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ON EQUIVALENCE OF SUPER LOG SOBOLEV AND NASH TYPE INEQUALITIES

BY

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Abstract. We prove the equivalence of Nash type and super log Sobolev inequalities for Dirichlet forms. We also show that both inequalities are equivalent to Orlicz–Sobolev type inequalities. No ultracontractivity of the semigroup is assumed. It is known that there is no equivalence between super log Sobolev or Nash type inequalities and ultracontractivity. We discuss Davies–Simon's counterexample as the borderline case of this equivalence and related open problems.

1. Introduction. Let $(T_t)_{t>0} = (e^{-At})_{t>0}$ be a symmetric submarkovian semigroup with infinitesimal generator $-A$ on $L^2(X,\mu)$ where (X,μ) is a σ-finite measure space. The symmetry means

$$(T_t f, g) = (f, T_t g), \qquad f, g \in L^2,\ t > 0,$$

and the submarkovian property reads

$$0 \le f \le 1 \ \Rightarrow\ 0 \le T_t f \le 1, \qquad f \in L^2.$$

Moreover, $(T_t)_{t>0}$ is a C_0-contraction semigroup on L^2 which extends to a C_0-contraction semigroup $T_t := T_t^{(p)}$ on each $L^p = L^p(X,\mu)$ with $1 \le p < \infty$ and acts as a contraction on L^∞. The infinitesimal generator $-A$ is defined by

$$Af := \lim_{t\to 0^+} \frac{f - T_t f}{t},$$

for $f \in \mathcal{D}(A)$, i.e. $f \in L^2$ such that the above limit exists in L^2. In particular, the operator A is non-negative and self-adjoint on L^2.

The associated Dirichlet form $\mathcal{E}$ is defined as follows. Let $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{A})$ where $\sqrt{A}$ is the positive square root of A. We set $\mathcal{E}(f,g) = (\sqrt{A}\,f, \sqrt{A}\,g)$ for $f, g \in \mathcal{D}(\mathcal{E})$. Then the bilinear form $\mathcal{E}$ is a positive, symmetric, bilinear, closed, densely defined form on $L^2(\mu)$. Moreover, $\mathcal{E}$ has the following contraction property:

$$\forall f \in \mathcal{D}(\mathcal{E}),\ g = (f \wedge 1) \vee 0 \ \Rightarrow\ g \in \mathcal{D}(\mathcal{E}) \text{ and } \mathcal{E}(g) \le \mathcal{E}(f) \tag{1.1}$$

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where $\mathcal{E}(f) := \mathcal{E}(f,f)$. Furthermore, there is a bijective correspondence between Dirichlet forms and submarkovian symmetric semigroups (see [F]).

A fundamental property of operator semigroups $(T_t)_{t>0}$ is *ultracontractivity*, that is, the existence of a non-increasing function a from $(0,\infty)$ to itself such that for any $f \in L^1$,

$$\|T_t f\|_\infty \le a(t)\|f\|_1, \qquad t>0. \tag{1.2}$$

Under some conditions described for instance in [GH, Section 3.3] (see also [W2, Proposition 3.3.11]), ultracontractivity of a semigroup (1.2) implies the existence of a *heat kernel* $h_t(x,y)$ for this semigroup, that is,

$$T_t f(x) = \int_X h_t(x,y) f(y)\, d\mu(y),$$

and uniform bounds on this kernel,

$$\sup_{x,y\in X} h_t(x,y) \le a(t), \quad t>0.$$

Conversely, the existence of a heat kernel and uniform bounds obviously imply ultracontractivity of the semigroup.

Before going further, let us discuss some equivalent formulation of ultracontractivity. By interpolation, the semigroup property, duality and symmetry, inequality (1.2) is equivalent to the existence of a non-increasing function $c:(0,\infty)\to(0,\infty)$ such that for any $f\in L^1$,

$$\|T_t f\|_2 \le c(t)\|f\|_1, \qquad t>0, \tag{1.3}$$

or equivalently, for any $f \in L^2$,

$$\|T_t f\|_\infty \le c(t)\|f\|_2, \quad t>0. \tag{1.4}$$

More precisely, (1.2) implies (1.3)&(1.4) with $c(t)\le \sqrt{a(t)}$, and conversely (1.3) or (1.4) implies (1.2) with $a(t)\le c^2(t/2)$ (see [D]).

It is known that some regularization properties (for instance ultracontractivity) of the semigroup $(T_t)_{t>0}$ can be quantified by functional inequalities satisfied by the infinitesimal generator $-A$. Let us recall two fundamental results in that direction.

Let $M:(0,\infty)\to\mathbb{R}$ be a function. For any $y\in\mathbb{R}$, we set

$$\Lambda(y) = \sup_{t>0}\{ty - 2tM(1/2t)\} \in (-\infty,\infty]. \tag{1.5}$$

The function Λ is the Legendre transform of $t\mapsto 2tM(1/2t)$.

In [C, Proposition II.2], T. Coulhon proved that if $(T_t)_{t>0}$ is ultracontractive with $c(t)=e^{M(t)}$ in (1.3), then the Nash type inequality

$$\Theta(\|f\|_2^2) \le \mathcal{E}(f), \qquad f\in\mathcal{D}(\mathcal{E}),\ \|f\|_1\le 1, \tag{1.6}$$

holds true with $\Theta(x) = x\Lambda(\log x)$, $x>0$, and Λ given by (1.5). This specific form of Θ will be of importance for the formulation of our main result, Theorem 1.1, as already noticed in [BM] in a particular case. This inequality can be seen as a weak form of Sobolev inequality (see for instance [C], [BCLS], [VSC], [W2], [BGL]).

On the other hand, E. B. Davies and B. Simon [D, Theorem 2.2.3] proved that if $(T_t)_{t>0}$ is ultracontractive with $c(t) = e^{\beta(t)}$ in (1.3), then for any non-negative function f in $\mathcal{D}(\mathcal{E}) \cap L^1 \cap L^\infty$, we have $f^2 \log f \in L^1$ and the following super log Sobolev inequality (1.7) holds true:

$$\int_X f^2 \log f \, d\mu \le t\mathcal{E}(f) + \beta(t)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \qquad t > 0. \tag{1.7}$$

This inequality is modeled on the celebrated Gross inequality [G1] and equivalent to supercontractivity of the semigroup $(T_t)_{t>0}$, i.e. for each $t > 0$, T_t is bounded from L^2 to L^4 (see [G2, Theorem 3.7]).

T. Coulhon's and Davies–Simon's results assert that ultracontractivity of the semigroup implies the Nash type inequality (1.6) and the super log Sobolev inequality (1.7) respectively for the generator of the semigroup. Then it is natural to ask whether there exist direct relationships between Nash type and super log Sobolev inequalities without the ultracontractivity assumption.

Our main theorem provides a positive answer to that question.

Main Theorem 1.1. *Let $\mathcal{E}$ be a Dirichlet form with domain $\mathcal{D}(\mathcal{E})$. Then the following statements are equivalent:*

(1) *There exists $M_1 : (0,\infty) \to \mathbb{R}$ such that, for any $f \in \mathcal{D}(\mathcal{E})$,*

$$\int_X f^2 \log\left(\frac{|f|}{\|f\|_2}\right) d\mu \le t\mathcal{E}(f) + M_1(t)\|f\|_2^2, \quad t > 0. \tag{1.8}$$

(2) *There exists $M_2 : (0,\infty) \to \mathbb{R}$ such that, for any $f \in \mathcal{D}(\mathcal{E}) \cap L^1$ with $\|f\|_1 \le 1$,*

$$\|f\|_2^2 \Lambda(\log \|f\|_2^2) \le \mathcal{E}(f) \tag{1.9}$$

where Λ is defined by (1.5) with $M = M_2$.

(3) *There exists $M_3 : (0,\infty) \to \mathbb{R}$ and constants $c_1, c_2 > 0$ such that, for any $f \in \mathcal{D}(\mathcal{E}) \cap L^1$ with $\|f\|_2^2 = 1$,*

$$c_1 \int_X f^2 \Lambda_+(\log c_2^2 f^2)\, d\mu \le \mathcal{E}(f) \tag{1.10}$$

with $\Lambda_+ = \sup(\Lambda, 0)$ where Λ is defined by (1.5) with $M = M_3$.

The proof is given in Section 2.

Remark 1.2. (1) The super log Sobolev inequality (1.8) implies the Nash type inequality (1.9) with $M = M_2 = M_1$ in the definition (1.5) of Λ.

(2) The Nash type inequality (1.9) implies the Orlicz–Sobolev inequality (1.10) with $M_3 = M_2$, $c_1 = 16^{-1}$, $c_2 = 8^{-1}$.

(3) The Orlicz–Sobolev inequality (1.10) implies the super log Sobolev inequality (1.8) with $M_1(t) = M_3(c_3 t) + c_4$ and $c_3 = c_1$, $c_4 = -\log c_2$.

(4) We do not need to assume that $\mathcal{E}$ is a Dirichlet form for the implications (1.8)⇒(1.9) and (1.10)⇒(1.8) to hold. But the assumption that $\mathcal{E}$ is a Dirichlet form is fundamental for (1.9)⇒(1.10).

(5) Inequalities similar to (1.10) (called the F-Sobolev inequality) have been obtained by F.-Y. Wang under a super Poincaré inequality assumption (see [W1], [W2]). See also the comments at the end of Section 3.2.

As a direct consequence of Theorem 1.1, we can provide an alternative proof of the Nash type inequality (1.6) under an ultracontractivity assumption. Indeed, we just need to apply successively Davies–Simon's result to deduce the super log Sobolev inequality (1.7), and Theorem 1.1 to get exactly the Nash type inequality (1.6). Likewise, we can provide an alternative proof of the super log Sobolev inequality (1.7) (with some loss on β) under an ultracontractivity assumption. Here again, we just need to apply successively Coulhon's result to deduce (1.6), and Theorem 1.1 to get (1.7).

In practice, functional inequalities are used to prove ultracontractivity, and hence to get bounds on the heat kernel of the semigroup, for instance, under a Nash type inequality or a super log Sobolev inequality assumption. Here, we briefly discuss these aspects of the theory by first quoting a result of T. Coulhon which assumes a Nash type inequality. We restrict the statement to the class of submarkovian semigroups. For a more general statement, we refer to [C, Proposition II.1].

Assume that a quadratic form $\mathcal{E}(f) = (Af, f)$ satisfies (1.6) for a continuous function $\Theta : [0,\infty) \to [0,\infty)$ such that $1/\Theta$ is integrable at infinity. Then the semigroup $(T_t)_{t>0}$ is ultracontractive and satisfies

$$\|T_t f\|_\infty \le m(t)\|f\|_1, \qquad 0 < t < t_0, \tag{1.11}$$

where m is the inverse function of $y \mapsto p(y) = \int_y^\infty \frac{dx}{\Theta(x)}$, $y > 0$, and $t_0 = \frac{1}{2}\int_0^\infty \frac{dx}{\Theta(x)} \in (0,\infty]$ (see also [BGL, Theorem 7.4.5] for a variant).

On the other hand, Theorem 2.2.7 and its Corollary 2.2.8 in [D], which assume a super log Sobolev inequality, allow us to deduce ultracontractivity under additional conditions on β in (1.7). This method has been refined by D. Bakry [B] (see also [BGL, Theorem 7.1.2]). We will not go into details of this refinement. Indeed, at the level of functional inequalities Theorem 1.1 says that super log Sobolev inequalities imply (in fact are equivalent to) Nash type inequalities with exactly the same formula (1.6) for Θ as in T. Coulhon's result. So, we will emphasize this first method to get ultracontractive bounds for various classes of behaviour in Section 3. We refer to [B], [BGL], [C], [D],

[DS], [W2] (and references therein) for more details on ultracontractivity. Note that sharp Euclidean ultracontractive bounds (heat kernel bounds) may be deduced from the sharp super log Sobolev inequality in $\mathbb{R}^n$ (see [BCL], [B], and also [BGL, Corollary 7.1.6] for a precise statement).

In Section 3, our goal is to briefly describe some classes of ultracontractivity and the corresponding functional inequalities using Theorem 1.1 and known results such as (1.6), (1.7), (1.11). In particular, we examine two classes of ultracontractivity where ultracontractivity and the corresponding functional inequalities are equivalent. We also exhibit a third class of ultracontractivity where the equivalence between ultracontractivity and the functional inequalities (1.8), (1.9) and (1.10) does not hold. In Section 4, we provide examples for these three particular classes of ultracontractivity. The non-equivalence will be confirmed by the counterexample of E. B. Davies and B. Simon described in Section 4.3. This leads us to discuss open problems in Section 5.

The paper is organized as follows.

In Section 2, we prove our main result, Theorem 1.1. In Section 3, we exhibit the functional inequalities for the polynomial, one-exponential and double-exponential classes of ultracontractivity. In Section 4, we briefly describe some examples belonging to these classes and the Davies–Simon counterexample. In Section 5, we suggest some open problems on the double-exponential class and beyond.

2. Proof of Theorem 1.1

2.1. Super log Sobolev inequality implies Nash type inequality. We prove that (1.8) implies (1.9) with $M_2 = M_1$ using the following convexity argument.

LEMMA 2.1. *If $f \in L^1 \cap L^2$ with $f \geq 0$ and $\|f\|_1 \leq 1$ then*

$$\|f\|_2^2 \log \|f\|_2 \leq \int_X f^2 \log(|f|/\|f\|_2)\, d\mu. \tag{2.1}$$

Proof. First, assume $\|f\|_1 = 1$. We deduce (2.1) by applying Jensen's inequality to the convex function $\Psi(x) = x \log x$ and the probability measure $d\nu = |f| d\mu$. For any $f \in L^1 \cap L^2$ with f not identically zero, by homogeneity we get

$$\|f\|_2^2 \log(\|f\|_2/\|f\|_1) \leq \int_X f^2 \log(|f|/\|f\|_2)\, d\mu, \tag{2.2}$$

or equivalently,

$$\|f\|_2^2 \log \|f\|_2 \leq \int_X f^2 \log(|f|/\|f\|_2)\, d\mu + \|f\|_2^2 \log \|f\|_1. \tag{2.3}$$

If we assume $\|f\|_1 \leq 1$ then $\log \|f\|_1 \leq 0$ and we get (2.1) immediately. ∎

Now, we assume that the super log Sobolev inequality (1.8) holds true. By Lemma 2.1, we deduce that for any function f in $\mathcal{D}(\mathcal{E})$ with $\|f\|_1 \le 1$,

$$\|f\|_2^2 \log \|f\|_2 \le s\mathcal{E}(f) + M_1(s)\|f\|_2^2, \quad s > 0. \tag{2.4}$$

Hence, for any $s > 0$,

$$\|f\|_2^2 \left(\frac{1}{2s} \log \|f\|_2^2 - \frac{1}{s} M_1(s) \right) \le \mathcal{E}(f).$$

Taking the supremum over $s > 0$ yields

$$\|f\|_2^2 \Lambda(\log \|f\|_2^2) \le \mathcal{E}(f)$$

where Λ is defined by (1.5) with $M = M_1$. Thus the Nash type inequality (1.9) and Remark 1.2(1) are proved.

REMARK 2.2. Note that we *do not* use any assumption on the functional $\mathcal{E}$ in this proof. Moreover, the function Λ is automatically finite on the set $\{\log \|f\|_2^2 : f \in \mathcal{D}(\mathcal{E}) \cap L^1,\ \|f\|_1 \le 1\}$.

2.2. Nash type inequality implies Orlicz–Sobolev type inequality. We use the cut-off method developed in [BCLS] to show that the Nash type inequality (1.9) implies the Orlicz–Sobolev type inequality (1.10). The fact that $\mathcal{E}$ is a Dirichlet form is fundamental, due to the next lemma.

LEMMA 2.3. *Let $\mathcal{E}$ be a Dirichlet form with domain $\mathcal{D}(\mathcal{E})$. Let f be any non-negative function in $\mathcal{D}(\mathcal{E})$. For any $\rho > 1$ and $k \in \mathbb{Z}$, set*

$$f_{\rho,k} = (f - \rho^k)^+ \wedge \rho^k(\rho - 1).$$

Then $f_{\rho,k} \in \mathcal{D}(\mathcal{E})$ and

$$\sum_{k\in\mathbb{Z}} \mathcal{E}(f_{\rho,k}) \le \mathcal{E}(f). \tag{2.5}$$

This lemma can be compared with Corollary 2.3 of [BCLS] and Lemma 3.3.2 of [W2]. In what follows, we will write $\mathcal{E}(f)$ or $\mathcal{E}(f,f)$. The starting point is the following important remark ([1]).

REMARK 2.4. Let $\lambda \ge 0$. Let $g \in \mathcal{D}(\mathcal{E})$ with $g \ge 0$. Denote the support of g by $\mathrm{supp}(g) = \{x \in X : g(x) \ne 0\}$. If $h \in \mathcal{D}(\mathcal{E})$ with $0 \le h \le \lambda$ satisfies $h = \lambda$ on $\mathrm{supp}(g)$, then $\mathcal{E}(h,g) \ge 0$ (see [A, (ii) p. 2]).

We prove the remark as follows. For any $\varepsilon > 0$, we have $(h + \varepsilon g) \wedge \lambda = h$. By the Dirichlet property,

$$\mathcal{E}(h) = \mathcal{E}((h + \varepsilon g) \wedge \lambda) \le \mathcal{E}(h + \varepsilon g) = \mathcal{E}(h) + 2\varepsilon\mathcal{E}(h,g) + \varepsilon^2\mathcal{E}(g).$$

Subtracting $\mathcal{E}(h)$ from both sides and dividing by $\varepsilon > 0$ yields

$$0 \le 2\mathcal{E}(h,g) + \varepsilon\mathcal{E}(g).$$

When ε goes to 0, we get $\mathcal{E}(h,g) \ge 0$ as stated.

([1]) The main argument was pointed out to us by G. Allain (see [A, pp. 2, 5]).

Proof of Lemma 2.3. Let $f \in \mathcal{D}(\mathcal{E})$ be non-negative. Fix $\rho > 0$. For any $n \in \mathbb{N}$, we set

$$f_\rho^n = f \wedge \rho^{n+1} = \rho^{n+1}\Big(\Big(\frac{f}{\rho^{n+1}} \wedge 1\Big) \vee 0\Big).$$

By the Dirichlet property, f_ρ^n, $f_\rho^{-(n+1)}$ and $f_{\rho,k}$ belong to $\mathcal{D}(\mathcal{E})$. Indeed, each one of these expressions is a normal contraction of f of the form $\phi(f)$ with $\phi(u) = ((u-a) \vee 0) \wedge b$ where $a, b > 0$ (see [F, p. 5]).

Applying Remark 2.4 to $h = f_\rho^{-(n+1)}$, $\lambda = \rho^{-n}$ and $g = f_{\rho,k}$ with $k = -n, \dots, n$, we deduce $\mathcal{E}(f_{\rho,k}, f_\rho^{-(n+1)}) \geq 0$ for $k = -n, \dots, n$.

Let $(p,k) \in \mathbb{Z}^2$ with $p < k$. Once again, we apply Remark 2.4 with $h = f_{\rho,p}$, $\lambda = \rho^p(\rho - 1)$ and $g = f_{\rho,k}$ to get $\mathcal{E}(f_{\rho,p}, f_{\rho,k}) \geq 0$. From the relation

$$f_\rho^n = \sum_{k=-\infty}^{n} f_{\rho,k} = \sum_{k=-n}^{n} f_{\rho,k} + f_\rho^{-(n+1)}$$

and the Dirichlet property, we obtain

$$\mathcal{E}(f) \geq \mathcal{E}(f_\rho^n) = \mathcal{E}\Big(\sum_{k=-n}^{n} f_{\rho,k}\Big) + 2\sum_{k=-n}^{n} \mathcal{E}(f_{\rho,k}, f_\rho^{-(n+1)}) + \mathcal{E}(f_\rho^{-(n+1)}).$$

Since the last two terms are non-negative, and by developing the third term, we get

$$\mathcal{E}(f) \geq \sum_{k,p=-n}^{n} \mathcal{E}(f_{\rho,p}, f_{\rho,k}) = \sum_{k=-n}^{n} \mathcal{E}(f_{\rho,k}, f_{\rho,k}) + \sum_{k \neq p,\, k,p \geq -n}^{n} \mathcal{E}(f_{\rho,p}, f_{\rho,k}).$$

Since the last term of the right-hand side is non-negative, we arrive at

$$\mathcal{E}(f) \geq \sum_{k=-n}^{n} \mathcal{E}(f_{\rho,k}, f_{\rho,k}).$$

The lemma is proved by letting $n \to \infty$. ∎

We are now in a position to prove the Orlicz–Sobolev inequality (1.10). We divide the proof into two steps.

Step 1. Let $f \in L^2$ be a non-negative function such that $\|f\|_2 = 1$. Let $k \in \mathbb{Z}$. We define $f_k := f_{2,k} = (f - 2^k)^+ \wedge 2^k$. By Hölder's inequality,

$$\|f_k\|_1^2 \leq \|f_k\|_2^2 \mu(f \geq 2^k). \tag{2.6}$$

On the other hand, by Bienaymé–Chebyshev's inequality we have

$$2^{2(k-1)}\mu(f \geq 2^k) \leq \|f_{k-1}\|_2^2 \leq \|f\|_2^2 \leq 1. \tag{2.7}$$

Combining inequalities (2.6) and (2.7), we obtain

$$2^{k-1} \leq \|f_k\|_2 / \|f_k\|_1. \tag{2.8}$$

Again by Bienaymé–Chebyshev's inequality, we have

$$2^{2k}\mu(f \geq 2^{k+1}) \leq \|f_k\|_2^2. \tag{2.9}$$

Step 2. Assume that the Nash type inequality (1.9) holds true. For convenience set $B(x) = \Lambda(\log x^2)$. For a function H defined on $[0,\infty)$, we define its non-negative part H_+ by $H_+(x) = \max(H(x), 0)$, $x \geq 0$. As the quadratic form $\mathcal{E}$ is non-negative, for any $g \in \mathcal{D}(\mathcal{E}) \cap L^1$ with $\|g\|_1 = 1$ we have

$$\|g\|_2^2 \Lambda_+(\log \|g\|_2^2) = \|g\|_2^2 B_+(\|g\|_2) \leq \mathcal{E}(g).$$

By homogeneity, this implies that for any $g \in \mathcal{D}(\mathcal{E}) \cap L^1$ and $g \neq 0$,

$$\|g\|_2^2 \Lambda_+(\log \|g\|_2^2/\|g\|_1^2) = \|g\|_2^2 B_+(\|g\|_2/\|g\|_1) \leq \mathcal{E}(g). \tag{2.10}$$

Let f and $f_k := f_{2,k}$ be as in Lemma 2.3. Applying (2.10) to $g = f_k$ yields

$$\|f_k\|_2^2 B_+(\|f_k\|_2/\|f_k\|_1) \leq \mathcal{E}(f_k).$$

Since B_+ is a non-negative non-decreasing function, from (2.8) and (2.9) we obtain

$$2^{2k} B_+(2^{k-1})\mu(f \geq 2^{k+1}) \leq \mathcal{E}(f_k), \qquad k \in \mathbb{Z}, \tag{2.11}$$

for any non-negative function $f \in \mathcal{D}(\mathcal{E}) \cap L^1$. Let $\lambda > 0$, to be chosen later. We discretize the integral

$$\begin{aligned}\int_X f^2 B_+(\lambda f)\, d\mu &= \sum_{k\in\mathbb{Z}} \int_{\{2^k \leq f < 2^{k+1}\}} f^2 B_+(\lambda f)\, d\mu \\ &\leq \sum_{k\in\mathbb{Z}} 2^{2(k+1)} B_+(\lambda 2^{k+1})\mu(f \geq 2^k) \\ &\leq \sum_{k\in\mathbb{Z}} 2^{2(k+2)} B_+(\lambda 2^{k+2})\mu(f \geq 2^{k+1}).\end{aligned}$$

We choose $\lambda = 2^{-3}$ to get

$$\int_X f^2 B_+(f/8)\, d\mu \leq 2^4 \sum_{k\in\mathbb{Z}} 2^{2k} B_+(2^{k-1})\mu(f \geq 2^{k+1}).$$

Applying (2.11) and Lemma 2.3 with $\rho = 2$ leads to

$$\int_X f^2 B_+(f/8)\, d\mu \leq 2^4 \sum_{k\in\mathbb{Z}} \mathcal{E}(f_k) \leq 2^4 \mathcal{E}(f). \tag{2.12}$$

So, the Orlicz–Sobolev type inequality (1.10) is proved for $f \geq 0$, $\|f\|_2 = 1$ with $c_1 = 1/16$ and $c_2 = 1/8$. For real-valued functions $f \in \mathcal{D}(\mathcal{E})$, we obtain the same conclusion by using $\mathcal{E}(|f|) \leq \mathcal{E}(f)$, which is a consequence of the Dirichlet property. This completes the proof of (1.10) and of Remark 1.2(2).

2.3. Orlicz–Sobolev type inequalities implies super log Sobolev inequalities. In this section, we prove the super log Sobolev inequality (1.8) under the assumption of the Orlicz–Sobolev type inequality (1.10).

From the definition (1.5) of Λ we have, for any $t > 0$ and $y \geq 0$,

$$ty/2 - tM_3(1/t) \leq \Lambda(y) \leq \Lambda_+(y).$$

By the change of variable $t = (c_1 s)^{-1}$ for $s > 0$ with $c_1 > 0$ as in (1.10), this yields

$$y/2 \leq c_1 s \Lambda_+(y) + M_3(c_1 s).$$

Let $f \in \mathcal{D}(\mathcal{E})$. Set $y = \log(c_2^2 f^2)$ with $c_2 > 0$ as in (1.10). Hence

$$\log(c_2|f|) \leq c_1 s \Lambda_+(\log(c_2^2 f^2)) + M_3(c_1 s).$$

Multiplying this expression by f^2 and integrating on X with respect to μ leads to

$$\begin{aligned}\int_X f^2 \log|f|\, d\mu + (\log c_2)\|f\|_2^2 &\leq sc_1 \int_X f^2 \Lambda_+(\log(c_2^2 f^2))d\mu + M_3(c_1 s)\|f\|_2^2 \\ &\leq s\mathcal{E}(f) + M_3(c_1 s)\|f\|_2^2.\end{aligned}$$

The last inequality follows from the assumption (1.10). Thus we easily deduce (1.8) with $M_1(s) = M_3(c_3 s) + c_4, s > 0$, and $c_3 = c_1$, $c_4 = -\log c_2$.

2.4. Remarks and comments

2.4.1. *A remark about the cut-off method.* For the proof of Orlicz–Sobolev type inequalities of Theorem 1.1, we have applied the cut-off method with f_k, i.e. $f_{\rho,k}$ with $\rho = 2$. A similar proof of (1.10) under the assumption of (1.9) can be performed with $f_{\rho,k}$ of Lemma 2.3 for any $\rho > 1$. In that case, we apply successively the circle of implications $\{(1.8) \text{ with } M_1\} \Rightarrow (1.9) \Rightarrow (1.10) \Rightarrow \{(1.8) \text{ with } \tilde{M}_1\}$. For any $\rho > 1$, we have

$$\tilde{M}_1(t) = M_1(t(\rho-1)^2\rho^{-4}) + \log(\rho^3(\rho-1)^{-1}).$$

In terms of equivalence of these functional inequalities, it is of interest to obtain $\tilde{M}_1$ as close to M_1 as possible, up to the additive constant $\log(\rho^3(\rho-1)^{-1})$. Since in applications we can always assume that M_1 is a non-increasing function, we need to optimize the choice of $\rho > 1$ of the expression $M_1(t(\rho-1)^2\rho^{-4})$ for any $t > 0$, i.e. to optimize the function $\rho \mapsto (\rho-1)^2\rho^{-4}$ over $(1,\infty)$. This function attains exactly its supremum at $\rho = 2$. This justifies our choice of $\rho = 2$ for the proof in Section 2.2.

2.4.2. *Comments.* After the submission of this paper, it was brought to our attention that the relationship between the statement (1) of our main Theorem 1.1 for a fixed t and a closed version of (2), namely (2.4) for a fixed t, is implicitly described in Proposition 5.1.8, p. 241, of the very recent book [BGL]. This is proved under the stronger assumptions that the

Dirichlet form $\mathcal{E}$ is given by a carré du champ, satisfies the diffusion property, is ergodic and the measure is invariant for the associated semigroup. These conditions are imposed by the use of [BGL, Proposition 3.1.17]. Note that, unlike Proposition 3.1.17 of [BGL], our Lemma 2.3 is valid for *all* Dirichlet forms without additional assumptions.

We also refer the reader to [BGL] for an introduction to the subject of functional inequalities treated here (see, in particular, Chapters 5 and 7) and also for related topics, in particular, for the measure-capacity formulations of Nash type inequalities which are not considered in this paper (see [BGL, Chap. 8]). Similarly, one can consult the book by F.-Y. Wang [W2].

3. Examples of classes of ultracontractivity. In this section, we consider three special classes of ultracontractivity. For the first two classes, we state the equivalence between the specific ultracontractivity bound and the corresponding functional inequalities. For the third class, ultracontractivity is a stronger property than the corresponding functional inequalities.

3.1. Polynomial class of ultracontractivity. We say that a semigroup $(T_t)_{t>0}$ with generator A belongs to the *polynomial class of ultracontractivity* of order $\nu > 0$ if $c(t) = c_1 t^{-\nu/4}$ in (1.3) with $c_1 > 0$. This case is most common in applications: see for instance [N], [CKS], [VSC], [G], [BGL]. We recall that $\mathcal{E}(f) = (\sqrt{A}\, f, \sqrt{A}\, f)$ with $f \in \mathcal{D}(\sqrt{A})$. The following statements are equivalent:

(1) The semigroup $(T_t)_{t>0}$ belongs to the polynomial class of ultracontractivity of order ν.

(2) The generator A satisfies the Nash inequality
$$\|f\|_2^{2+4/\nu} \le c_2 \mathcal{E}(f) \|f\|_1^{4/\nu}.$$

(3) The generator A satisfies the super log Sobolev inequality
$$\int_X f^2 \log(|f|/\|f\|_2)\, d\mu \le t\mathcal{E}(f) + \log(c_3 t^{-\nu/4}) \|f\|_2^2, \qquad t > 0.$$

(4) The generator A satisfies the Orlicz–Sobolev type inequality
$$\int_X f^{2+4/\nu}\, d\mu \le c_4 \mathcal{E}(f) \|f\|_2^{4/\nu}.$$

(5) The generator A satisfies the Sobolev inequality ($\nu > 2$)
$$\|f\|_{2\nu/(\nu-2)}^2 \le c_5 \mathcal{E}(f).$$

(6) The generator A satisfies the super Poincaré inequality
$$\|f\|_2^2 \le t\mathcal{E}(f) + c_6 t^{-\nu/2} \|f\|_1^2, \quad t > 0,$$

for some positive constants c_i, $i = 1, \dots, 6$.

The equivalences between polynomial ultracontractivity, a Nash inequality, a super log Sobolev inequality and an Orlicz–Sobolev type inequality are deduced from (1.6), (1.7), (1.11) and Theorem 1.1. A Nash inequality is a super Poincaré inequality optimized over $t > 0$. Historically, the first proof of equivalence between ultracontractivity and a Sobolev inequality can be found in [V] and the one between ultracontractivity and a Nash inequality (and implicitly a super Poincaré inequality) in [CKS] inspired by [N]. A direct proof of the fact that a Nash inequality implies a Sobolev inequality using the cut-off method can be found in [BCLS]. The Orlicz–Sobolev inequality (4) is also called *Moser's inequality.*

3.2. One-exponential class of ultracontractivity. We say that a semigroup $(T_t)_{t>0}$ with generator A belongs to the *one-exponential class of ultracontractivity* of order $\alpha > 0$ if $c(t) = \exp(c_1 t^{-\alpha})$ in (1.3) with $c_1 > 0$. The following statements are equivalent:

(1) The semigroup $(T_t)_{t>0}$ belongs to the one-exponential class of ultracontractivity of order α.
(2) The generator A satisfies the Nash type inequality (1.9) with

$$\Lambda(\log x) = c_2[\log_+(c_3 x)]^{1+1/\alpha}. \tag{3.1}$$

(3) The generator A satisfies the super log Sobolev inequality (1.7) with $\beta(t) = c_4 t^{-\alpha}$.
(4) The generator A satisfies the Orlicz–Sobolev type inequality (1.10) with Λ as in (3.1).

Here c_i are positive constants for $i = 1, \ldots, 4$.

The equivalences between one-exponential ultracontractivity, a Nash type inequality, a super log Sobolev inequality and a Orlicz–Sobolev type inequality can be deduced from (1.6), (1.7), (1.11) and Theorem 1.1.

These results apply to the family of examples of Section 4.1 described below.

Comments. First, we notice the following fact. The usual semigroup proof of a super Poincaré inequality under the ultracontractivity assumption

$$\|T_t f\|_2 \le c(t)\|f\|_1, \qquad t > 0,$$

is as follows. Let $f \in \mathcal{D}(\mathcal{E}) \cap L^1$ and $t > 0$. By symmetry and semigroup properties, we obtain

$$\begin{aligned}\|f\|_2^2 - \|T_{t/2} f\|_2^2 = (f - T_t f, f) &= \int_0^t (A T_{s/2} f, T_{s/2} f)\, ds \\ &\le \int_0^t (Af, f)\, ds = t\mathcal{E}(f)\end{aligned}$$

because $s \mapsto (AT_{s/2}f, T_{s/2}f)$ is non-increasing for all $f \in \mathcal{D}(\mathcal{E})$. Hence, the following super Poincaré inequality is satisfied:

$$\|f\|_2^2 \le t\mathcal{E}(f) + \gamma(t)\|f\|_1^2, \quad t > 0, \tag{3.2}$$

with $\gamma(t) = c^2(t/2)$. In the case where $c(t) = \exp[(c/2)(1 + (2t)^{-1/\delta})]$ for some $\delta > 0$, we get $\gamma(t) = \exp[c(1 + t^{-1/\delta})]$. By [W1, Corollary 3.3], this implies the so-called *F-Sobolev inequality* of index δ,

$$\int_X f^2[\log(1 + f^2)]^\delta \, d\mu \le c_1\mathcal{E}(f) + c_2\|f\|_2^2 \tag{3.3}$$

for some constants $c_1, c_2 > 0$.

But below we follow another route and deduce an improved F-Sobolev inequality under the same assumption of ultracontractivity by using Theorem 1.1. In particular, this will show that (3.2) and (3.3) are not necessarily sharp for the Dirichlet form $\mathcal{E}$, despite the fact that they are equivalent in a general framework [W1, Corollary 3.3]. Indeed, in Section 1 we have seen that the super log Sobolev inequality

$$\int_X f^2 \log|f| \, d\mu \le t\mathcal{E}(f) + (\log c(t))\|f\|_2^2 + \|f\|_2^2 \log\|f\|_2, \quad t > 0, \tag{3.4}$$

holds true by using [D, Theorem 2.2.3]. Now by applying Theorem 1.1, we deduce the following F-Sobolev inequality of index $\tilde{\delta} = \delta + 1$:

$$\int_X f^2[\log(1 + f^2)]^{\delta+1} \, d\mu \le c_1\mathcal{E}(f) + c_2\|f\|_2^2. \tag{3.5}$$

More precisely, we obtain (3.5) from the Orlicz–Sobolev type inequality associated to (3.1) with $\alpha = 1/\delta$ and by adding the term $c_2\|f\|_2^2$. Since we are considering the one-exponential class of order $1/\delta$, the index $\tilde{\delta} = 1 + \delta$ in (3.5) is now sharp by the results described before these comments. Moreover, by applying [W1, Corollary 3.3] again, we obtain an improved super Poincaré inequality (3.2) for small t with rate function

$$\tilde{\gamma}(t) = \exp[\tilde{c}(1 + t^{-1/\tilde{\delta}})], \quad \tilde{\delta} = 1 + \delta, \quad t > 0,$$

in place of $\gamma(t)$ since $\lim_{t\to 0} \tilde{\gamma}(t)/\gamma(t) = 0$.

The discussion above reveals some weakness of what we have called the "usual" semigroup proof of (3.2). But note that the phenomenon described above does not occur for the polynomial class studied in Section 3.1.

3.3. Double-exponential class of ultracontractivity. We say that a semigroup $(T_t)_{t>0}$ with generator A belongs to the *double-exponential class of ultracontractivity* of order $\gamma > 0$ if $c(t) = \exp_2(c_1 t^{-\gamma})$ in (1.3) where $\exp_2 = \exp \circ \exp$ and $c_1 > 0$. For this class, the situation is quite different. Ultracontractivity is now strictly stronger than the other functional inequalities introduced in this paper. More precisely, if the semigroup $(T_t)_{t>0}$ belongs to the double-exponential class of order $\gamma > 0$ then:

(1) The generator A satisfies the Nash type inequality (1.9) with

$$\Lambda(\log x) = k_1 \log_+(k_2 x)[(\log_+)_2(k_2 x)]^{1/\gamma}. \tag{3.6}$$

(2) The generator A satisfies the super log Sobolev inequality (1.7) with $\beta(t) = \exp(k_0 t^{-\gamma})$.

(3) The generator A satisfies the Orlicz–Sobolev type inequality (1.10) with Λ as in (3.6).

Here k_i are positive constants for $i = 0, 1, 2$.

Ultracontractivity implies a Nash type inequality by (1.6) and a super log Sobolev inequality by (1.7). Theorem 1.1 says that an Orlicz–Sobolev type inequality is equivalent to a Nash type inequality and a super log Sobolev inequality. The use of (1.11) fails for the converse. We postpone the discussion of the non-equivalence of one of these functional inequalities with ultracontractivity to Section 5. Note that these functional inequalities are equivalent to each other by Theorem 1.1, independently of the ultracontractivity assumption.

These results apply to the family of examples of Section 4.2 described below.

4. Examples of ultracontractive semigroups. In this section, we briefly describe examples of semigroups belonging to the one-exponential and double-exponential classes of ultracontractivity for which the results of Section 3 apply. The polynomial class of ultracontractivity is classical and many examples of operators can be found in the literature. So, we will not provide a detailed account of this class but just indicate some examples. The first is the Laplacian on $\mathbb{R}^n$ and examples elaborated on this model (see [N]). Sub-Laplacians on Lie groups of polynomial growth also provide many examples (see [VSC] and references therein), as well as Laplace–Beltrami operators on some Riemannian manifolds (see [G, p. 368]), and Laplacians on fractals (see e.g. [K]). The list above is not exhaustive.

Here, we focus on examples in the one- and double-exponential classes. The examples are taken from [B] and concern convolution semigroups on the infinite-dimensional torus $\mathbb{T}^\infty$. Other examples can be found for instance in [BCS, Section 8]. Note that the study of the convolution of distributions of probability measures on topological groups is an old and vast subject and goes back at least to [ST]. See also the selected open problems gathered in the recent paper [S] and references therein.

Let $X = \mathbb{T}^\infty$ be the product of countably many copies of the torus $\mathbb{T}$ with its ordinary product structure. The group $\mathbb{T}^\infty$ is an abelian compact group. We denote by 0 the neutral element and by μ the normalized Haar measure of the group. This measure is the countable product of the normalized Haar measure on $\mathbb{T}$.

Let μ_t be the Brownian semigroup on $\mathbb{T}$. To a sequence $\mathcal{A} = \{a_k\}_{k=1}^{\infty}$ of positive numbers, we associate the product measure $\mu_t^{\mathcal{A}}$ on $\mathbb{T}^\infty$ defined by

$$\mu_t^{\mathcal{A}} = \otimes_{k=1}^{\infty} \mu_{a_k t}, \qquad t > 0.$$

The family $(\mu_t^{\mathcal{A}})_{t>0}$ of probability measures defines a symmetric convolution semigroup on $\mathbb{T}^\infty$ denoted by $(T_t^{\mathcal{A}})_{t>0}$. Ultracontractivity reads

$$\|T_t^{\mathcal{A}}\|_{1\to\infty} = \mu_t^{\mathcal{A}}(0) \in (0, \infty]$$

where $\mu_t^{\mathcal{A}}(0)$ denotes the density of $\mu_t^{\mathcal{A}}$ (when it exists) evaluated at 0. The infinitesimal generator A of $(T_t^{\mathcal{A}})_{t>0}$ acts on cylindrical functions, i.e. functions depending on a finite number of variables, as

$$Af = \sum_{k=1}^{\infty} a_k \frac{\partial^2 f}{\partial x_k^2}.$$

The associated counting function $\mathcal{N}^{\mathcal{A}}$ defined by

$$\mathcal{N}^{\mathcal{A}}(x) = \sharp\{k \geq 1 : a_k \leq x\}, \qquad x > 0,$$

is of fundamental importance for the study of ultracontractivity, as can be seen from the following examples.

4.1. Examples in the one-exponential class. Let $\alpha > 0$. If the sequence $(a_k)_{k\geq 1}$ is chosen such that $\mathcal{N}^{\mathcal{A}}(x) \sim x^\alpha$ as $x \to \infty$ then

$$\log \mu_t^{\mathcal{A}}(0) \sim k(\alpha) t^{-\alpha} \qquad \text{as } t \searrow 0 \tag{4.1}$$

(see [B, Theorem 3.18]). Hence, the semigroup $(T_t^{\mathcal{A}})_{t>0}$ belongs to the one-exponential class of ultracontractivity, and the results of Section 3.2 hold true for such families of $(a_k)_{k\geq 1}$. For example, one can take $a_k = k^{1/\alpha}$.

4.2. Examples in the double-exponential class. Let $\gamma > 0$. If the sequence $(a_k)_{k\geq 1}$ is chosen such that

$$\log \mathcal{N}^{\mathcal{A}}(x) \sim x^{\gamma/(\gamma+1)} \qquad \text{as } x \to \infty,$$

then

$$\log\log \mu_t^{\mathcal{A}}(0) \sim c(\gamma) t^{-\gamma} \qquad \text{as } t \searrow 0 \tag{4.2}$$

(see [B, Theorem 3.27]). Hence, the semigroup $(T_t^{\mathcal{A}})_{t>0}$ belongs to the double-exponential class of ultracontractivity and the results of Section 3.3 hold true for such families of $(a_k)_{k\geq 1}$. For example, one can take $a_k = [\log(k+2)]^\delta$ with $\delta = (\gamma+1)/\gamma$.

4.3. A borderline case of the double-exponential class: Davies–Simon's counterexample. In this section, we show that ultracontractivity and a super log Sobolev inequality are not equivalent properties in the double-exponential class. For that purpose, we describe Davies–Simon's counterexample, i.e. the generator of a submarkovian semigroup satisfying the super log Sobolev inequality (1.7) with $\beta(t) = \exp(c/t)$ but not ultracontractive (see [DS, Theorem 6.1(b) and Remark 1, p. 359]). Later on, a more detailed study was provided for a family of examples including this one in [KKR], [BCL], [BGL, Sect. 7.3]. See also the comments after Proposition 7.3.1 of [BGL] on the examples treated by this proposition concerning the one- and double-exponential classes.

Let A be the operator $Af = \Delta f + \nabla U.\nabla f$ defined on smooth functions f on the real line $\mathbb{R}$ with $\Delta = -d^2/dx^2$ and let $U : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Let μ be the invariant measure $d\mu(x) = e^{-U(x)}\,dx$ where dx denotes the Lebesgue measure on $\mathbb{R}$. The Dirichlet form associated to A is given by $\mathcal{E}(f) = \int_{\mathbb{R}} |\nabla f|^2\,d\mu$. The expected counterexample corresponds to the choice $U(x) = (1+x^2)\log(1+x^2)$. In the following, we denote by $\|\cdot\|_2$ the L^2-norm with respect to the measure μ.

THEOREM 4.1. *Let $(T_t)_{t>0}$ be the semigroup associated with the infinitesimal generator A defined above. Then:*

(1) *The following log Sobolev inequality holds true: for any $f \in \mathcal{D}(\mathcal{E})$,*

$$\int_{\mathbb{R}} f^2 \log\left(\frac{f^2}{\|f\|_2^2}\right) d\mu \le t\mathcal{E}(f) + H(t)\|f\|_2^2, \qquad t > 0, \tag{4.3}$$

where $H(t)$ is such that there are constants $c_3, c_4, c_3', c_4' > 0$, with

$$c_3 e^{c_4 t^{-1}} \le H(t) \le c_3' e^{c_4' t^{-1}} \tag{4.4}$$

for t small enough.

(2) *The Nash type inequality* (1.9) *and the Orlicz–Sobolev inequality* (1.10) *hold true with Λ given by* (3.6) *where $\gamma = 1$.*

(3) *The semigroup $(T_t)_{t>0}$ is not ultracontractive.*

Gathering the arguments of the proof given in [DS] is rather difficult. Therefore we propose a direct but different proof of the super log Sobolev inequality following [BCL]. Here, we do not pretend to novelty. A simple proof of non-ultracontractivity (using the subsolution method for instance) is known and well detailed in [KKR, Example 5.3]. Statement (2) is obtained from (1) and Theorem 1.1.

Proof of Theorem 4.1. Here we only prove the first statement. We divide the proof into two steps. In the first step, we prove the super log Sobolev inequality (4.3) with

$$H(t) = -\frac{1}{2}\log(\pi e^2 t) + \sup_{x\in\mathbb{R}} V_t(x) \tag{4.5}$$

where $V_t(x) = -t\big[-\frac{1}{2}U''(x) + \frac{1}{4}(U'(x))^2\big] + U(x) = -tV(x) + U(x)$. In the second step, we provide the estimates (4.4) of $H(t)$.

STEP 1. We follow the arguments of Proposition 3.1 of [BCL] (see also [BGL, Prop. 7.3.1]). We start from Gross' inequality which reads

$$\int_{\mathbb{R}} v^2 \log\left(\frac{v^2}{\|v\|^2_{L^2(d\gamma)}}\right) d\gamma \leq 2\int_{\mathbb{R}} |\nabla v|^2\, d\gamma$$

where v is a smooth function with compact support and $d\gamma(x) = (2\pi)^{-1/2}e^{-x^2/2}\,dx$ is the Gaussian measure. We set $G(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $g = v\sqrt{G}$. By integration by parts, we obtain

$$\int_{\mathbb{R}} g^2 \log\left(\frac{g^2}{\|g\|^2_{L^2(dx)}}\right) dx \leq 2\int_{\mathbb{R}} |\nabla g|^2\, dx - \frac{1}{2}\log(2\pi e^2)\int_{\mathbb{R}} |g|^2\, dx.$$

Let h be a smooth function with compact support on $\mathbb{R}$. We set $g(x) = h(x\sqrt{2^{-1}t})$ with $t > 0$. The previous inequality becomes

$$\int_{\mathbb{R}} h^2 \log\left(\frac{h^2}{\|h\|^2_{L^2(dx)}}\right) dx \leq t\int_{\mathbb{R}} |\nabla h|^2\, dx - \frac{1}{2}\log(\pi e^2 t)\int_{\mathbb{R}} |h|^2\, dx.$$

Now, let f be a smooth function with compact support on $\mathbb{R}$. Choose $h = fe^{-U/2}$ in the preceding inequality. Again by integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}} f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(d\mu)}}\right) d\mu \leq t\int_{\mathbb{R}} |\nabla f|^2\, d\mu - \frac{1}{2}\log(\pi e^2 t)\|f\|^2_{L^2(d\mu)} \\ + \int_{\mathbb{R}}\left(-t\left[-\frac{1}{2}U''(x) + \frac{1}{4}(U'(x))^2\right] + U(x)\right) f^2\, d\mu.\end{aligned}$$

This immediately implies (4.3) with H given by (4.5).

STEP 2. *Upper bound on $H(t)$.* Let $0 < t < 1$. To obtain the upper bound (4.4) on $H(t)$, it is enough to find a similar upper bound on V_t where $V_t(x)$ is given by

$$V_t(x) = t\big[-x^2\log^2(e(1+x^2)) + \log(e(1+x^2))\big] + \frac{2tx^2}{1+x^2} + (1+x^2)\log(1+x^2).$$

The function $V_t(x)$ is clearly uniformly bounded by a constant on the set $\{x \in \mathbb{R} : |x| \leq 1\}$ when $0 < t < 1$. By symmetry, it suffices to bound the supremum of $V_t(x)$ for $x \in [1, \infty)$. For any $\delta > 0$ and $x \geq 1$, it is easily shown that

$$\delta e \log^2(e(1+x^2)) + \log(e(1+x^2)) \leq \left(\delta e + \frac{1}{\log(2e)}\right) x^2 \log^2(e(1+x^2)).$$

Now, choose δ such that $0 < \delta \le \frac{1}{2e} \frac{\log 2}{1+\log 2}$. This implies $\delta e + \frac{1}{1+\log 2} \le 1 - \delta e$ and

$$\delta e \log^2(e(1+x^2)) + \log(e(1+x^2)) \le (1-\delta e)x^2 \log^2(e(1+x^2)).$$

Then we deduce

$$-x^2 \log^2(e(1+x^2)) + \log(e(1+x^2)) \le \delta[-e(1+x^2)\log^2(e(1+x^2))]$$

for all $x \ge 1$. This yields

$$V_t(x) \le t\delta[-e(1+x^2)\log^2(e(1+x^2))] + 2t + e(1+x^2)\log(e(1+x^2)).$$

Define

$$W_s(y) := -sy\log^2 y + y\log y + 2t \qquad \text{with } y > 1.$$

The function W_s attains its supremum at $y_0 = \exp((1-2s+\sqrt{1+4s^2})(2s)^{-1})$. Since $V_t(x) \le W_s(y)$ for any x and y such that $y = e(1+x^2)$ and $s = t\delta$, we deduce that

$$\sup_{x\ge 1} V_t(x) \le W_s(y_0) \le \frac{e^{-1}}{2s+\sqrt{1+4s^2}} \exp\left(\frac{1}{2s} + \frac{\sqrt{1+4s^2}}{2s}\right) + 2t.$$

Thus for any $0 < t < \frac{\sqrt{3}}{2\delta}$, we have

$$\sup_{x\ge 1} V_t(x) \le \frac{e^{-1}}{2\delta t} \exp\left(\frac{3}{2\delta t}\right) + \frac{\sqrt{3}}{\delta}.$$

From this inequality, we deduce the upper bound (4.4) on $H(t)$ for t small enough.

Lower bound on H. Let $t > 0$ and $x_0 > 0$ be such that $\log(e(1+x_0^2)) = (2t)^{-1}$. Thus for any $0 < t < 1/8$,

$$\sup_{x\in\mathbb{R}} V_t(x) \ge V_t(x_0) = \left(\frac{1}{4t} - 1\right)e^{\frac{1}{2t}-1} + \frac{1}{4t} + \frac{1}{2} + 2t(1 - e^{1-\frac{1}{2t}}).$$

Thus

$$\sup_{x\in\mathbb{R}} V_t(x) \ge \left(\frac{1}{4t} - 1\right)e^{\frac{1}{2t}-1} \ge \frac{1}{8t}e^{\frac{1}{2t}-1} \ge e^{-1}e^{\frac{1}{2t}}.$$

This proves the lower bound (4.4) on $H(t)$ and completes the proof of the first statement of Theorem 4.1. ∎

5. Open problems and concluding remarks. In this section, we address questions about equivalence between ultracontractivity and the functional inequalities introduced in this paper for the double-exponential class and beyond this class. These problems deserve to be studied due to the existence of many different ultracontractivity behaviours (see for instance [BCS, Sections 6 and 8]).

Here is a list of questions and open problems.

Equivalence theorems.

(i) To the authors' knowledge, the characterization of the largest class of functions Θ in the Nash type inequality or Λ in the Orlicz–Sobolev inequality or β in the super log Sobolev inequality for a generator to be equivalent to ultracontractivity of the semigroup remains an open problem.

(ii) It would be of interest to describe new stable classes of ultracontractivity as in Sections 3.1 and 3.2, in view of (i).

Sharpness of known theorems.

(iii) By (1.6), ultracontractivity for a semigroup with $c(t) = \exp_2(kt^{-\gamma})$ in (1.3) implies a Nash type inequality for the generator with Θ of the form

$$\Theta(x) \simeq x(\log x)(\log\log x)^{1/\gamma} \quad \text{for } x \text{ large enough.}$$

It would be interesting to know whether there are (Dirichlet) operators such that both ultracontractivity and a Nash type inequality are sharp with $c(t)$ and $\Theta(x)$ as above, for all or some $\gamma > 0$. If the answer is positive, this would show that (1.6) is sharp for the double-exponential class.

(iv) In the opposite direction, if an operator satisfies a Nash type inequality with $\Theta(x) \simeq x(\log x)(\log\log x)^{1/\gamma}$ for x large enough then by (1.11) the semigroup is ultracontractive with $c(t) = \exp_2(kt^{-\alpha})$ and the *defective exponent* $\alpha = \gamma/(1-\gamma)$ when $\gamma \in (0,1)$. It would be interesting to know whether there are (Dirichlet) operators such that both ultracontractivity and a Nash type inequality are sharp with $\Theta(x)$ and $c(t)$ as above for some or all $\gamma \in (0,1)$. If the answer is positive, this would show that (1.11) is sharp for the double-exponential class.

(v) Also in another direction, if an operator satisfies a stronger Nash type inequality with $\Theta(x) \simeq x(\log x)(\log\log x)^{1/\gamma+1}$ for x large enough then by (1.11) the semigroup is ultracontractive with $c(t) = \exp_2(kt^{-\gamma})$. A similar question to the one of (iv) arises for some or all $\gamma > 0$. If the answer is positive, this would show that (1.11) is sharp but (1.6) is not.

Concluding remarks. At present, we are not able to conjecture a general equivalence theorem. In this regard, it would be interesting to compare Theorem 2.2.7 of [D] and Coulhon's result (1.11) in light of our Theorem 1.1 (see also [BGL, Theorems 7.1.2 and 7.4.5]). Indeed, such a relation is not clear despite the fact that ultracontractivity (1.11) appears to be more direct for the double-exponential class. In that direction, the contribution of Theorem 1.1 is important since it describes the exact correspondence between super log Sobolev inequalities and Nash type inequalities. As a consequence, it should be possible to compare the ultracontractive bounds obtained by both methods discussed above. The authors of this paper do not know if such a comparison is really possible due to the fact that Theorem 7.1.2 of

[BGL] treats also hypercontractivity and Theorem 7.4.5 of [BGL] apparently does not. A restricted open problem is to find what general conditions on β in the super log Sobolev inequality (1.7) and Λ in the Nash type inequality (1.6) related by (1.5) lead to comparable ultracontractive bounds by applying both methods.

To conclude, we conjecture that the super log Sobolev profile H_0 defined by

$$H_0(t) = \sup\Big\{ \int_X f^2 \log f^2 \, d\mu - t\mathcal{E}(f) : f \in \mathcal{D}(\mathcal{E}), \|f\|_2^2 = 1 \Big\}, \qquad t > 0, \tag{5.1}$$

of Davies–Simon's counterexample satisfies the same lower estimate as H in (4.4). Note that the upper bound holds trivially by minimality of H_0.

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