COLLOQUIUM MATHEMATICUM

VOL. 114 2009 NO. 2

DEGENERATIONS IN THE MODULE VARIETIES OF ALMOST CYCLIC COHERENT AUSLANDER–REITEN COMPONENTS

BY

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Abstract. We establish when the partial orders $\leq_{\text{ext}}$ and $\leq_{\text{deg}}$ coincide for all modules of the same dimension from the additive category of a generalized standard almost cyclic coherent component of the Auslander–Reiten quiver of a finite-dimensional algebra.

1. Introduction and the main results. Throughout the paper by an algebra we mean a basic, finite-dimensional k-algebra over a fixed algebraically closed field k. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable modules. We denote by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. Moreover, we denote by Γ_A the Auslander–Reiten quiver of A, and by $\tau_A = \tau$ and $\tau_A^- = \tau^-$ the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We do not distinguish between a module in $\operatorname{ind} A$ and the corresponding vertex of Γ_A. For a family $\mathcal{F}$ of A-modules, we denote by $\operatorname{add}(\mathcal{F})$ the additive category given by $\mathcal{F}$, that is, the full subcategory of $\operatorname{mod} A$ formed by all modules isomorphic to direct sums of modules from $\mathcal{F}$.

For an algebra A with basis $a_1 = 1, \dots, a_n$, we have the constant structures a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\operatorname{mod}_A(d)$ of d-dimensional A-modules consists of n-tuples $m = (m_1, \dots, m_n)$ of $d \times d$-matrices with coefficients in k such that m_1 is the identity matrix and $m_i m_j = \sum m_k a_{ijk}$ for all indices i and j. The general linear group $\mathrm{Gl}_d(k)$ acts on $\operatorname{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [17]). We identify a d-dimensional A-module M with the corresponding point of $\operatorname{mod}_A(d)$. We denote by $\mathcal{O}(M)$ the $\mathrm{Gl}_d(k)$-orbit of a module M in $\operatorname{mod}_A(d)$. Then one says that a module N in $\operatorname{mod}_A(d)$ is a *degeneration* of a module M in

2000 *Mathematics Subject Classification*: 14L30, 16G10, 16G70.

Key words and phrases: degenerations of modules, Auslander–Reiten quivers.

DOI: 10.4064/cm114-2-8

$\mathrm{mod}_A(d)$, and writes $M \leq_{\mathrm{deg}} N$, if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\mathrm{mod}_A(d)$. Thus $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. An interesting problem is to describe connections of $\leq_{\mathrm{deg}}$ with other partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{virt}}$ and $\leq$ on the isomorphism classes in $\mathrm{mod}_A(d)$, where

- $M \leq_{\mathrm{ext}} N :\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\mathrm{mod}\, A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.
- $M \leq_{\mathrm{virt}} N :\Leftrightarrow M \oplus X \leq_{\mathrm{deg}} N \oplus X$ for some A-module X.
- $M \leq N :\Leftrightarrow [M, X] \leq [N, X]$ holds for all modules X.

Here and later on we abbreviate $\dim_k \mathrm{Hom}_A(X, Y)$ by $[X, Y]$. There are many results on approximation of $\leq_{\mathrm{deg}}$ by the above partial orders (see [1], [12], [13], [24], [36]).

In general, for all modules M and N in $\mathrm{mod}_A(d)$ the following implications hold:

$$M \leq_{\mathrm{ext}} N \;\Rightarrow\; M \leq_{\mathrm{deg}} N \;\Rightarrow\; M \leq_{\mathrm{virt}} N \;\Rightarrow\; M \leq N$$

(see [13], [24]). Unfortunately the reverse implications are not always true, and it would be interesting to find out when they are. We know that $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$, $\leq$ coincide for modules over the path algebras of Dynkin and extended Dynkin quivers [12], [13], [36], and $\leq_{\mathrm{deg}}$ and $\leq$ coincide for modules over arbitrary algebras of finite representation type [37]. More comprehensive information about degenerations of modules can be found in [12], [13], [24], [38].

For a module M in $\mathrm{mod}\, A$, we shall denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if M and N have the same simple composition factors including multiplicities. Observe that, if M and N have the same dimension and $M \leq N$, then $[M] = [N]$.

We are interested in the relationship between the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$, $\leq_{\mathrm{virt}}$, $\leq$ for modules from the additive category $\mathrm{add}(\varGamma)$ of a connected component $\varGamma$ of $\varGamma_A$. Then the following order on the isomorphism classes of modules in $\mathrm{add}(\varGamma)$ is natural [31]: for $M, N \in \mathrm{add}(\varGamma)$,

$$M \leq_{\varGamma} N :\Leftrightarrow\; [M, X] \leq [N, X] \text{ for all } X \text{ in } \mathrm{add}(\varGamma).$$

Clearly, for M, N in $\mathrm{add}(\varGamma)$, $M \leq N$ implies $M \leq_{\varGamma} N$. Moreover, by [31], $\leq_{\varGamma}$ is a partial order on the isomorphism classes of modules in $\mathrm{add}(\varGamma)$ having the same composition factors.

In the representation theory of algebras an important role is played by generalized standard Auslander–Reiten components. Recall that following A. Skowroński [30], a connected component $\varGamma$ in $\varGamma_A$ is called *generalized standard* if $\mathrm{rad}^{\infty}(X, Y) = 0$ for all $X, Y \in \varGamma$. The Auslander–Reiten quiver $\varGamma_A$ of any algebra A of finite representation type is generalized standard.

Examples of generalized standard components include: preprojective components, preinjective components, connecting components of tilted algebras, and tubes over tame tilted, tubular and canonical algebras [25]. The generalized standard components without oriented cycles have been described in [29]. The structure of arbitrary generalized standard components is not yet well understood. In general we only know from [30] that if Γ is a generalized standard component in Γ_A, then all but finitely many τ-orbits in Γ are periodic. It is known that $\leq_{\mathrm{ext}}$ and $\leq_\Gamma$ coincide when Γ is preprojective (preinjective) [13] or a generalized standard quasi-tube [31]. From [33] and [38] we know that $\leq_{\mathrm{deg}}$ and $\leq_\Gamma$ coincide for each generalized standard component Γ of Γ_A. Moreover, there are generalized standard components for which $\leq_{\mathrm{ext}}$ and $\leq_{\mathrm{deg}}$ do not coincide (see [24], [31]).

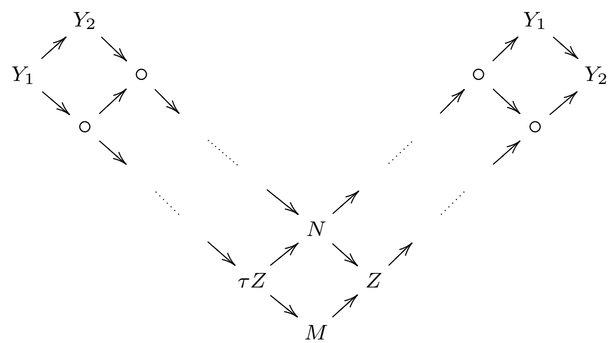
Recall that a component Γ of Γ_A is called *almost cyclic* if all but finitely many modules in Γ lie on oriented cycles contained entirely in Γ. Moreover, a component Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots$ in Γ (that is, $X_i \neq \tau X_{i+2}$ for any $i \geq 1$).

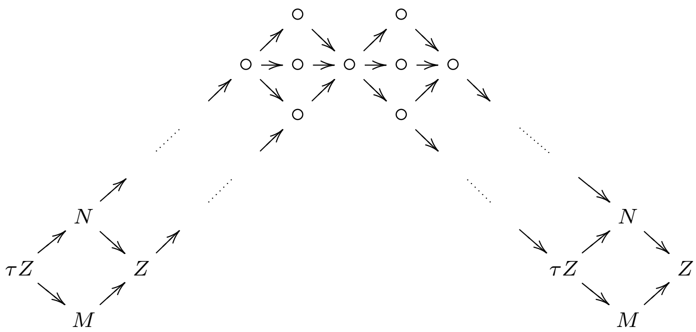
(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_2 \to Y_1 = I$ in Γ (that is, $Y_{j+2} \neq \tau Y_j$ for any $j \geq 1$).

Following [14], a component Γ is said to be *standard* if the full subcategory of $\operatorname{mod} A$ formed by the modules from Γ is equivalent to the mesh category $K(\Gamma)$ of Γ. It is known [18] that every standard component of Γ_A is generalized standard but the converse is not true in general. However, it was shown recently in [23, Proposition 2.7] that for an almost cyclic coherent component of Γ_A the converse implication is also true. Note that the class of algebras with generalized standard almost cyclic coherent Auslander–Reiten components is large (see [23, Proposition 2.9]).

In order to formulate our main result we define two kinds of full translation subquivers of Γ_A. A translation subquiver of Γ_A of the form



is said to be a *Möbius configuration*, and one of the form



is a *coil configuration.*

The main aim of the paper is to prove the following theorem.

THEOREM 1.1. *Let A be an algebra and Γ a generalized standard almost cyclic coherent component of Γ_A. The following conditions are equivalent*:

(i) *Γ contains neither a Möbius configuration nor a coil configuration.*
(ii) *The partial orders $\leq_{\mathrm{deg}}$ and $\leq_{\mathrm{ext}}$ coincide on* $\mathrm{add}(\Gamma)$.

In the representation theory of algebras a prominent role is played by algebras with separating families of components. A family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be *separating* in $\mathrm{mod}\,A$ if the modules in $\mathrm{ind}\,A$ split into three disjoint classes $\mathcal{P}_A$, $\mathcal{C}_A = \mathcal{C}$ and $\mathcal{Q}_A$ such that:

(S1) $\mathcal{C}_A$ is a sincere generalized standard family of components;
(S2) $\mathrm{Hom}_A(\mathcal{Q}_A, \mathcal{P}_A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}_A, \mathcal{C}_A) = 0$, $\mathrm{Hom}_A(\mathcal{C}_A, \mathcal{P}_A) = 0$;
(S3) any morphism from $\mathcal{P}_A$ to $\mathcal{Q}_A$ factors through $\mathrm{add}(\mathcal{C}_A)$.

We then say that $\mathcal{C}_A$ *separates* $\mathcal{P}_A$ *from* $\mathcal{Q}_A$ and write $\mathrm{ind}\,A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. We also note that $\mathcal{P}_A$ and $\mathcal{Q}_A$ are then uniquely determined by $\mathcal{C}_A$ (see [5, (2.1)] or [25, (3.1)]). Recall also that $\mathcal{C}_A$ is called *sincere* if every simple A-module occurs as a composition factor of a module in $\mathcal{C}_A$.

From Drozd's Tame and Wild Theorem [16] the algebras may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all algebras. Hence, we may realistically hope to classify the indecomposable modules only for the tame algebras. More precisely, following [16], an algebra A is called *tame* if, for each dimension d, there exist a finite number of $k[x]$-A-bimodules M_i which are finitely generated and free as left $k[x]$-modules, and all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form

$k[x]/(x-\lambda)\otimes_{k[x]} M_i$ for some i and some $\lambda \in k$ (see [27, Chapter XIX] for more details).

As an immediate consequence of Theorem 1.1, [22, Theorem C] and [32, Theorem 3] we obtain the following fact.

Corollary 1.2. *Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A. Then the orders $\le_{\mathrm{deg}}$ and $\le_{\mathrm{ext}}$ coincide on* $\operatorname{mod} A$ *if and only if A is tame and $\mathcal{C}_A$ contains neither a Möbius configuration nor a coil configuration.*

For basic background on the representation theory of algebras we refer to the books [2], [8], [25], [26], [27]. We also refer to [22] for description of the module category, homological properties and representation type of algebras with separating families of almost cyclic coherent Auslander–Reiten components.

2. Preliminaries on partial orders of modules. Following [24], for M, N from $\operatorname{mod} A$, we write $M \le N$ if and only if $[X,M] \le [X,N]$ for all A-modules X. The fact that $\le$ is a partial order on the isomorphism classes of A-modules follows from a result of M. Auslander (see [6], [11]). M. Auslander and I. Reiten [7] have shown that, if $[M]=[N]$ for some modules M and N, then for all nonprojective A-modules X and all noninjective modules Y the following formulas hold:

$$(*)\qquad \begin{aligned} [X,M]-[M,\tau X] &= [X,N]-[N,\tau X],\\ [M,Y]-[\tau^- Y,M] &= [N,Y]-[\tau^- Y,N]. \end{aligned}$$

Hence, if $[M]=[N]$, then $M \le N$ if and only if $[M,X] \le [N,X]$ for all A-modules X.

Let M and N be A-modules with $[M]=[N]$, and

$$\Sigma:\ 0 \to D \to E \to F \to 0$$

an exact sequence in $\operatorname{mod} A$. Following [24], we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$, δ_Σ and δ'_Σ on A-modules X as follows:

$$\begin{gathered} \delta_{M,N}(X)=[N,X]-[M,X], \qquad \delta'_{M,N}(X)=[X,N]-[X,M],\\ \delta_\Sigma(X)=\delta_{E,D\oplus F}(X)=[D\oplus F,X]-[E,X],\\ \delta'_\Sigma(X)=\delta'_{E,D\oplus F}(X)=[X,D\oplus F]-[X,E]. \end{gathered}$$

From the Auslander–Reiten formulas $(*)$ we get the following useful equalities:

$$\begin{aligned} \delta_{M,N}(X)&=\delta'_{M,N}(\tau^- X), &\quad \delta_{M,N}(\tau X)&=\delta'_{M,N}(X),\\ \delta_\Sigma(X)&=\delta'_\Sigma(\tau^- X), & \delta_\Sigma(\tau X)&=\delta'_\Sigma(X), \end{aligned}$$

for all A-modules X. Observe also that $\delta_{M,N}(I) = 0$ for any injective A-module I, and $\delta'_{M,N}(P) = 0$ for any projective A-module P. In particular, the following conditions are equivalent:

(1) $M \leq N$,
(2) $\delta_{M,N}(X) \geq 0$ for all $X \in \operatorname{ind} A$,
(3) $\delta'_{M,N}(X) \geq 0$ for all $X \in \operatorname{ind} A$.

For an A-module M and an indecomposable A-module Z, we denote by $\mu(M,Z)$ the multiplicity of Z as a direct summand of M. For a noninjective indecomposable A-module U we denote by $\Sigma(U)$ the Auslander–Reiten sequence

$$\Sigma(U):\ 0 \to U \to E(U) \to \tau^{-}U \to 0.$$

We need the following lemmas.

Lemma 2.1 (see [34, Lemma 2.5]). *Let M, N be A-modules with $[M] = [N]$ and U an indecomposable A-module.*

(i) *If U is noninjective, then $\delta_{\Sigma(U)}(M) = \mu(M,U)$ and*

$$\mu(N,U) - \mu(M,U) = \delta'_{M,N}(U) - \delta'_{M,N}(E(U)) + \delta'_{M,N}(\tau^{-}U).$$

(ii) *If U is injective, then $[U,M] - [U/\operatorname{soc}(U),M] = \mu(M,U)$ and*

$$\mu(N,U) - \mu(M,U) = \delta'_{M,N}(U) - \delta'_{M,N}(U/\operatorname{soc}(U)).$$

(iii) *If U is nonprojective, then $\delta'_{\Sigma(\tau U)}(M) = \mu(M,U)$ and*

$$\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) - \delta_{M,N}(E(\tau U)) + \delta_{M,N}(\tau U).$$

(iv) *If U is projective, then $[M,U] - [M,\operatorname{rad} U] = \mu(M,U)$ and*

$$\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) - \delta_{M,N}(\operatorname{rad} U).$$

Lemma 2.2. *Let Γ_1 denote the set of arrows in Γ_A. Let $M,N,U \in \operatorname{mod} A$ with $[M] = [N]$. Then for any $U \in \Gamma_A$,*

$$\begin{aligned}\mu(N,U) - \mu(M,U) &= \delta_{M,N}(U) + \delta'_{M,N}(U) - \sum_{(X\to U)\in\Gamma_1} \delta_{M,N}(X)\\ &= \delta_{M,N}(U) + \delta'_{M,N}(U) - \sum_{(U\to Y)\in\Gamma_1} \delta'_{M,N}(Y).\end{aligned}$$

Proof. Since $\delta'_{M,N}(U) = \delta_{M,N}(\tau U)$ and $\delta_{M,N}(U) = \delta'_{M,N}(\tau^{-}U)$, the formula is a consequence of Lemma 2.1 and the definition of Γ_1. ∎

Let Γ be a connected component of Γ_A. For M and N in $\operatorname{add}(\Gamma)$ we set

$$M \leq_\Gamma N \ \Leftrightarrow\ [X,M] \leq [X,N] \text{ for all } X \in \operatorname{add}(\Gamma).$$

Clearly, $M \leq N$ implies $M \leq_\Gamma N$. By [31], $\leq_\Gamma$ is a partial order on the isomorphism classes of modules in $\operatorname{add}(\Gamma)$ having the same dimension vectors.

COROLLARY 2.3. *Let* $M, N \in \operatorname{add}(\Gamma)$ *with* $[M] = [N]$. *Then* $M \simeq N$ *if and only if* $M \leq_\Gamma N$ *and* $N \leq_\Gamma M$.

Moreover, if $M, N \in \operatorname{add}(\Gamma)$ *and* $[M] = [N]$ *then the following conditions are equivalent*:

(i) $M \leq_\Gamma N$.
(ii) $\delta_{M,N}(X) \geq 0$ *for all* X *in* Γ.
(iii) $\delta'_{M,N}(X) \geq 0$ *for all* X *in* Γ.

We also need the following facts.

PROPOSITION 2.4 (see [34, Proposition 4.2]). *Let* Γ *be a generalized standard component of* Γ_A *and assume that* $M, N \in \operatorname{add}(\Gamma)$ *with* $[M] = [N]$ *and* $M \leq_\Gamma N$. *Then* $\delta_{M,N}(X) = 0$ *and* $\delta'_{M,N}(X) = 0$ *for all but finitely many* X *in* Γ *and all* X *in* $\Gamma_A \setminus \Gamma$.

LEMMA 2.5 (see [33, Lemma 4.5]). *Let* Γ *be a generalized standard component of* Γ_A *and assume that* $M, N \in \operatorname{add}(\Gamma)$ *with* $[M] = [N]$ *and* $M \leq_\Gamma N$. *Then* $\delta_{M,N}(N) > 0$ *and* $\delta'_{M,N}(N) > 0$.

PROPOSITION 2.6. *Let* Γ *be a generalized standard component in* Γ_A *which contains a coil configuration. Then there exist indecomposables* $M, N \in \Gamma$ *such that* $[M] = [N]$ *and* $M <_{\deg} N$.

Proof. Follows from the proof of [31, Theorem 4]. ■

In the proof of the next proposition we shall need the following direct consequence of the lemma in [3, (2.1)].

LEMMA 2.7. *Let*

$$0 \to M_1 \xrightarrow{[f_1, u_1]^t} N_1 \oplus M_2 \xrightarrow{[u_2, f_2]} N_2 \to 0$$

and

$$0 \to M_2 \xrightarrow{[f_2, v_1]^t} N_2 \oplus M_3 \xrightarrow{[v_2, f_3]} N_3 \to 0$$

be short exact sequences in $\operatorname{mod} A$. *Then the sequence*

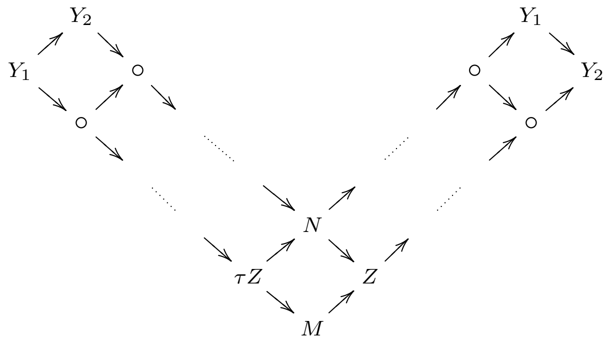
$$0 \to M_1 \xrightarrow{[f_1, v_1 u_1]^t} N_1 \oplus M_3 \xrightarrow{[-v_2 u_2, f_3]} N_3 \to 0$$

is exact.

We have the following fact.

PROPOSITION 2.8. *Let* Γ *be a generalized standard component in* Γ_A *which contains a Möbius configuration. Then there exist indecomposables* $M, N \in \Gamma$ *such that* $[M] = [N]$ *and* $M <_{\deg} N$.

Proof. We know that Γ has a full translation subquiver of the form



Applying Lemma 2.7 to the short exact sequences given by the meshes of the above quiver we get exact sequences

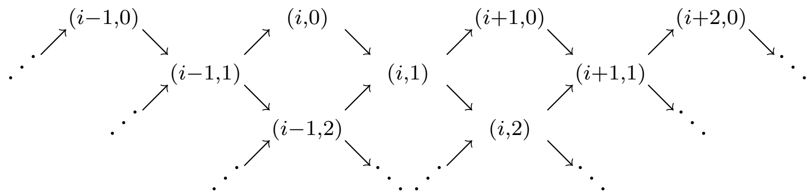
$$0 \to N \to Y_1 \oplus Z \to Y_2 \to 0, \qquad 0 \to Y_1 \to Y_2 \oplus M \to Z \to 0.$$

Applying Lemma 2.7 again we obtain an exact sequence

$$0 \to N \to M \oplus Z \to Z \to 0.$$

Observe that $[M] = [N]$. Finally, by [24, (3.4)], we infer that $M \leq_{\deg} N$. Then $M <_{\deg} N$, since $M \not\simeq N$. This finishes the proof. ∎

3. Generalized multicoils. The aim of this section is to recall the concept of a generalized multicoil playing a fundamental role in the proof of our main theorem. Recall that if $\mathbb{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$ (with the trivial valuations $(1,1)$), then $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver of the form



with $\tau(i,j) = (i-1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver Γ obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i,j) of $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^r(i,j)$, and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^r x \to \tau^r y$. The translation quivers $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, are called *stable tubes of rank* r. The *rank* of a stable tube Γ is the least positive integer r such that $\tau^r x = x$ for all x in Γ. A stable tube of rank 1 is said to be *homogeneous*. The τ-orbit of a stable tube Γ formed by all vertices having exactly one predecessor is said to be the *mouth* of Γ.

For $r \geq 1$, we denote by $T_r(k)$ the $r \times r$-lower triangular matrix algebra

$$\begin{bmatrix} k & 0 & 0 & \dots & 0 & 0 \\ k & k & 0 & \dots & 0 & 0 \\ k & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & k & \dots & k & 0 \\ k & k & k & \dots & k & k \end{bmatrix}.$$

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X,-)|_\Gamma$ is the k-linear category defined as follows [4]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X,M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X,f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the k-linear category $\mathcal{S}(X)$ with its quiver.

From now on let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable module X in Γ, called the *pivot*, one defines five admissible operations (ad 1)–(ad 5) and their duals (ad 1*)–(ad 5*) modifying the translation quiver $\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ') and the algebra A to a new algebra A', depending on the shape of the support $\mathcal{S}(X)$ (see [21, Section 2] for figures illustrating Γ').

(ad 1) Assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to \cdots .$$

In this case, we let $t \geq 1$ be a positive integer, $D = T_t(k)$ and $Y_1, \dots, Y_t$ denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra* A' to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the *modified translation quiver* Γ' to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ or Γ_D, respectively.

If $t = 0$ we define A' to be the one-point extension $A[X]$, and Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X_i', $i \geq 0$.

The nonnegative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [15].

(ad 2) Suppose that $\mathcal{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. We define $A' = A[X]$, and Γ' to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \big(k, X_i \oplus Y_j, \big[\begin{smallmatrix}1\\1\end{smallmatrix}\big]\big)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: X_0' is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geq 2$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 2$, $\tau' X_1' = Y_t$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The parameter $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to \cdots$ in the inserted rectangle equals $t + 1$.

(ad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two parallel sectional paths:

$$\begin{array}{ccccccccc}
Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & \\
\uparrow & & \uparrow & & & & \uparrow & & \\
X = X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t & \to \cdots
\end{array}$$

where $t \geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ obtained from Γ by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of the connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define $A' = A[X]$, and Γ' to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \big(k, X_i \oplus Y_j, \big[\begin{smallmatrix}1\\1\end{smallmatrix}\big]\big)$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: X_0' is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X_i' = Y_i$ if $1 \leq i \leq t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq t+1$, $\tau' Y_j = X_{j-2}'$ if $2 \leq j \leq t$, $\tau'(\tau^{-1} X_i) = X_i'$, if $i \geq t$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X_{t-1}' is injective.

The parameter $t \geq 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to \cdots$ in the inserted rectangle equals $t+1$.

(ad 4) Suppose that $\mathcal{S}(X)$ consists of an infinite sectional path, starting at X,

$$X = X_0 \to X_1 \to \cdots$$

and let

$$Y = Y_1 \to \cdots \to Y_t$$

with $t \geq 1$, be a finite sectional path in Γ_A. Let r be a positive integer. Consider the translation quiver Γ obtained from $\overline{\Gamma}$ by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of the connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we define $A' = A[X \oplus Y]$, and Γ' to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X_0' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \geq 1$, let $G = T_r(k)$, $U_{1,t+1}, \ldots, U_{r,t+1}$ denote the indecomposable projective G-modules, and $U_{r,t+1}, \ldots, U_{r,t+r}$ the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define A' to be the triangular matrix algebra

$$A' = \begin{bmatrix} A & 0 & 0 & \ldots & 0 & 0 \\ Y & k & 0 & \ldots & 0 & 0 \\ Y & k & k & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & k & k & \ldots & k & 0 \\ X \oplus Y & k & k & \ldots & k & k \end{bmatrix}$$

with $r+2$ columns and rows, and Γ' to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{kl} = Y_l \oplus U_{k,t+k}$ for $1 \leq k \leq r$, $1 \leq l \leq t$, and $Z_{ij} = \left(k, X_i \oplus U_{rj}, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 0$, $1 \leq j \leq t+r$, and $X_i' = (k, X_i, 1)$ for $i \geq 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = U_{r,j-1}$ if $2 \leq j \leq t+r$, $Z_{01}, U_{k1}, 1 \leq k \leq r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \leq k \leq r$, $2 \leq l \leq t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \leq l \leq t+1$, $\tau' X_0' = U_{r,t+r}, \tau' X_i' = Z_{i-1,t+r}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i'

is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^* or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r+1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \geq 1$ and $r \geq 0$ are such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangles equals $t+r+1$. We call $t+r$ the *parameter* of the operation.

To define the next admissible operation we also need the finite versions of the admissible operations (ad 1)–(ad 4), denoted by (fad 1)–(fad 4), respectively. In order to obtain the latter operations we replace all infinite sectional paths of the form $X_0 \to X_1 \to \cdots$ in the definitions of (ad 1)–(ad 4) by finite sectional paths $X_0 \to X_1 \to \cdots \to X_s$. We have $s \geq 0$ for the operation (fad 1), $s \geq 1$ for (fad 2) and (fad 4), and $s \geq t-1$ for (fad 3). In all above operations X_s is injective (see [21] or [22] for the details).

(ad 5) We define A' to be the iteration of the extensions described in the definitions of (ad 1)–(ad 4), and their finite versions (fad 1)–(fad 4). The Γ' is obtained in the following three steps: first we perform on Γ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly zero) of (fad 4)'s and finally (ad 4), in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

Finally, together with each of the operations (ad 1)–(ad 5), we consider its dual, denoted by (ad 1*)–(ad 5*). These ten operations are called the *admissible operations.*

Clearly, the admissible operations can be defined as operations on translation quivers rather than on Auslander–Reiten components (see [21] for the details).

Following [21] a connected translation quiver Γ is said to be a *generalized multicoil* if it can be obtained from a finite family $\mathcal{T}_1, \ldots, \mathcal{T}_s$ of stable tubes by an iterated application of (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s = 1$, then Γ is said to be a *generalized coil.* The admissible operations (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*) and (ad 3*) have been introduced in [3, 4, 5], and (ad 4) and (ad 4*) for $r = 0$ in [19].

Observe that any stable tube is trivially a generalized coil. A *tube* (in the sense of [15]) is a generalized coil such that each admissible operation used to define it is of the form (ad 1) or (ad 1*). If we only apply (ad 1) (respectively, (ad 1*)) operations then the generalized coil is called a *ray tube* (respectively, a *coray tube*). Observe that a generalized coil without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A *quasi-tube* (in the sense of [28]) is a generalized coil such that each operation used to define it is of type (ad 1), (ad 1*), (ad 2) or (ad 2*). Finally, following [4], a *coil* is a generalized coil obtained by applying (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*) only. We note that any generalized multicoil Γ is a coherent translation quiver with trivial valuations, and the translation subquiver of Γ obtained by removing all acyclic vertices and the attached arrows is infinite, connected and cofinite in Γ, and so Γ is almost cyclic.

We have the following characterization of generalized multicoils established in [21, Theorem A].

THEOREM 3.1. *Let Γ be a connected component of Γ_A. Then Γ is coherent and almost cyclic if and only if Γ is a generalized multicoil.*

4. Degenerations in generalized multicoils. Let A be an algebra. Note that from Proposition 2.6 we know that if Γ is a generalized standard coil of Γ_A which is not a quasi-tube (this means that we use at least one (ad 3) or (ad 3*)) then there exist indecomposable modules M and N in Γ such that $[M] = [N]$ and $M <_{\mathrm{deg}} N$, and clearly $M \not\leq_{\mathrm{ext}} N$. On the other hand, from [31, Corollary 2] we know that if Γ is a generalized standard quasi-tube of Γ_A then the partial orders $\leq_{\mathrm{ext}}$ and $\leq_{\mathrm{deg}}$ coincide on $\mathrm{add}(\Gamma)$.

By an *exceptional chain* in a connected component Γ of Γ_A we mean a full translation subquiver of Γ of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
 & A_1 \arrow[dr] & & A_2 \arrow[dr] & & \arrow[dr] & & A_t \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & B_1 \arrow[r] & \circ \arrow[ur] \arrow[r] \arrow[dr] & B_2 \arrow[r] & \circ \arrow[ur] \arrow[r] \arrow[dr] & \cdots \arrow[r] & \circ \arrow[ur] \arrow[r] \arrow[dr] & B_t \arrow[r] & \circ \\
 & \circ \arrow[ur] & & \circ \arrow[ur] & & \arrow[ur] & & \circ \arrow[ur] &
\end{tikzcd}$$

where $t \geq 2$, A_1 or B_1 is projective, and A_t or B_t is injective. Note that if Γ is a generalized multicoil in Γ_A then for any $1 \leq i \leq t$ there are a ray starting at A_i and a coray ending in A_i.

An exceptional chain in Γ is said to be *proper* if for any $1 \leq i \leq t$ a ray starting at A_i has infinitely many common vertices with a coray ending in A_j, $1 \leq j \leq t$.

Note that if Γ is a generalized multicoil in Γ_A without an exceptional chain then in the whole process of creating Γ none of the operations (ad 3), (ad 3*), (fad 3), (fad 3*) appears.

COROLLARY 4.1. *Let Γ be a generalized standard generalized multicoil in Γ_A. Assume that either Γ contains a Möbius configuration, or there exists in Γ a proper exceptional chain. Then there exist indecomposable modules M and N in Γ such that $[M] = [N]$ and $M <_{\mathrm{deg}} N$.*

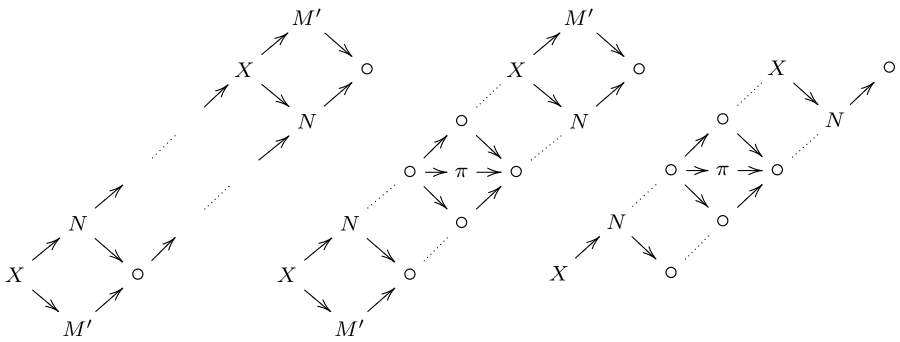
Proof. In this case Γ admits a Möbius configuration or a coil configuration. Therefore, repeating the arguments used in the proofs of Proposition 2.8 or [31, Theorem 4] we get $M <_{\deg} N$. ∎

Note that the definitions of admissible operations imply that if Γ is a generalized multicoil with a Möbius configuration then in any sequence of admissible operations leading from a finite family of stable tubes to Γ, we need at least one (ad 4), (ad 4*), (ad 5) or (ad 5*) with $t \geq 2$.

Let Γ be a generalized standard generalized multicoil in Γ_A with a Möbius configuration $\mathcal{C}$. It is easy to see that if $\mathcal{C}$ was created by applying (ad 4) or (ad 4*) then any two modules M, N from the finite sectional path occurring in the definition of (ad 4) or (ad 4*) are such that $M <_{\deg} N$ (see Example 4.6).

PROPOSITION 4.2. *Let Γ be a generalized standard generalized multicoil in Γ_A without an exceptional chain. Then the following conditions are equivalent*:

(i) *There exists a sectional path in Γ of the form $X_0 \to X_1 \to \cdots \to X_k = X_0$ consisting of noninjective modules.*
(ii) *There exists a sectional path in Γ of the form $X_0 \to X_1 \to \cdots \to X_k = X_0$ consisting of nonprojective modules.*
(iii) *There is in Γ a full translation subquiver of one of the forms*



where $X \to N \to \cdots \to X$ is a sectional path consisting of noninjective modules.

Proof. Clearly, (iii) implies (i). That (i) is equivalent to (ii) follows from the fact that $X_0 \to X_1 \to \cdots \to X_k = X_0$ is a sectional path in Γ_A consisting of nonprojective modules if and only if $\tau X_0 \to \tau X_1 \to \cdots \to \tau X_k = \tau X_0$ is a sectional path in Γ_A consisting of noninjective modules. From the definition of generalized multicoil and the assumption we know that the middle term of any Auslander–Reiten sequence of A-modules in Γ has at most two indecomposable direct summands which are not projective-injective. Therefore, (i)⇒(iii) follows from the proof of [35, Theorem 1(iii)⇒(v)]. ∎

Let Γ be a generalized standard component of Γ_A, and M and N be two modules in $\mathrm{add}(\Gamma)$ such that $M < N$. We denote by $\mathcal{F}(M,N)$ the full subquiver of Γ consisting of the vertices X such that $\delta_{M,N}(X) > 0$. Dually one can define the subquiver $\mathcal{F}'(M,N)$ consisting of the vertices X such that $\delta'_{M,N}(X) > 0$. Then the translations τ and τ^- induce mutually inverse isomorphisms of the translation quivers $\mathcal{F}(M,N)$ and $\mathcal{F}'(M,N)$.

Lemma 4.3. *Let M and N be modules in* $\mathrm{add}(\Gamma)$ *such that $M < N$, $[M] = [N]$ and N is indecomposable. Then for each vertex $V \in \mathcal{F}(M,N)$ there is an arrow in $\mathcal{F}(M,N)$ starting at V and an arrow in $\mathcal{F}(M,N)$ ending at V.*

Proof. Suppose that V is a vertex of $\mathcal{F}(M,N)$ having no predecessors in $\mathcal{F}(M,N)$. Applying Lemma 2.2 we get

$$\mu(N,V) - \mu(M,V) = \delta_{M,N}(V) + \delta'_{M,N}(V) \geq \delta_{M,N}(V) > 0.$$

Since N is indecomposable, we have $V \simeq N$. Furthermore, since $M \not\simeq N$ and $[M] = [N]$, it follows that $\mu(M,V) = 0$. Consequently,

$$1 = \mu(N,N) - \mu(M,N) = \delta_{M,N}(N) + \delta'_{M,N}(N) \geq 1 + 1 = 2,$$

by Lemma 2.5. This contradiction shows that for each V in $\mathcal{F}(M,N)$ there is an arrow in $\mathcal{F}(M,N)$ ending at V. By duality, for each V' in $\mathcal{F}'(M,N)$ there is an arrow in $\mathcal{F}'(M,N)$ starting at V'. This proves the lemma since the quivers $\mathcal{F}(M,N)$ and $\mathcal{F}'(M,N)$ are isomorphic. ∎

Lemma 4.4. *Let Γ be a generalized standard generalized multicoil in Γ_A without an exceptional chain, and M, N be two modules in* $\mathrm{add}(\Gamma)$ *with $M < N$. Let*

$$(*) \qquad X_t \to X_{t-1} \to \cdots \to X_1 \to X_0$$

be a sectional path in $\mathcal{F}(M,N)$ for some $t \geq 1$. Then

$$\text{(i)} \qquad \sum_{i=1}^{t-1} (\mu(N,X_i) - \mu(M,X_i)) = \delta_{M,N}(X_1) - \delta_{M,N}(X_t) + \delta'_{M,N}(X_{t-1}) - \delta'_{M,N}(X_0).$$

Furthermore, if $(*)$ is maximal among the sectional paths in $\mathcal{F}(M,N)$ ending with the arrow $X_1 \to X_0$ then

$$\text{(ii)} \qquad \mu(N,X_t) - \mu(M,X_t) = \delta_{M,N}(X_t) + \delta'_{M,N}(X_t) - \delta'_{M,N}(X_{t-1}),$$

$$\text{(iii)} \qquad \sum_{i=1}^{t} (\mu(N,X_i) - \mu(M,X_i)) = \delta_{M,N}(X_1) + \delta'_{M,N}(X_t) - \delta'_{M,N}(X_0).$$

Proof. Let U be a module in Γ. Applying Lemma 2.2 we get

$$(**)\quad \mu(N,U)-\mu(M,U)=\delta_{M,N}(U)+\delta'_{M,N}(U)-\sum_{(X\to U)\in\mathcal{F}(M,N)}\delta_{M,N}(X).$$

Let $\widetilde{\Gamma}$ denote the translation quiver obtained from Γ by removing all projective-injective vertices and the attached arrows. Since Γ is a generalized multicoil without an exceptional chain and $\mathcal{F}(M,N)$ is a subquiver of $\widetilde{\Gamma}$, there are at most two arrows in $\mathcal{F}(M,N)$ ending at U.

Let $1\le i<t$. We claim that

$$\sum_{(X\to X_i)\in\mathcal{F}(M,N)}\delta_{M,N}(X)=\delta_{M,N}(X_{i+1})+\delta'_{M,N}(X_{i-1}).$$

Indeed, $X_{i+1}\to X_i$ is an arrow in $\mathcal{F}(M,N)$ and the other possible arrow in $\mathcal{F}(M,N)$ ending at X_i is $\tau X_{i-1}\to X_i$ provided the module X_{i-1} is not projective and $\delta'_{M,N}(X_{i-1})=\delta_{M,N}(\tau X_{i-1})>0$. Conversely, if $\delta'_{M,N}(X_{i-1})>0$ then X_{i-1} is not a projective module, $\delta_{M,N}(\tau X_{i-1})=\delta'_{M,N}(X_{i-1})$ is a positive integer and $\tau X_{i-1}\to X_i$ is an arrow in $\mathcal{F}(M,N)$. Combining the claim and $(**)$ we get

$$\mu(N,X_i)-\mu(M,X_i)=\delta_{M,N}(X_i)-\delta_{M,N}(X_{i+1})+\delta'_{M,N}(X_i)-\delta'_{M,N}(X_{i-1}),$$

where $1\le i<t$. Summing up and reducing we obtain (i).

Assume now that the sectional path $(*)$ is maximal. Observe that (iii) is a consequence of (i) and (ii). Applying $(**)$ for $U=X_t$, it remains to prove that

$$\sum_{(X\to X_t)\in\mathcal{F}(M,N)}\delta_{M,N}(X)=\delta'_{M,N}(X_{t-1}).$$

The only possible arrow in $\mathcal{F}(M,N)$ ending at X_t is $\tau X_{t-1}\to X_t$ provided the module X_{t-1} is not projective and $\delta'_{M,N}(X_{t-1})=\delta_{M,N}(\tau X_{t-1})>0$. Conversely, if $\delta'_{M,N}(X_{t-1})>0$ then X_{t-1} is not a projective module, $\delta_{M,N}(\tau X_{t-1})=\delta'_{M,N}(X_{t-1})$ is a positive integer and $\tau X_{t-1}\to X_t$ is an arrow in $\mathcal{F}(M,N)$. ∎

The proof of the following fact grew out of discussions with Grzegorz Zwara.

PROPOSITION 4.5. *Let Γ be a generalized standard generalized multicoil in Γ_A that contains neither a Möbius configuration nor an exceptional chain, and M, N two modules in* $\mathrm{add}(\Gamma)$ *with $[M]=[N]$. Then $M\le_{\mathrm{deg}}N$ if and only if $M\le_{\mathrm{ext}}N$.*

Proof. Suppose that there is a proper degeneration $M<_{\mathrm{deg}}N$. We denote by $\mathcal{F}(M,N)$ the full subquiver of Γ consisting of the vertices X with $\delta_{M,N}(X)>0$. By Proposition 2.4, $\mathcal{F}(M,N)$ has only finitely many vertices. Furthermore, since $\mathcal{F}(M,N)$ contains no injective modules, there are at most two arrows in $\mathcal{F}(M,N)$ starting at X and at most two arrows in $\mathcal{F}(M,N)$ ending at X, for any vertex X in $\mathcal{F}(M,N)$. From Proposition 4.2 and the definition of Γ we deduce that there is no sectional path in Γ of the form $X_0 \to X_1 \to \cdots \to X_t = X_0$. This implies that there is no such path in $\mathcal{F}(M,N)$ nor in $\mathcal{F}'(M,N)$. By Lemma 2.5, N belongs to $\mathcal{F}(M,N)$ and to $\mathcal{F}'(M,N)$.

We claim that for any arrow $Y_1 \to N$ in $\mathcal{F}(M,N)$ ending at N,

$$\delta_{M,N}(N) > \delta_{M,N}(Y_1).$$

From [9] (see also [10]) we know that there is no sectional path in Γ_A of the form $X_1 \to \cdots \to X_n \to X_1 \to X_2$, $n \geq 2$. As a consequence, there is no infinite sectional path in Γ_A provided Γ_A is a finite quiver. So, we may take a maximal sectional path in $\mathcal{F}(M,N)$

$$Y_p \to Y_{p-1} \to \cdots \to Y_1 \to N$$

ending with the arrow $Y_1 \to N$. We set $Y_0 = N$. From our assumption, $Y_p \neq N$. Applying Lemma 4.4(ii) we get

$$\begin{aligned} 0 \geq \mu(N,Y_p) - \mu(M,Y_p) &= \delta_{M,N}(Y_p) + \delta'_{M,N}(Y_p) - \delta'_{M,N}(Y_{p-1}) \\ &> \delta'_{M,N}(Y_p) - \delta'_{M,N}(Y_{p-1}). \end{aligned}$$

By the maximality of $Y_p \to Y_{p-1} \to \cdots \to Y_1 \to N$ and from Lemma 4.3 we conclude that Y_{p-1} is not a projective module and $\tau Y_{p-1} \to Y_p$ is a unique arrow in $\mathcal{F}(M,N)$ ending at Y_p. Let $Z_0 = Y_p$ and $Z_1 = \tau Y_{p-1}$. Then

$$\delta_{M,N}(Z_1) - \delta'_{M,N}(Y_p) = \delta'_{M,N}(Y_{p-1}) - \delta'_{M,N}(Y_p) > 0.$$

Again (using arguments as above), we may extend the arrow $Z_1 \to Y_p$ to a maximal sectional path in $\mathcal{F}(M,N)$

$$Z_q \to Z_{q-1} \to \cdots \to Z_1 \to Y_p.$$

Applying Lemma 4.4(iii) we get

$$\begin{aligned} \sum_{i=1}^{q}(\mu(N,Z_i) - \mu(M,Z_i)) &= \delta_{M,N}(Z_1) + \delta'_{M,N}(Z_q) - \delta'_{M,N}(Y_p) \\ &\geq \delta_{M,N}(Z_1) - \delta'_{M,N}(Y_p) > 0. \end{aligned}$$

Thus $N = Z_h$ for some $1 \leq h \leq q$. We take h minimal possible. Applying Lemma 4.4(i) to the sectional path $N \to Z_{h-1} \to \cdots \to Z_1 \to Y_p$ we get

$$\begin{aligned} 0 &\geq \sum_{i=1}^{h-1}(\mu(N,Z_i) - \mu(M,Z_i)) \\ &= \delta_{M,N}(Z_1) - \delta_{M,N}(N) + \delta'_{M,N}(Z_{h-1}) - \delta'_{M,N}(Y_p) \\ &> \delta'_{M,N}(Z_{h-1}) - \delta_{M,N}(N). \end{aligned}$$

By our assumption the path

$$Y_p \to \cdots \to Y_1 \to N \to Z_{h-1} \to \cdots \to Z_1 \to Y_p$$

in $\mathcal{F}(M,N)$ is not sectional. Hence $Y_1 = \tau Z_{h-1}$, and consequently

$$\delta_{M,N}(N) > \delta'_{M,N}(Z_{h-1}) = \delta_{M,N}(Y_1),$$

which proves the claim. By dual considerations in $\mathcal{F}'(M,N)$ we get $\delta'_{M,N}(N) > \delta'_{M,N}(Y'_1)$ for any arrow $N \to Y'_1$ in $\mathcal{F}'(M,N)$ starting at N. Since N has at most two direct predecessors in $\mathcal{F}(M,N)$ and at most two direct successors in $\mathcal{F}'(M,N)$ we conclude that

$$2 \le 2\delta_{M,N}(N) - \sum_{(X\to N)\in\mathcal{F}(M,N)} \delta_{M,N}(X),$$
$$2 \le 2\delta'_{M,N}(N) - \sum_{(N\to Y)\in\mathcal{F}'(M,N)} \delta'_{M,N}(Y).$$

From Lemma 2.2 we obtain

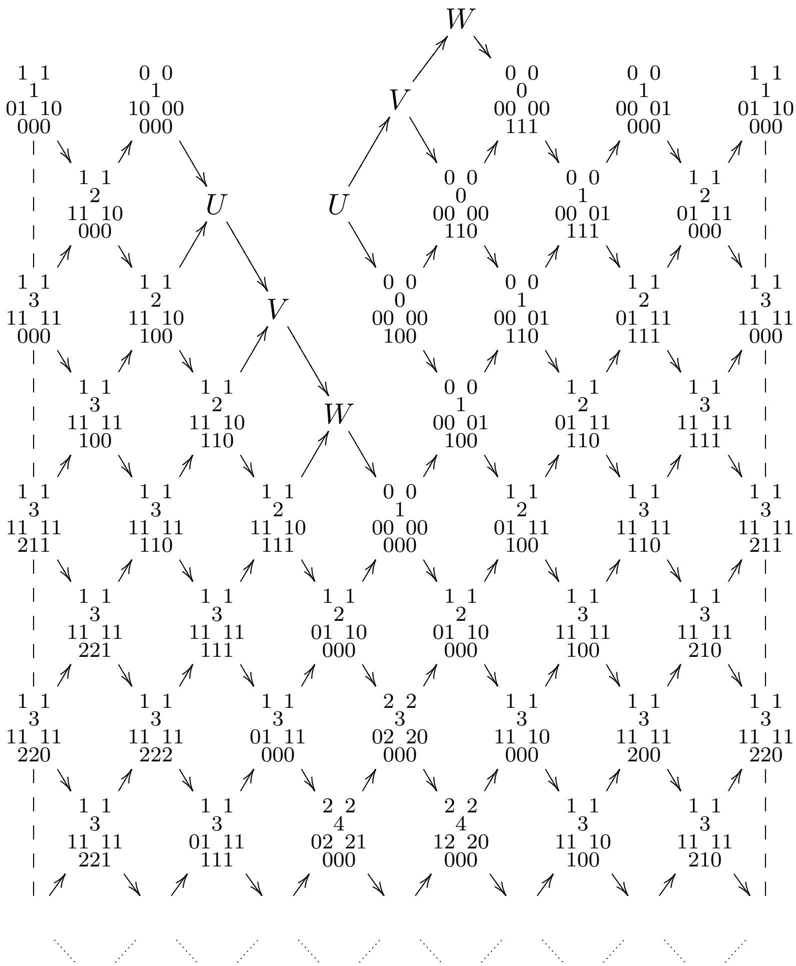
$$\begin{aligned} 2 &= 2(\mu(N,N) - \mu(M,N)) \\ &= 2\delta_{M,N}(N) + 2\delta'_{M,N}(N) \\ &\quad - \sum_{(X\to N)\in\mathcal{F}(M,N)} \delta_{M,N}(X) - \sum_{(N\to Y)\in\mathcal{F}'(M,N)} \delta'_{M,N}(Y) \ge 2+2, \end{aligned}$$

a contradiction. So, N is decomposable. Thus there is no proper degeneration $M <_{\deg} N$ to an indecomposable A-module and our claim follows from [38, Corollary 5]. ■

Example 4.6. Consider the algebra A given by the quiver

$$\begin{tikzcd}
& 1 & & 4 \arrow[dl, "\alpha"'] & \\
& & 3 \arrow[ul, "\beta"'] \arrow[dll, "\lambda"'] \arrow[dl, "\delta"] & & \\
6 & 2 & & 5 \arrow[ul, "\gamma"'] & 10 \arrow[ull, "\varrho"'] \arrow[dlll, "\sigma"'] \\
& 7 \arrow[ul, "\mu"] & 8 \arrow[l, "\nu"] & 9 \arrow[l, "\omega"] &
\end{tikzcd}$$

bound by $\alpha\lambda = 0, \gamma\lambda = 0, \varrho\lambda = 0, \varrho\beta = 0, \varrho\delta = 0, \sigma\mu = 0$. The Auslander–Reiten quiver Γ_A has as a generalized standard component a generalized multicoil Γ of the form



where the indecomposables are represented by their dimension-vectors and one identifies along the vertical dashed lines to form the generalized multicoil. Moreover,

$$U = \begin{smallmatrix} 0\ 0 \\ 1 \\ 10\ 00 \\ 100 \end{smallmatrix}, \qquad V = \begin{smallmatrix} 0\ 0 \\ 1 \\ 10\ 00 \\ 110 \end{smallmatrix}, \qquad W = \begin{smallmatrix} 0\ 0 \\ 1 \\ 10\ 00 \\ 111 \end{smallmatrix},$$

the indecomposable A-module with dimension-vector W is injective, the indecomposable A-modules with dimension-vectors $\begin{smallmatrix} 0\ 0 \\ 1 \\ 00\ 01 \\ 100 \end{smallmatrix}$, $\begin{smallmatrix} 0\ 0 \\ 0 \\ 00\ 00 \\ 100 \end{smallmatrix}$ are projective and we identify the two copies with dimension-vector U, the two copies with dimension-vector V, and also the two copies with dimension-vector W. We have

$$\begin{smallmatrix} 1 & & 1 \\ & 3 & \\ 11 & & 11 \\ & 210 & \end{smallmatrix} = M_1 <_{\deg} N_1 = \begin{smallmatrix} 1 & & 1 \\ & 3 & \\ 11 & & 11 \\ & 210 & \end{smallmatrix},$$

$$\begin{smallmatrix} 1 & & 1 \\ & 3 & \\ 11 & & 11 \\ & 221 & \end{smallmatrix} = M_2 <_{\deg} N_2 = \begin{smallmatrix} 1 & & 1 \\ & 3 & \\ 11 & & 11 \\ & 221 & \end{smallmatrix},$$

$$\begin{smallmatrix} 1 & & 1 \\ & 3 & \\ 11 & & 11 \\ & 211 & \end{smallmatrix} = M_3 <_{\deg} N_3 = \begin{smallmatrix} 1 & & 1 \\ & 3 & \\ 11 & & 11 \\ & 211 & \end{smallmatrix}$$

(see Proposition 2.8).

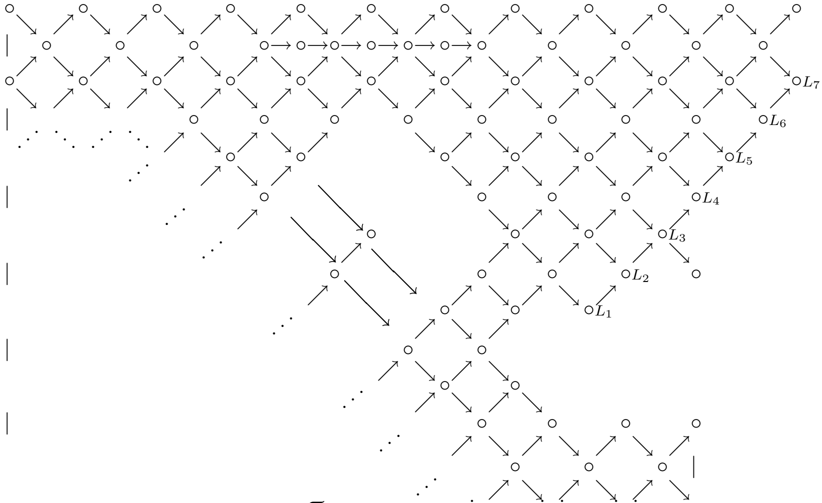
Example 4.7. Consider the algebra A given by the quiver

$$\begin{tikzcd}
 & & & & & & & 23 \arrow[r] \arrow[d] & 24 \arrow[r] & 25 \\
 & & & & & & 35 \arrow[r, "\sigma"] \arrow[d] & 22 \arrow[r, "\theta"] & 21 \arrow[r, "\varphi"] & 26 \arrow[u] \arrow[dd, "\psi"] \\
1 \arrow[d] & & 5 & & & & 34 \arrow[d, "\nu"] & & & \\
2 \arrow[r, "\alpha"] & 4 \arrow[r] \arrow[d, "\beta"] & 6 \arrow[u] \arrow[d] & & & & 33 \arrow[r, "\pi"] \arrow[d] & 29 \arrow[r, "\omega"] & 28 \arrow[r] & 27 \\
3 \arrow[u] & 8 \arrow[d] & 7 & & & & 32 \arrow[d, "\mu"] & 30 \arrow[u, "j"] \arrow[d] & & \\
 & 9 \arrow[dd, "\delta"] & 10 \arrow[l, "\gamma"'] & & & & 20 \arrow[llll, "\zeta"'] \arrow[dd] & 31 & & \\
 & & 11 \arrow[u] & 15 \arrow[d, "\eta"] & & & & & & \\
 & 12 & 13 \arrow[l, "\varrho"'] & 14 \arrow[l, "\lambda"'] \arrow[d, "\varepsilon"] & 17 \arrow[l, "\kappa"'] & 18 \arrow[l, "\xi"'] & 19 \arrow[l] & & & \\
 & & & 16 & & & & & &
\end{tikzcd}$$

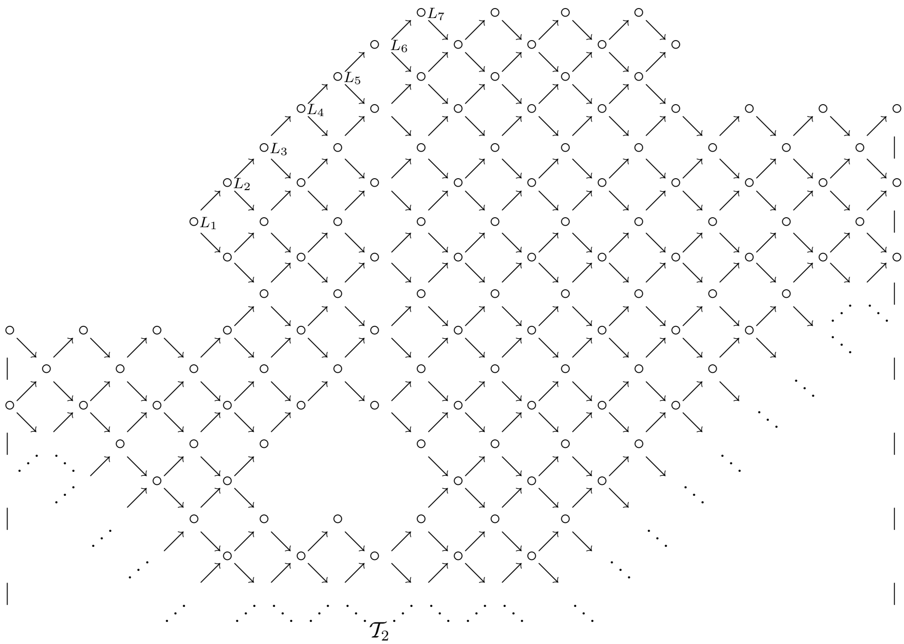
bound by

$$\begin{array}{llll} \alpha\beta = 0, & \gamma\delta = 0, & \eta\varepsilon = 0, & \kappa\lambda\varrho = 0, \\ \zeta\gamma = 0, & \mu\zeta = 0, & \nu\pi = 0, & \xi\kappa\lambda = 0, \\ \pi\omega = 0, & \sigma\theta = 0, & \varphi\psi = 0. & \end{array}$$

The Auslander–Reiten quiver Γ_A has as a generalized standard component a generalized multicoil Γ obtained by identifying (along the sectional path $L_1 \to \cdots \to L_7$) the following translation quivers $\mathcal{T}_1$ and $\mathcal{T}_2$ (see details in [21, Examples in Sections 2 and 3]):



$\mathcal{T}_1$



$\mathcal{T}_2$

where the vertical dashed lines have to be identified.

Note that the exceptional chain in Γ is not proper.

5. Proof of Theorem 1.1

(i)$\Rightarrow$(ii). From Theorem 3.1 we know that a component of Γ_A is almost cyclic and coherent if and only if it is generalized multicoil. Since Γ contains no coil configuration, it is a generalized standard generalized multicoil in Γ_A

without a proper exceptional chain. If Γ does not contain an exceptional chain then the statement follows from Proposition 4.5. Assume that Γ has an exceptional chain $\mathcal{C}$ which is not proper. So, $\mathcal{C}$ was created by applying (ad 5) or (ad 5*) to at least two components (of algebra or algebras). From [20, Lemma 3.3] we know that, for a fixed $x \in K_0(A)$, each ray and coray in Γ contains at most one module X with $[X] = x$. Therefore, from the definition of the generalized multicoil we conclude that the set of new vertices appearing after applying the above admissible operation contains no two modules V, W such that $[V] = [W]$. Using Proposition 4.5 again we complete the proof of the implication.

(ii)⇒(i). If Γ contains a Möbius configuration or a coil configuration then Proposition 2.8 or Proposition 2.6 implies that there exist indecomposable modules M and N in Γ such that $[M] = [N]$ and $M <_{\deg} N$, and clearly $M \not\leq_{\mathrm{ext}} N$.

Acknowledgments. The author gratefully acknowledges support from the Research Grant No. N N201 269 135 of Polish Ministry of Science and High Education.

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Received 13 May 2008;
revised 19 June 2008 (5048)