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A NOTE ON GROUPS OF INFINITE RANK WHOSE PROPER SUBGROUPS ARE ABELIAN-BY-FINITE

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Abstract. It is proved that if G is a locally (soluble-by-finite) group of infinite rank in which every proper subgroup of infinite rank contains an abelian subgroup of finite index, then all proper subgroups of G are abelian-by-finite.

1. Introduction. Let $\mathfrak{X}$ be a class of groups. A group G is said to be a *minimal non-*$\mathfrak{X}$ group if G is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. Clearly, *Tarski groups* (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) are minimal non-abelian, and this example suggests that some further restriction is necessary in order to investigate the behavior of minimal non-$\mathfrak{X}$ groups for a group class $\mathfrak{X}$. A group G is called *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Locally graded groups form a wide class, containing in particular all locally (soluble-by-finite) groups, and the assumption for a group to be locally graded is sufficient to avoid Tarski groups and other similar pathologies. In fact, it is easy to show that any locally (soluble-by-finite) group whose proper subgroups are abelian is either abelian or finite.

The structure of groups whose proper subgroups are abelian-by-finite was studied by B. Bruno and R. E. Phillips (see [1] and [2]) within the universe of imperfect (and so also soluble-by-finite) groups. They proved in particular that any minimal non-(abelian-by-finite) group with proper commutator subgroup is periodic and has infinite rank. Recall here that a group G is said to have *finite* (*Prüfer*) *rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. In recent years a series of papers has been published on the behavior of groups of infinite rank in which every proper subgroup of infinite rank has a suitable property (see for instance [4], [6], [7]).

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The aim of this short paper is to provide a further contribution to this topic, investigating the structure of groups in which every proper subgroup of infinite rank contains an abelian subgroup of finite index. Our main result is the following.

THEOREM. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. Then all proper subgroups of G are abelian-by-finite.*

It is an open question whether an arbitrary locally graded group of infinite rank must contain a proper subgroup of infinite rank, and this seems to be the main obstacle to extending our theorem to the case of locally graded groups. However, we will point out that this is at least possible for a special class of locally graded groups.

Most of our notation is standard and can be found in [10].

2. Proof of the Theorem. As we mentioned in the introduction, all imperfect minimal non-(abelian-by-finite) groups are periodic. Our first lemma collects some of the main properties of these groups, which were proved in [1] and [2].

LEMMA 2.1. *Let G be a group whose proper subgroups are abelian-by-finite. If $G \neq G'$, then either G is abelian-by-finite, or G/G' is a group of type p^∞ and G' is an abelian q-group, where p and q are prime numbers.*

For our purposes, we will also need the following result of [2]. Here for any group X, the symbol $\pi(X)$ denotes the set of all prime numbers p such that X has elements of order p.

LEMMA 2.2. *Let G be a group, and let A be a torsion-free abelian non-trivial normal subgroup of G such that G/A is periodic. Then for any finite set π of prime numbers, there exists a G-invariant subgroup N of A such that A/N is periodic and π is contained in $\pi(A/N)$.*

LEMMA 2.3. *Let G be a group whose proper subgroups of infinite rank are abelian-by-finite. If the factor group G/G' has infinite rank, then every proper subgroup of G is abelian-by-finite.*

Proof. Let X be any subgroup of finite rank of G. Then XG' is properly contained in G, and so there exists a proper subgroup Y of G of infinite rank such that $XG' \leq Y$. Therefore X is abelian-by-finite. ∎

Our next result allows us to restrict considerations to the case of groups whose lower central series terminates with the commutator subgroup.

LEMMA 2.4. *Let G be a group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. If G contains a proper subgroup which is not abelian-by-finite, then $G' = [G', G]$ and G is either perfect or metabelian.*

Proof. Since G/G' has finite rank by Lemma 2.3 and $G'/[G',G]$ is a homomorphic image of the tensor product

$$(G/G') \otimes (G/G'),$$

it follows that also the nilpotent group $G/[G',G]$ has finite rank. Then $[G',G]$ must have infinite rank, and hence all proper subgroups of $G/[G',G]$ are abelian-by-finite. Assume for a contradiction that $[G',G]$ is properly contained in G'. Then $G/[G',G]$ cannot be abelian-by-finite, since G has no proper subgroups of finite index, and hence Lemma 2.1 shows that G/G' is a group of type p^∞ for some prime number p. It follows that the nilpotent group $G/[G',G]$ is abelian, and this contradiction shows that $G' = [G',G]$.

Suppose now that G' is a proper subgroup of G. Then G' is abelian-by-finite, and so it contains an abelian characteristic subgroup A of finite index. As the factor group G/A has finite commutator subgroup, it is nilpotent-by-finite (see [10, Part 1, Theorem 4.25]). On the other hand, G has no proper subgroups of finite index, so that G/A is nilpotent and hence even abelian by the first part of the proof. Therefore $G' = A$ is abelian. ∎

Lemma 2.5. *Let G be a locally (soluble-by-finite) group containing a proper normal subgroup N such that G/N has finite rank. Then G has a non-trivial homomorphic image which is either finite or abelian.*

Proof. Assume that G has no proper subgroups of finite index. As the factor group, G/N is locally (soluble-by-finite) by a result of N. S. Chernikov [3], it is even locally soluble. Then there exists a positive integer k such that $G^{(k)}N/N$ is hypercentral (see [10, Part 2, Lemma 10.39]), and it follows that the commutator subgroup of G is a proper subgroup. ∎

The next lemma is the main step in the proof of our Theorem. It improves the result by Bruno and Phillips [2] on periodicity of soluble-by-finite minimal non-(abelian-by-finite) groups.

Lemma 2.6. *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G are abelian-by-finite, then G is either abelian-by-finite or periodic.*

Proof. Assume for a contradiction that the group G is neither abelian-by-finite nor periodic. In particular, G has no proper subgroups of finite index and so it cannot be finitely generated. As finitely generated abelian-by-finite groups have obviously finite rank, it follows from the hypotheses that every finitely generated subgroup of G has finite rank. Moreover, it is also clear that each proper subgroup of G is abelian-by-(finite rank). Then either G is soluble or it contains an abelian normal subgroup A such that G/A has finite rank (see [8, Theorem 8]), so that an application of Lemma 2.5 shows that G' is a proper subgroup of G.

It follows from Lemma 2.1 that G contains proper subgroups which are not abelian-by-finite. Then G/G' must have finite rank by Lemma 2.3, and G' is an abelian group of infinite rank by Lemma 2.4. Suppose first that the group G/G' is not periodic. Since G/G' is divisible, G can be decomposed into the product of two proper normal subgroups of infinite rank. Then G is nilpotent-by-finite, and so nilpotent. Again Lemma 2.4 gives now the contradiction that G is abelian. Therefore G/G' must be periodic, and hence G' is not periodic.

Let T be the subgroup consisting of all elements of finite order of G'. If T has infinite rank, then all proper subgroups of G/T are abelian-by-finite, and hence the non-periodic group G/T is abelian-by-finite by Lemma 2.1. Then G/T is abelian, a contradiction, because T is properly contained in G'. Therefore T has finite rank, and the factor group G/T is likewise a counterexample, so that without loss of generality it can be assumed that G' is a torsion-free abelian group. If q_1 and q_2 are distinct prime numbers, by Lemma 2.2 there exists a G-invariant subgroup N of G' such that G'/N is periodic and q_1, q_2 belong to $\pi(G'/N)$. As G' is torsion-free, the subgroup N has infinite rank, and hence all proper subgroups of G/N are abelian-by-finite. On the other hand, G/N is not abelian-by-finite, and its commutator subgroup is not a primary group, contradicting Lemma 2.1. ∎

We are now in a position to prove our main result.

Proof of the Theorem. Assume for a contradiction that the statement is false. In particular, G is not abelian-by-finite, and so it has no proper subgroups of finite index. Moreover, G is locally finite by Lemma 2.6. Suppose that G contains a proper normal subgroup K such that G/K is simple. Then G/K must have infinite rank by Lemma 2.5, and hence it is isomorphic either to $\mathrm{PSL}(2, F)$ or to $\mathrm{Sz}(F)$ for some infinite locally finite field F containing no infinite proper subfields (see [8, Lemma 2]). On the other hand, each of the last two groups contains a proper subgroup of infinite rank which is not abelian-by-finite (see for instance [9]). This contradiction shows that G has no infinite simple homomorphic images, so that it contains an abelian normal subgroup N such that G/N has finite rank (see [8, Theorem 5]), and hence G' is a proper subgroup of G by Lemma 2.5. It follows from Lemmas 2.3 and 2.4 that G' is an abelian group of infinite rank.

Let X be a subgroup of finite rank of G which is not abelian-by-finite. Clearly, X cannot be contained in any proper subgroup of infinite rank of G, and so $XG' = G$. Then

$$X/X \cap G' \simeq G/G'$$

has no proper subgroups of finite index, and hence it is divisible. Moreover, the abelian normal subgroup $X \cap G'$ of G has finite rank, and hence it is covered by finite characteristic subgroups. It follows that $X \cap G'$ is contained in the center of X, and so X is nilpotent. Consider now a maximal abelian normal subgroup A of X containing $X \cap G'$. Then $C_X(A) = A$, and hence the divisible group X/A is isomorphic to a group of automorphisms of A. On the other hand, A is covered by finite characteristic subgroups, so that its full automorphism group is residually finite. Therefore $X = A$ is abelian, and this contradiction proves the statement. ■

3. Concluding remarks. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\bar{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ under the operators $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, **R, L** (we are using here the first chapter of the monograph [10] as a general reference for definitions and properties of closure operations on group classes). It is easy to prove that any $\bar{\mathfrak{D}}$-group is locally graded, and that the class $\bar{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, Chernikov [3] proved that any $\bar{\mathfrak{D}}$-group of finite rank is a finite extension of a locally soluble subgroup. Obviously, all residually finite groups belong to $\bar{\mathfrak{D}}$, and hence the consideration of any free non-abelian group shows that the class $\bar{\mathfrak{D}}$ is not closed with respect to homomorphic images. Following [5], we shall say that a group G is *strongly locally graded* if every section of G is a $\bar{\mathfrak{D}}$-group. Thus strongly locally graded groups form a large class $\mathcal{U}$ of generalized soluble groups, which is closed with respect to subgroups and homomorphic images, and contains all locally (soluble-by-finite) groups. Although it is not clear whether our main theorem can be extended to arbitrary locally graded groups, we can at least prove that it remains true for strongly locally graded groups.

LEMMA 3.1. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are locally* (*soluble-by-finite*), *then G is locally soluble-by-finite.*

Proof. Suppose first that G is a finitely generated non-trivial group, so that it contains a proper subgroup H of finite index. Then H is a finitely generated subgroup of infinite rank, hence it is soluble-by-finite and therefore G itself is soluble-by-finite. Assume now that G is not finitely generated, so that all its finitely generated subgroups of infinite rank are soluble-by-finite. On the other hand, it follows from Chernikov's result on $\bar{\mathfrak{D}}$-groups that any finitely generated subgroup X of G of finite rank must be soluble-by-finite. Therefore G is locally (soluble-by-finite). ■

COROLLARY 3.2. *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite. Then all proper subgroups of G are abelian-by-finite.*

Proof. The group G is locally (soluble-by-finite) by Lemma 3.1, and so the statement is a direct consequence of our main Theorem. ■

REFERENCES

[1] B. Bruno, *On groups with abelian-by-finite proper subgroups*, Boll. Un. Mat. Ital. 3 (1984), 797–807.

[2] B. Bruno and R. E. Phillips, *A note on groups with nilpotent-by-finite proper subgroups*, Arch. Math. (Basel) 65 (1995), 369–374.

[3] N. S. Chernikov, *Theorem on groups of finite special rank*, Ukrain. Math. J. 42 (1990), 855–861.

[4] M. De Falco, F. de Giovanni and C. Musella, *Groups whose proper subgroups of infinite rank have a transitive normality relation*, Mediterr. J. Math. 10 (2013), 1999–2006.

[5] M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi, *Groups with restrictions on subgroups of infinite rank*, Rev. Mat. Iberoamer. 30 (2014), 535–548.

[6] M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi, *Groups whose proper subgroups of infinite rank have finite conjugacy classes*, Bull. Austral. Math. Soc. 89 (2014), 41–48.

[7] M. R. Dixon, M. J. Evans and H. Smith, *Locally (soluble-by-finite) groups with all proper non-nilpotent subgroups of finite rank*, J. Pure Appl. Algebra 135 (1999), 33–43.

[8] M. R. Dixon, M. J. Evans and H. Smith, *Groups with all proper subgroups nilpotent-by-finite rank*, Arch. Math. (Basel) 75 (2000), 81–91.

[9] J. Otal and J. M. Peña, *Infinite locally finite groups of type* $\mathrm{PSL}(2, K)$ *or* $\mathrm{Sz}(K)$ *are not minimal under certain conditions*, Publ. Mat. 32 (1988), 43–47.

[10] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin, 1972.

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