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ON THE PROLONGATION OF RESTRICTIONS OF BAIRE 1 FUNCTIONS TO FUNCTIONS WHICH ARE QUASICONTINUOUS AND APPROXIMATELY CONTINUOUS

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Abstract. Let $I \subset \mathbb{R}$ be an open interval and let $A \subset I$ be any set. Every Baire 1 function $f : I \to \mathbb{R}$ coincides on A with a function $g : I \to \mathbb{R}$ which is simultaneously approximately continuous and quasicontinuous if and only if the set A is nowhere dense and of Lebesgue measure zero.

Let μ be the Lebesgue measure on $\mathbb{R}$. For a (Lebesgue) measurable set $A \subset \mathbb{R}$ and a point x we define the *upper density* $D_u(A, x)$ and *lower density* $D_l(A, x)$ of A at x as

$$D_u(A, x) = \limsup_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h},$$
$$D_l(A, x) = \liminf_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h}.$$

A point x is called a *density point* of a set B if there is a Lebesgue measurable set $A \subset B$ such that $D_l(A, x) = 1$.

The family T_d of all sets A such that every $x \in A$ is a density point of A is a topology on $\mathbb{R}$, called the *density topology* ([1, 7]). All sets in T_d are Lebesgue measurable [1].

Moreover, let T_e denote the Euclidean topology on $\mathbb{R}$. The continuity of functions from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called the *approximate continuity* ([1, 7]).

Let I be an open interval. A function $f : I \to \mathbb{R}$ is called *quasicontinuous* at a point x if for each positive real r and for each $U \in T_e$ contained in I and containing x there is an open interval $J \subset U$ such that $|f(t) - f(x)| < r$ for all $t \in J$ ([2, 4]).

The following theorem is shown in [5].

THEOREM 1. *Let $A \subset I$. Every Baire* 1 *function $f : I \to \mathbb{R}$ coincides on A with an approximately continuous function $g : I \to \mathbb{R}$ if and only if $\mu(A) = 0$.*

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We have the following simple observation concerning quasicontinuity.

REMARK 1. *Let $A \subset I$. Every Baire 1 function (resp. every function) $f : I \to \mathbb{R}$ coincides on A with a Baire 1 quasicontinuous (resp. quasicontinuous) function $g : I \to \mathbb{R}$ if and only if the set A is nowhere dense.*

Proof. If A is dense in some open interval J then we pick an $x \in J \cap A$ and put $f(x) = 1$ and $f(t) = 0$ for $t \neq x$. Then f is of the first Baire class and for each quasicontinuous function $g : I \to \mathbb{R}$ with $g(x) = 1$ there is a point $t \in A \cap I$ such that $t \neq x$ and $g(t) \neq 0$. So, $f|A \neq g|A$, and the proof of the necessity is complete.

To prove the sufficiency, enumerate all components of $I \setminus \mathrm{cl}(A)$, where cl denotes closure, in a sequence $((a_n, b_n))_n$ of pairwise disjoint intervals. If $a_n \in I$ we find a sequence of points $c_{k,n} \in (a_n, b_n)$ such that

$$c_{k,n} > c_{k+1,n} \quad \text{for } k \geq 1 \quad \text{and} \quad \lim_{k\to\infty} c_{k,n} = a_n.$$

Similarly if $b_n \in I$ we find a sequence of points $d_{k,n} \in (a_n, b_n)$ such that

$$d_{k,n} < d_{k+1,n} \quad \text{for } k \geq 1 \quad \text{and} \quad \lim_{k\to\infty} d_{k,n} = b_n.$$

Moreover, if $a_n, b_n \in I$ then we assume that $c_{n,1} < d_{n,1}$. If $a_n, b_n \in I$ then we define a continuous function $g_n : (a_n, b_n) \to \mathbb{R}$ such that $g_n([c_{n,2k}, c_{n,2k-1}]) = g_n([d_{n,2k-1}, d_{n,2k}]) = [-k, k]$ for each $k \geq 1$. If a_n or b_n is not in I then we define a continuous function $g_n : (a_n, b_n) \to \mathbb{R}$ such that for each $k \geq 1$, $g_n([d_{n,2k-1}, d_{n,2k}]) = [-k, k]$ or $g_n([c_{n,2k}, c_{n,2k-1}]) = [-k, k]$. To finish the proof it suffices to observe that the function

$$g(x) = \begin{cases} g_n(x) & \text{for } x \in (a_n, b_n),\ n \geq 1, \\ f(x) & \text{for } x \in \mathrm{cl}(A), \end{cases}$$

is quasicontinuous and of Baire class 1 (resp. quasicontinuous) and $g|\mathrm{cl}(A) = f|\mathrm{cl}(A)$. ∎

REMARK 2. *If in Remark 1 we suppose that the function f is bounded, i.e. $c \leq f \leq d$, then we may find g such that $c \leq g \leq d$.*

Proof. It suffices to put $g_1(x) = \min(d, \max(c, g(x)))$ for $x \in I$. ∎

THEOREM 2. *Let $A \subset I$. Every Baire 1 function $f : I \to \mathbb{R}$ coincides on A with an approximately continuous and quasicontinuous function $g : I \to \mathbb{R}$ if and only if the set A is nowhere dense and $\mu(A) = 0$.*

Proof. The necessity follows from Theorem 1 and Remark 1. The sufficiency follows from Theorem 1 and Theorem 3 below. ∎

THEOREM 3. *Assume that $f : I \to \mathbb{R}$ is an approximately continuous function and $A \subset I$ is a nowhere dense set. Then there is an approximately continuous and quasicontinuous function $g : I \to \mathbb{R}$ such that $g|A = f|A$.*

Proof. Let $A_1 = \{x \in I; \operatorname{osc} f(x) \geq 1/2\}$ and for $n > 1$ let

$$A_n = \{x \in I; 1/2^{n-1} > \operatorname{osc} f(x) \geq 1/2^n\}.$$

Since the set $C(f)$ of all continuity points of f is residual, the sets A_n, $n \geq 1$, are nowhere dense. Evidently the sets

$$B_1 = A_1 \quad \text{and} \quad B_n = \{x \in I; \operatorname{osc} f(x) \geq 1/2^n\}, \quad n \geq 2,$$

are closed in I. We will construct a sequence of functions (g_n) by induction.

STEP 1. Let $((a_{1,k}, b_{1,k}))_k$ be a sequence of all components of $I \setminus A_1$ such that $(a_{1,k}, b_{1,k}) \cap (a_{1,m}, b_{1,m}) = \emptyset$ for $k \neq m$. If $a_{1,k} \in I$ then we find a sequence of points $c_{1,k,i} \in (a_{1,k}, b_{1,k}) \cap C(f)$ such that:

- $[c_{1,k,2i}, c_{1,k,2i-1}] \cap A = \emptyset$ for $i \geq 1$,
- $c_{1,k,i} > c_{1,k,i+1}$ for $i \geq 1$ and $\lim_{i\to\infty} c_{1,k,i} = a_{1,k}$,
- $\displaystyle \lim_{h\to 0^+} \frac{\mu([a_{1,k}, a_{1,k}+h] \cap \bigcup_{i=1}^{\infty}[c_{1,k,2i}, c_{1,k,2i-1}])}{h} = 0,$
- $\displaystyle \frac{\mu(\bigcup_{i=1}^{\infty}[c_{1,k,2i}, c_{1,k,2i-1}])}{b_{1,k} - a_{1,k}} < \frac{1}{8k}.$

Similarly if $b_{1,k} \in I$ then we find a sequence of points $d_{1,k,i} \in (a_{1,k}, b_{1,k}) \cap C(f)$ with $d_{1,k,1} > c_{1,k,1}$ such that:

- $[d_{1,k,2i-1}, d_{1,k,2i}] \cap A = \emptyset$ for $i \geq 1$,
- $d_{1,k,i} < d_{1,k,i+1}$ for $i \geq 1$ and $\lim_{i\to\infty} d_{1,k,i} = b_{1,k}$,
- $\displaystyle \lim_{h\to 0^+} \frac{\mu([b_{1,k}, b_{1,k}-h] \cap \bigcup_{i=1}^{\infty}[d_{1,k,2i-1}, d_{1,k,2i}])}{h} = 0,$
- $\displaystyle \frac{\mu(\bigcup_{i=1}^{\infty}[d_{1,k,2i-1}, d_{1,k,2i}])}{b_{1,k} - a_{1,k}} < \frac{1}{8k}.$

If $a_{1,k}$ or $b_{1,k}$ is not in I then we find only one monotone sequence satisfying the above conditions convergent to that endpoint of $(a_{1,k}, b_{1,k})$ which belongs to I.

Now for each $k \geq 1$ we define an approximately continuous function $g_{1,k} : (a_{1,k}, b_{1,k}) \to \mathbb{R}$ such that

- for each $i \geq 1$ the restrictions $g_{1,k}|[c_{1,k,2i}, c_{1,k,2i-1}]$ and $g_{1,k}|[d_{1,k,2i-1}, d_{1,k,2i}]$ are continuous and $g_{1,k}([c_{1,k,2i}, c_{1,k,2i-1}]) \cap g_{1,k}([d_{1,k,2i-1}, d_{1,k,2i}]) \supset [-i, i]$,
- $g_{1,k}(x) = f(x)$ for $x \in I \setminus \bigcup_{i\geq 1}((c_{1,k,2i}, c_{1,k,2i-1}) \cup (d_{1,k,2i-1}, d_{1,k,2i}))$.

Putting

$$g_1(x) = \begin{cases} g_{1,k}(x) & \text{for } x \in (a_{1,k}, b_{1,k}),\ k \geq 1, \\ f(x) & \text{elsewhere on } I, \end{cases}$$

we obtain an approximately continuous function $g_1 : I \to \mathbb{R}$ which is quasi-continuous at each $x \in C(f) \cup A_1$ and $g_1|A = f|A$.

Step 2. Let (K_n) be a sequence of bounded closed nondegenerate intervals such that

$$K_n \subset K_{n+1} \quad \text{for } n \geq 1 \quad \text{and} \quad I = \bigcup_{n=1}^{\infty} K_n.$$

Observe that $\operatorname{osc} g_1(x) < 1/2$ for each $x \in A_2$, so for each $x \in A_2 \cap K_2$ there is an open bounded interval $I(x) \subset I$ such that $x \in I(x)$ and $\operatorname{diam}(g_1(I(x))) < 1/2$. Since $A_2 \cap K_2$ is compact, there are $x_1, \dots, x_m \in A_2 \cap K_2$ such that

$$A_2 \cap K_2 \subset \bigcup_{i=1}^{m} I(x_i).$$

But $A_2 \cap K_2$ is nowhere dense and $C(g_1)$ is dense, so there are pairwise disjoint open intervals $J_1, \dots, J_r \subset \bigcup_{i=1}^{m} I(x_i)$ such that

$$A_2 \cap K_2 \subset \bigcup_{j=1}^{r} J_j$$

and the endpoints of all J_j, $j \geq r$, belong to $C(g_1)$.

Fix $j \leq r$ and enumerate all components of $I_j \setminus (A_2 \cap K_2)$ in a sequence $((a_{2,j,k}, b_{2,j,k}))_k$ with $(a_{2,j,k}, b_{2,j,k}) \cap (a_{2,j,l}, b_{2,j,l}) = \emptyset$ for $k \neq l$. Now for each $k \geq 1$ we find a sequence of points $c_{2,j,k,i}, d_{2,j,k,i} \in (a_{2,j,k}, b_{2,j,k}) \cap C(g_1)$ such that:

- $[c_{2,j,k,2i}, c_{2,j,k,2i-1}] \cap A = \emptyset$ for $i \geq 1$,
- $c_{2,j,k,i} > c_{2,j,k,i+1}$ for $i \geq 1$ and $\lim_{i\to\infty} c_{2,j,k,i} = a_{2,j,k}$,
- $\displaystyle\lim_{h\to 0^+} \frac{\mu([a_{2,j,k}, a_{2,j,k}+h] \cap \bigcup_{i=1}^{\infty}[c_{2,j,k,2i}, c_{2,j,k,2i-1}])}{h} = 0,$
- $\displaystyle\frac{\mu(\bigcup_{i=1}^{\infty}[c_{2,j,k,2i}, c_{2,j,k,2i-1}])}{b_{2,j,k} - a_{2,j,k}} < \frac{1}{8(k+j)},$

and similarly a sequence of points $d_{2,j,k,i} \in (a_{2,j,k}, b_{2,j,k}) \cap C(g_1)$ with $d_{1,k,1} > c_{1,k,1}$ such that:

- $[d_{2,j,k,2i-1}, d_{2,j,k,2i}] \cap A = \emptyset$ for $i \geq 1$,
- $d_{2,j,k,i} < d_{2,j,k,i+1}$ for $i \geq 1$ and $\lim_{i\to\infty} d_{2,j,k,i} = b_{2,j,k}$,
- $\displaystyle\lim_{h\to 0^+} \frac{\mu([b_{2,j,k}, b_{2,j,k}-h] \cap \bigcup_{i=1}^{\infty}[d_{2,j,k,2i-1}, d_{2,j,k,2i}])}{h} = 0,$
- $\displaystyle\frac{\mu(\bigcup_{i=1}^{\infty}[d_{2,j,k,2i-1}, d_{2,j,k,2i}])}{b_{2,j,k} - a_{2,j,k}} < \frac{1}{8(k+j)}.$

Now for each $k \geq 1$ we define an approximately continuous function $g_{2,j,k} : (a_{2,j,k}, b_{2,j,k}) \to \mathbb{R}$ such that

- for each $i \geq 1$ the restrictions

$$g_{2,j,k}|[c_{2,j,k,2i}, c_{2,j,k,2i-1}] \quad \text{and} \quad g_{2,j,k}|[d_{2,j,k,2i-1}, d_{2,j,k,2i}]$$

are continuous and their images both equals the smallest closed interval containing $g_1(J_r)$,

- $g_{2,j,k}(x)=g_1(x)$ for $x\in J_r\setminus\bigcup_{i\geq 1}((c_{2,j,k,2i},c_{2,j,k,2i-1})\cup(d_{2,j,k,2i-1},d_{2,j,k,2i}))$.

Putting

$$g_2(x)=\begin{cases} g_{2,j,k}(x) & \text{for } x\in(a_{2,j,k},b_{2,j,k}),\ k\geq 1,\ j\leq r,\\ g_1(x) & \text{elsewhere } I,\end{cases}$$

we obtain an approximately continuous function $g_2 : I\to\mathbb{R}$ which is quasicontinuous at each $x\in C(f)\cup(B_2\cap K_2)$, and such that $g_2|A=f|A$ and $|g_2-g_1|\leq 1/2$.

Similarly in step $n\geq 2$ we construct an approximately continuous function $g_n : I\to\mathbb{R}$ which is quasicontinuous at each $x\in C(f)\cup(B_n\cap K_n)$ and such that $g_n|A=f|A$ and $|g_n-g_{n-1}|\leq 1/2^{n-1}$. The sequence (g_n) uniformly converges to an approximately continuous and quasicontinuous function such that $g|A=f|A$. ■

Remark 3. *Let $A\subset\mathbb{R}$. For each approximately continuous function $f:\mathbb{R}\to\mathbb{R}$ there is an approximately continuous and quasicontinuous function $g:\mathbb{R}\to\mathbb{R}$ with $g|A=f|A$ if and only if A is a nowhere dense subset of $\mathbb{R}$.*

Proof. If A is nowhere dense and $f:\mathbb{R}\to\mathbb{R}$ is approximately continuous then by Theorem 3 there is an approximately continuous and quasicontinuous function $g:\mathbb{R}\to\mathbb{R}$ such that $f|A=g|A$. Conversely, if A is dense in an open interval I then we find two countable disjoint subsets $B,C\subset I\cap A$ dense in I and a G_δ-set $E\supset B$ with $E\cap C=\emptyset$ and $\mu(E)=0$. By Zahorski's theorem ([8, Lem. 11]) there is an approximately continuous function $f:\mathbb{R}\to[0,1]$ such that $f(E)=\{0\}$ and $f(x)>0$ for $x\in\mathbb{R}\setminus E$. To finish the proof it suffices to observe that no function $h:\mathbb{R}\to\mathbb{R}$ with $h|A=f|A$ is quasicontinuous at any $x\in I\cap C$. ■

Remark 4. *Let $A\subset\mathbb{R}$. For each Baire 1 quasicontinuous function $f:\mathbb{R}\to\mathbb{R}$ there is an approximately continuous and quasicontinuous function $g:\mathbb{R}\to\mathbb{R}$ with $g|A=f|A$ if and only if A is a nowhere dense subset of $\mathbb{R}$ and $\mu(A)=0$.*

Proof. The sufficiency follows from Theorem 2. For the proof of the necessity we consider two cases.

1. If A is not of measure zero then we find a measurable set $G\supset A$ such that each measurable subset of $G\setminus A$ is of measure zero. Let $a\in A$ be a density point of G and let

$$f(x)=\begin{cases} 0 & \text{for } x\in(-\infty,a),\\ 1 & \text{for } x\in[a,\infty).\end{cases}$$

Ewidently f is a quasicontinuous Baire 1 function and no function $h : \mathbb{R} \to \mathbb{R}$ with $h|A = f|A$ is approximately continuous at a.

2. If A is dense in an open interval J then we find a countable subset $B = \{x_n;\ n \geq 1\} \subset J \cap A$ dense in J and define

$$f(x) = \begin{cases} 0 & \text{for } x \in (-\infty, \inf B), \\ \sum_{x_n \leq x} \frac{1}{2^n} & \text{for } x \geq \inf B. \end{cases}$$

The function f is monotone and right-continuous, so it is Baire 1 quasi-continuous. Moreover, it is continuous at each point of $\mathbb{R} \setminus B$ and discontinuous at all points of B. Suppose towards a contradiction that there is an approximately continuous and quasicontinuous function $g : \mathbb{R} \to \mathbb{R}$ with $g|A = f|A$. Then $E = C(f) \cap C(g)$ is residual, where $C(h)$ denotes the set of all continuity points of h. Since B is dense in J and $f|B = g|B$, we have $f|(J \cap E) = g|(J \cap E)$. Fix $x_k \in B$. Observe that

$$g(x_k) = f(x_k) = \lim_{x \to x_k^-} f(x) + \frac{1}{2^k} \quad \text{and} \quad f(x) \leq g(x_k) - \frac{1}{2^k} \quad \text{for} \quad x \leq x_k.$$

Since g is approximately continuous at x_k, there is a point $u \in (\inf J, x_k)$ such that $g(u) > g(x_k) - 1/2^{k+1}$. But g is quasicontinuous at u, so there is an open interval $K \subset (\inf J, x_k)$ such that $g(x) > g(x_k) - 1/2^{k+1}$ for $x \in K$. Pick $w \in K \cap E$. Observe that

$$f(w) = g(w) > g(x_k) - \frac{1}{2^{k+1}} > f(x_k) - \frac{1}{2^k}.$$

Since $w < x_k$, we obtain a contradiction which completes the proof. ∎

As an application of the above observations to transfinite sequences of functions ([6]) we observe that the function

$$f(x) = \sum_{r_n \leq x} \frac{1}{2^n},$$

where (r_n) is an enumeration of all rationals such that $r_n \neq r_m$ for $n \neq m$, is quasicontinuous and of Baire class 1, but is not the transfinite limit of any sequence of functions $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha < \omega_1$, which are simultaneously approximately continuous and quasicontinuous. Indeed, if there are approximately continuous and quasicontinuous functions $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha < \omega_1$, such that $f = \lim_{\alpha<\omega_1} f_\alpha$ then there is a countable ordinal β such that for all countable ordinals $\alpha \geq \beta$ and all rationals r_n we have $f_\alpha(r_n) = f(r_n)$ and $f_\alpha(r_n + 2^{1/2}) = f(r_n + 2^{1/2})$. This means that f may be extended from $\mathbb{Q} \cup (\mathbb{Q} + 2^{1/2})$ to a function f_β which is simultaneously approximately continuous and quasicontinuous. The reasoning from the proof of Remark 4 shows that this is impossible.

On the other hand, we recall that the transfinite limit of quasicontinuous (resp. Baire 1) functions $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha < \omega_1$, is quasicontinuous (resp. Baire 1), and each Baire 1 function $f : \mathbb{R} \to \mathbb{R}$ is the limit of a transfinite sequence of approximately continuous functions f_α, $\alpha < \omega_1$ ([6, 4, 3]).

Finally, consider a general problem. Let Φ, Φ_1 and Φ_2 be some classes of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\Phi_1 \cap \Phi_2 \neq \emptyset$ and $\Phi \supset \Phi_1 \cup \Phi_2$. For $i = 1, 2$ let H_{Φ_i} denote the class of all subsets $A \subset \mathbb{R}$ such that for each $f \in \Phi$ there is a $g \in \Phi_i$ with $f|A = g|A$. A natural question is whether

$$H_{\Phi_1 \cap \Phi_2} = H_{\phi_1} \cap H_{\phi_2}.$$

In the next example we show that the answer is negative.

Example. Let Φ be the class of all functions from $\mathbb{R}$ to $\mathbb{R}$, let Φ_1 denote the family of all polynomials and let Φ_2 be the family of all trigonometric polynomials. Then $\Phi_1 \cap \Phi_2$ is the family of all constant functions, $H_{\Phi_1} \cap H_{\phi_2}$ is the class containing all finite subsets of $\mathbb{R}$ and $H_{\Phi_1 \cap \Phi_2}$ is the family composed of all singletons and $\emptyset$.

Remark 5. *If for each $A \in H_{\Phi_1} \cap H_{\Phi_2}$ there is $i \leq 2$ such that for each $f \in \Phi_i$ there is $g \in \Phi_1 \cap \Phi_2$ with $f|A = g|A$ then $H_{\Phi_1 \cap \Phi_2} = H_{\Phi_1} \cap H_{\Phi_2}$.*

Proof. The proof is evident.

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