies that

$$\begin{aligned}&\iint\frac{1}{|x-y|^4}\Big|\sum_{j=1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{x}{\lambda_{j,n}}\Big)\Big|^2\\ &\quad\times\Big|\sum_{j=1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{y}{\lambda_{j,n}}\Big)\Big|^2\,dx\,dy\\ &=\sum_{j=1}^{J}\iint\frac{1}{|x-y|^4}\,\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},x\Big)\Big|^2\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},y\Big)\Big|^2\,dx\,dy+o_n(1).\end{aligned}\tag{4.7}$$

By reindexing, we can arrange that there is $J_0\le J$ such that

(1) if $1\le j\le J_0$, then $|t_{j,n}/\lambda_{j,n}^2|\le C$ in n;
(2) if $J_0+1\le j\le J$, then $|t_{j,n}/\lambda_{j,n}^2|\to\infty$ as $n\to\infty$.

By passing to a subsequence and adjusting the profile $V_{0,j}$, we may assume that

$$\forall 1\le j\le J_0,\quad t_{j,n}/\lambda_{j,n}^2=0.$$

From case (2), we have

$$\lim_{n\to\infty}\iint\frac{1}{|x-y|^4}\,\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},x\Big)\Big|^2\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},y\Big)\Big|^2\,dx\,dy=0,\qquad \forall J_0+1\le j\le J.\tag{4.8}$$

Indeed, using the Hardy inequality and the decay estimates for the free Schrödinger equation (similarly to Lemma 4.1 in [5] and Corollary 2.3.7

in [1]), we have for $J_0+1\le j\le J$,

$$\iint \frac{1}{|x-y|^4}\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},x\Big)\Big|^2\Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},y\Big)\Big|^2\,dx\,dy \lesssim \Big\|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}\Big)\Big\|_{L^{2d/(d-2)}}^4 \to 0 \quad \text{as } n\to\infty.$$

By (4.1), if $1\le j<k\le J_0$, we have

$$\frac{\lambda_{j,n}}{\lambda_{k,n}}+\frac{\lambda_{k,n}}{\lambda_{j,n}}\to\infty \quad \text{as } n\to\infty. \tag{4.9}$$

This implies that

$$\begin{aligned}\iint \frac{1}{|x-y|^4}\Big|\sum_{j=1}^{J_0}\frac{1}{\lambda_{j,n}^{(d-2)/2}}V_{0,j}\Big(\frac{x}{\lambda_{j,n}}\Big)\Big|^2 &\times\Big|\sum_{j=1}^{J_0}\frac{1}{\lambda_{j,n}^{(d-2)/2}}V_{0,j}\Big(\frac{y}{\lambda_{j,n}}\Big)\Big|^2\,dx\,dy\\ &=\sum_{j=1}^{J_0}\iint\frac{1}{|x-y|^4}|V_{0,j}(x)|^2|V_{0,j}(y)|^2\,dx\,dy+o_n(1).\end{aligned} \tag{4.10}$$

Hence, from (4.8) and (4.10), we obtain

$$\begin{aligned}&\iint \frac{1}{|x-y|^4}\Big|\sum_{j=1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{x}{\lambda_{j,n}}\Big)\Big|^2\\ &\qquad\qquad\times\Big|\sum_{j=1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{y}{\lambda_{j,n}}\Big)\Big|^2\,dx\,dy\\ &=\iint\frac{1}{|x-y|^4}\Big|\sum_{j=1}^{J_0}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{x}{\lambda_{j,n}}\Big)\\ &\quad+\sum_{j=J_0+1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{x}{\lambda_{j,n}}\Big)\Big|^2\Big|\sum_{j=1}^{J_0}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{y}{\lambda_{j,n}}\Big)\\ &\quad+\sum_{j=J_0+1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{y}{\lambda_{j,n}}\Big)\Big|^2\,dx\,dy\\ &=\iint\frac{1}{|x-y|^4}\Big|\sum_{j=1}^{J_0}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_{0,j}\Big(\frac{x}{\lambda_{j,n}}\Big)+\sum_{j=J_0+1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{x}{\lambda_{j,n}}\Big)\Big|^2\\ &\quad\times\Big|\sum_{j=1}^{J_0}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_{0,j}\Big(\frac{y}{\lambda_{j,n}}\Big)+\sum_{j=J_0+1}^{J}\frac{1}{\lambda_{j,n}^{(d-2)/2}}\,V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2},\frac{y}{\lambda_{j,n}}\Big)\Big|^2\,dx\,dy\end{aligned}$$

$$= \iint \frac{1}{|x-y|^4} \Big| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_{0,j}\Big(\frac{x}{\lambda_{j,n}}\Big)\Big|^2 \Big| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_{0,j}\Big(\frac{y}{\lambda_{j,n}}\Big)\Big|^2 dx\,dy$$

$$+ \sum_{j=J_0+1}^{J} \iint \frac{1}{|x-y|^4} \Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x\Big)\Big|^2 \Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y\Big)\Big|^2 dx\,dy + o_n(1)$$

$$= \sum_{j=1}^{J} \iint \frac{1}{|x-y|^4} \Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x\Big)\Big|^2 \Big|V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y\Big)\Big|^2 dx\,dy + o_n(1),$$

which yields (4.7).

Second, we claim that

$$\lim_{n\to\infty} \|w_n^J(x)\|_{L_x^{2d/(d-2)}} = 0 \quad \text{as } J \to \infty. \tag{4.11}$$

Indeed, we have

$$\|w_n^J(x)\|_{L_x^{2d/(d-2)}} \lesssim \|e^{it\Delta} w_n^J(x)\|_{L_t^\infty(\mathbb{R}; L_x^{2d/(d-2)})},$$

which together with (4.6) implies the claim.

Note that (4.11) implies that $\{w_n^J\}$ is uniformly bounded in $L^{2d/(d-2)}(\mathbb{R}^d)$; the uniform boundedness of $\{v_{n,0}\}$ in $\dot H^1(\mathbb{R}^d)$ also implies uniform boundedness in $L^{2d/(d-2)}(\mathbb{R}^d)$. Thus we can choose $J_1 \geq J$ and N_1 such that for $n \geq N_1$, we have

$$\begin{aligned} &\Big| \iint \frac{|v_{n,0}(x)|^2 |v_{n,0}(y)|^2}{|x-y|^4}\,dx\,dy \\ &\qquad - \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\,dx\,dy \Big| \\ &+ \Big| \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\,dx\,dy \\ &\qquad - \iint \frac{|w_n^J(x)|^2 |w_n^J(y)|^2}{|x-y|^4}\,dx\,dy \Big| \\ &\leq C\big(\sup_n \|v_{n,0}(x)\|^3_{L^{2d/(d-2)}} + \sup_n \|w_n^J(x)\|^3_{L^{2d/(d-2)}}\big) \|w_n^{J_1}(x)\|_{L^{2d/(d-2)}} \\ &\quad + C\|w_n^{J_1}(x)\|^4_{L^{2d/(d-2)}} \leq \varepsilon. \end{aligned} \tag{4.12}$$

By (4.7), there exists $N_2 \geq N_1$ such that for $n \geq N_2$,

$$\begin{aligned} &\Big| \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\,dx\,dy \\ &\qquad - \sum_{j=1}^{J_1} \iint \frac{|V_j^l(-t_{j,n}/\lambda_{j,n}^2, x)|^2 |V_j^l(-t_{j,n}/\lambda_{j,n}^2, y)|^2}{|x-y|^4}\,dx\,dy \Big| \leq \varepsilon. \end{aligned} \tag{4.13}$$

Using (4.2), we have

$$w_n^J(x) - w_n^{J_1}(x) = \sum_{j=J+1}^{J_1} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_j^l\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\Big).$$

By (4.7), there exists $N_3 \geq N_2$ such that for $n \geq N_3$,

$$\Big| \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\, dx\, dy - \sum_{j=J+1}^{J_1} \iint \frac{|V_j^l(-t_{j,n}/\lambda_{j,n}^2, x)|^2 |V_j^l(-t_{j,n}/\lambda_{j,n}^2, y)|^2}{|x-y|^4}\, dx\, dy \Big| \leq \varepsilon.$$

Combining the above inequality with (4.12), (4.13), we deduce that for $n \geq N_3$,

$$\begin{aligned}
&\Big| \iint \frac{|v_{n,0}(x)|^2 |v_{n,0}(y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad - \sum_{j=1}^{J} \iint \frac{|V_j^l(-t_{j,n}/\lambda_{j,n}^2, x)|^2 |V_j^l(-t_{j,n}/\lambda_{j,n}^2, y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad - \iint \frac{|w_n^J(x)|^2 |w_n^J(y)|^2}{|x-y|^4}\, dx\, dy \Big| \\
&= \Big| \iint \frac{|v_{n,0}(x)|^2 |v_{n,0}(y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad - \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad + \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad - \sum_{j=1}^{J_1} \iint \frac{|V_j^l(-t_{j,n}/\lambda_{j,n}^2, x)|^2 |V_j^l(-t_{j,n}/\lambda_{j,n}^2, y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad + \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad - \iint \frac{|w_n^J(x)|^2 |w_n^J(y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad + \sum_{j=J+1}^{J_1} \iint \frac{|V_j^l(-t_{j,n}/\lambda_{j,n}^2, x)|^2 |V_j^l(-t_{j,n}/\lambda_{j,n}^2, y)|^2}{|x-y|^4}\, dx\, dy \\
&\qquad - \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4}\, dx\, dy \Big| \\
&\leq 3\varepsilon,
\end{aligned}$$

which completes the proof.

Lemma 4.2. *Let $\{z_{0,n}\} \in \dot{H}^1$ be radial with*

$$\|\nabla z_{0,n}\|_{L^2} < \|\nabla W\|_{L^2}, \quad E(z_{0,n}) \to E_c,$$

and with $\|e^{it\Delta} z_{0,n}\|_{X(\mathbb{R})} \geq \delta > 0$, where $\delta = \delta(\|\nabla W\|_{L^2})$ is as in Proposition 2.1. *Let $V_{0,j}$ be as in Lemma* 4.1. *Assume that either*

$$\liminf_{n\to\infty} E\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}^2}\right)\right) < E_c, \tag{4.14}$$

or after passing to a subsequence,

$$\liminf_{n\to\infty} E\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}^2}\right)\right) = E_c \tag{4.15}$$

with $s_{1,n} = -t_{1,n}/\lambda_{1,n}^2 \to s_* \in [-\infty, \infty]$, and if U_1 is the nonlinear profile associated to $(V_{0,1}, \{s_{1,n}\})$, then the maximal interval of existence of U_1 is $I = (-\infty, \infty)$ and $\|U_1\|_{X(\mathbb{R})} < \infty$.

Then, *after passing to a subsequence*, *for n large*, *if z_n is the solution of* (2.1) *with data at $t = 0$ equal to $z_{0,n}$, then* (SC)$(z_{0,n})$ *holds.*

Proof. The proof is similar to that of Lemma 4.9 in [12].

Along with the proof in [12], we can obtain a critical element, which has minimal energy and compactness property modulo scaling symmetry in $\dot{H}^1$.

Proposition 4.1 (Existence of a minimal energy blow-up solution). *There exists a radial solution u_c of* (2.1) *in $\dot{H}^1$ with data $u_{c,0}$ and maximal interval of existence I such that*

$$\|\nabla u_{c,0}\|_{L^2} < \|\nabla W\|_{L^2}, \quad E(u_{c,0}) = E_c, \quad \|u_c\|_{X(I)} = \infty.$$

Proposition 4.2 (Precompactness of the flow of the critical solution). *Let u_c be as in Proposition* 4.1, *and with $\|u_c\|_{X(I_+)} = \infty$, where $I_+ = (0, \infty) \cap I$. Then for $t \in I_+$, there exists $\lambda(t) \in \mathbb{R}^+$ such that K is precompact in $\dot{H}^1$ where*

$$K = \left\{ v(t,x) : v(t,x) = \frac{1}{\lambda(t)^{(d-2)/2}}\, u_c\left(t, \frac{x}{\lambda(t)}\right), \ t \in I_+ \right\}.$$

Remark 4.1. We refer to $\lambda(t)$ as the frequency scale function for the solution u_c because $\lambda(t)$ measures the frequency scale of the solution at time t and $1/\lambda(t)$ measures the spatial scale.

5. Rigidity theorem. In this section, we will prove the main theorem.

Theorem 5.1. *Assume that $u_0 \in \dot{H}^1$ is radial and satisfies*

$$E(u_0) < E(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}.$$

Let u be the solution of (2.1) *with maximal interval of existence $(-T_-(u_0), T_+(u_0))$. Assume that there exists $\lambda(t) > 0$, for $t \in [0, T_+(u_0))$, with the* *property that*

$$K = \left\{ v(t,x) = \frac{1}{\lambda(t)^{(d-2)/2}}\, u\Big(t, \frac{x}{\lambda(t)}\Big) : t \in [0, T_+(u_0)) \right\}$$

is precompact in $\dot{H}^1$. *Then* $T_+(u_0) = \infty$ *and* $u_0 \equiv 0$.

We start out with a special case of the strengthened form of Theorem 5.1.

PROPOSITION 5.1. *Assume that* $u, v, \lambda(t)$ *are as in Theorem* 5.1, *and that* $\lambda(t) \ge A_0 > 0$. *Then the conclusion of Theorem* 5.1 *holds.*

First we collect some useful facts:

LEMMA 5.1. *Let* u, v *be as in Theorem* 5.1.

(1) *Let* $\delta_0 > 0$ *be such that* $E(u_0) \le (1-\delta_0)E(W)$. *Then there exists* $\overline{\delta} > 0$ *such that for all* $t \in [0, T_+(u_0))$,

$$\int |\nabla u(t)|^2\, dx \le (1-\overline{\delta}) \int |\nabla W|^2\, dx,$$

$$\int |\nabla u(t,x)|^2\, dx - \iint \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^4}\, dx\, dy \ge \frac{\overline{\delta}}{2} \int |\nabla u|^2\, dx, \tag{5.1}$$

$$\int |\nabla u(t)|^2\, dx \approx E(u(t)) = E(u_0) \approx \int |\nabla u_0|^2\, dx.$$

(2) *For all* $t \in [0, T_+(u_0))$,

$$\|v(t,x)\|_{L^{2^*}}^2 \le C_1 \int |\nabla v(t,x)|^2\, dx \le C_2 \int |\nabla W(x)|^2\, dx.$$

(3) *For each* ε, *there exists* $R(\varepsilon) > 0$ *such that for* $t \in [0, T_+(u_0))$,

$$\int\limits_{|x|>R(\varepsilon)} \left(|\nabla v(t,x)|^2 + |v(t,x)|^{2^*} + \frac{|v(t,x)|^2}{|x|^2} \right) dx + \iint\limits_{\Omega} \frac{|v(t,x)|^2 |v(t,y)|^2}{|x-y|^4}\, dx\, dy \le \varepsilon, \tag{5.2}$$

where

$$\Omega = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| > R(\varepsilon)\} \cup \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |y| > R(\varepsilon)\}.$$

Proof. From the property of K, we can easily verify the above facts.

Proof of Proposition 5.1. We split the proof into two cases, the finite time blow up for u and the infinite time of existence for u.

CASE 1: $T_+(u_0) < +\infty$. This case corresponds to the finite time blow up case. The proof is similar to [12].

CASE 2: $T_+(u_0) = +\infty$. This case corresponds to a stationary solution or double low to high frequency cascade case (which means that the frequency scale function $\lambda(t)$ goes to infinity as $t \to \infty$, see [16]).

From $u(t,x)=\lambda(t)^{(d-2)/2}v(t,\lambda(t)x)$ and Lemma 5.1, we infer that for each $\varepsilon>0$, there exists $R(\varepsilon)>0$ such that

$$\int\limits_{|x|>R(\varepsilon)} \frac{|u(t,x)|^2}{|x|^2}\,dx+\int\limits_{|x|>R(\varepsilon)} |\nabla u(t,x)|^2\,dx + \iint\limits_{\Omega} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy \le \varepsilon, \tag{5.3}$$

where Ω is as in Lemma 5.1.

On the other hand, from Lemma 5.1 and (5.3), there exists R such that, for all $t\in[0,\infty)$,

$$8\int\limits_{|x|\le R} |\nabla u(t,x)|^2\,dx - 8\iint\limits_{\Omega_1} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy \ge C_{\delta_0}\int|\nabla u_0(x)|^2\,dx, \tag{5.4}$$

where

$$\Omega_1=\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : |x|\le R,\ |y|\le R\}.$$

Now let $\varphi\in C_0^\infty(\mathbb{R}^d)$ be radial, and

$$\varphi(x)=\begin{cases} |x|^2 & \text{for } |x|\le 1,\\ 0 & \text{for } |x|\ge 2.\end{cases}$$

Set

$$\varphi_R(x)=R^2\varphi(x/R),\qquad z_R(t)=\int\varphi_R(x)|u(t,x)|^2\,dx,\qquad t\in[0,T_+(u_0)).$$

We then have

$$\begin{aligned} &|z_R'(t)|\le CR^2\int|\nabla u_0|^2\,dx && \text{for } t>0,\\ &z_R''(t)\ge C_{\delta_0}\int|\nabla u_0|^2\,dx && \text{for } R \text{ large enough}, t>0.\end{aligned} \tag{5.5}$$

In fact, from Lemmas 5.1 and 3.2, we have

$$\begin{aligned} |z_R'(t)| &\le 2R\int|\overline{u}(t,x)\nabla u(t,x)\nabla\varphi(x/R)|\,dx\\ &\le CR\int\limits_{|x|\le 2R}|u|\,|\nabla u|\,dx \le CR^2\|\nabla u(t,x)\|_{L^2}\left\|\frac{|u|}{|x|}\right\|_{L^2}\\ &\le CR^2\int|\nabla u_0|^2\,dx.\end{aligned}$$

On the other hand, from Lemma 3.2, (5.3) and (5.4), we have, for sufficiently large R,

$$\begin{aligned}
z_R''(t) = & -\int \Delta\Delta\varphi\Big(\frac{x}{R}\Big)\frac{|u|^2}{R^2}\,dx + 4\,\mathrm{Re}\int \varphi_{jk}\overline{u}_j u_k\,dx \\
& - 4\,\mathrm{Re}\iint (a_j(x)-a_j(y))\,\frac{x_j-y_j}{|x-y|^6}\,|u(t,x)|^2|u(t,y)|^2\,dx\,dy \\
\approx & \ 8\int_{|x|\le R} |\nabla u(t,x)|^2\,dx - 8\iint_{\Omega_1} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy \\
& + O\bigg(\int_{|x|\approx R} \frac{|u(t,x)|^2}{R^2}\,dx + \int_{|x|\approx R} |\nabla u(t,x)|^2\,dx \\
& \qquad\qquad + \iint_{\Omega_2} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4}\,dx\,dy\bigg) \\
\ge & \ C_{\delta_0}\int |\nabla u_0|^2\,dx,
\end{aligned}$$

where

$$\Omega_1 = \{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : |x|\le R,\ |y|\le R\};$$
$$\Omega_2 = \{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : |x|\sim R\}\cup\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : |y|\sim R\}.$$

From (5.5), we have

$$C_{\delta_0} t\int |\nabla u_0|^2\,dx \le |z_R'(t)-z_R'(0)| \le 2CR^2\int |\nabla u_0|^2\,dx.$$

We have a contradiction for t large unless $u_0\equiv 0$.

Proof of Theorem 5.1. It is analogous to the proofs in [12], [22]. Assume that $u_0\not\equiv 0$. Then

$$\int |\nabla u_0|^2\,dx > 0. \tag{5.6}$$

From Lemma 5.1, we have

$$E(u_0) \ge C_{\delta_0}\int |\nabla u_0|^2\,dx > 0.$$

Because of Proposition 5.1, we only need to consider the case where there exists $\{t_n\}_{n=1}^\infty$, $t_n\ge 0$, such that $\lambda(t_n)\to 0$. We claim that

$$t_n \to T_+(u_0).$$

Indeed, if $t_n\to t_0\in[0,T_+(u_0))$, then for all $R>0$ we have

$$\begin{aligned}
\int_{|x|>R} |v(t_n,x)|^{2^*}\,dx &= \int_{|x|>R} \bigg|\frac{1}{\lambda(t_n)^{(d-2)/2}}\,u\Big(t_n,\frac{x}{\lambda(t_n)}\Big)\bigg|^{2^*}\,dx \\
&= \int_{|x|>R/\lambda(t_n)} |u(t_n,x)|^{2^*}\,dx.
\end{aligned}$$

Since $u \in C_t^0([0, T_+(u_0)); \dot{H}^1)$, we have

$$\int_{|x|>R} |v(t_0, x)|^{2^*} \, dx = 0, \quad \forall R > 0.$$

This contradicts the fact that

$$\int |\nabla v(t_0, x)|^2 \, dx = \int |\nabla u(t_0, x)|^2 \, dx > 0.$$

Now, after possibly redefining $\{t_n\}_{n=1}^{\infty}$, we can assume that

$$\lambda(t_n) \le 2 \inf_{t \in [0, t_n]} \lambda(t). \tag{5.7}$$

From the hypothesis, we have

$$w_n(x) = \frac{1}{\lambda(t_n)^{(d-2)/2}} u\Big(t_n, \frac{x}{\lambda(t_n)}\Big) \to w_0 \quad \text{in } \dot{H}^1.$$

By Proposition 3.1, we have

$$\int |\nabla w_n(x)|^2 \, dx = \int |\nabla u(t_n, x)|^2 \, dx < (1 - \overline{\delta}) \int |\nabla W(x)|^2 \, dx,$$
$$E(w_n) = E(u(t_n)) = E(u_0) < E(W).$$

Hence, we obtain

$$\int |\nabla w_0|^2 \, dx \le (1 - \overline{\delta}) \int |\nabla W(x)|^2 \, dx, \qquad 0 < E(w_0) = E(u_0) < E(W).$$

Thus $w_0 \not\equiv 0$. Let us now consider solutions $w_n(\tau, x), w_0(\tau, x)$ of (2.1) with data $w_n(x), w_0(x)$ at $\tau = 0$, defined in maximal intervals $\tau \in (-T_-(w_n), 0]$ and $\tau \in (-T_-(w_0), 0]$, respectively.

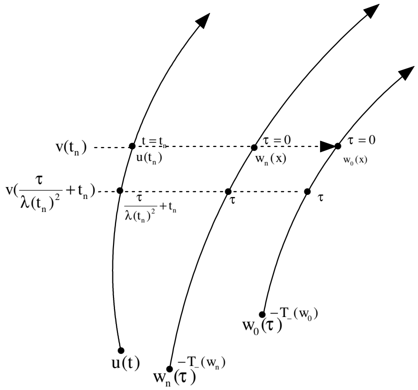

Fig. 2. A description of the normalization on $\lambda(t)$

Since $w_n \to w_0$ in $\dot{H}^1$, we see from Remark 2.6 that

$$\begin{aligned} &\liminf_{n \to \infty} T_-(w_n) \ge T_-(w_0), \\ &w_n(\tau, x) \to w_0(\tau, x) \quad \text{in } \dot{H}^1, \ \forall \tau \in (-T_-(w_0), 0]. \end{aligned} \tag{5.8}$$

By the uniqueness of solution of (2.1), we have

$$w_n(\tau,x)=\frac{1}{\lambda(t_n)^{(d-2)/2}}\,u\biggl(\frac{\tau}{\lambda(t_n)^2}+t_n,\frac{x}{\lambda(t_n)}\biggr)\quad\text{for }\frac{\tau}{\lambda(t_n)^2}+t_n\geq 0.$$

Now we claim that

$$\liminf_{n\to\infty} t_n\lambda(t_n)^2\geq T_-(w_0). \tag{5.9}$$

Indeed, if not, then $\liminf_{n\to\infty} t_n\lambda(t_n)^2=\tau_0<T_-(w_0)$, and from (5.8) we have, as $n\to\infty$,

$$w_n(-t_n\lambda(t_n)^2,x)=\frac{1}{\lambda(t_n)^{(d-2)/2}}\,u_0\biggl(\frac{x}{\lambda(t_n)}\biggr)\to w_0(-\tau_0,x)\quad\text{in }\dot H^1.$$

Note that from $\lambda(t_n)\to 0$, we have, as $n\to\infty$,

$$\frac{1}{\lambda(t_n)^{(d-2)/2}}\,u_0\biggl(\frac{x}{\lambda(t_n)}\biggr)\rightharpoonup 0\quad\text{in }\dot H^1,$$

and thus we obtain $w_0(-\tau_0)\equiv 0$, which yields a contradiction.

From (5.9), we see that for fixed $\tau\in(-T_-(w_0),0]$ and sufficiently large n,

$$0\leq\frac{\tau}{\lambda(t_n)^2}+t_n\leq t_n,$$

$v(\tau/\lambda(t_n)^2+t_n,x)$ and $\lambda(\tau/\lambda(t_n)^2+t_n)$ are defined, and we have

$$\begin{aligned}
v\biggl(\frac{\tau}{\lambda(t_n)^2}+t_n,x\biggr)
&=\frac{1}{\lambda(\tau/\lambda(t_n)^2+t_n)^{(d-2)/2}}\,u\biggl(\frac{\tau}{\lambda(t_n)^2}+t_n,\frac{x}{\lambda(\tau/\lambda(t_n)^2+t_n)}\biggr)\\
&=\frac{1}{\widetilde\lambda_n(\tau)^{(d-2)/2}}\,w_n\biggl(\tau,\frac{x}{\widetilde\lambda_n(\tau)}\biggr),
\end{aligned}$$

where

$$\widetilde\lambda_n(\tau)=\frac{\lambda(\tau/\lambda(t_n)^2+t_n)}{\lambda(t_n)}\geq\frac12$$

by (5.7). After passing to a subsequence, we can assume that

$$\widetilde\lambda_n(\tau)\to\widetilde\lambda_0(\tau)\in[1/2,\infty].$$

Hence,

$$v\biggl(\frac{\tau}{\lambda(t_n)^2}+t_n,x\biggr)\to\frac{1}{\widetilde\lambda_0(\tau)^{(d-2)/2}}\,w_0\biggl(\tau,\frac{x}{\widetilde\lambda_0(\tau)}\biggr)=v_0(\tau,x)\in\overline{K}.$$

Now we claim that

$$\widetilde\lambda_0(\tau)<\infty.$$

If not, then from

$$\frac{1}{\widetilde{\lambda}_n(\tau)^{(d-2)/2}}\, w_n\Big(\tau, \frac{x}{\widetilde{\lambda}_n(\tau)}\Big) \to \frac{1}{\lambda_0(\tau)^{(d-2)/2}}\, w_0\Big(\tau, \frac{x}{\lambda_0(\tau)}\Big) = v_0(\tau, x),$$

we have $w_0(\tau) = 0$, which yields a contradiction.

So $w_0(\tau)$, $v_0(\tau)$ and $\widetilde{\lambda}_0(\tau)$ satisfy the conditions of Proposition 5.1, and we deduce that $w_0 \equiv 0$, which yields a contradiction. This completes the proof.

Acknowledgements. C. Miao and G. Xu were partly supported by the NSF of China (No. 10725102, No. 10726053), and L. Zhao was supported by the China postdoctoral science foundation project. The authors thank the referees for their invaluable comments and suggestions.

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Received 12 May 2008 (5044)