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UPPER BOUNDS FOR THE COHOMOLOGICAL DIMENSIONS OF FINITELY GENERATED MODULES OVER A COMMUTATIVE NOETHERIAN RING

BY

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Abstract. Let R be a commutative Noetherian ring, I a proper ideal of R, and M be a finitely generated R-module. We provide bounds for the cohomological dimension of the R-module M with respect to the ideal I in several cases.

1. Introduction. Throughout, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. The notions of the cohomological dimension and the arithmetic rank of algebraic varieties have produced some interesting results and problems in local algebra. The local cohomology modules $H^i_I(M)$, $i = 0, 1, 2, \ldots$, of an R-module M with respect to I were introduced by Grothendieck [Ha1]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some powers of I, i.e., $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism

$$H^i_I(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [Ha1] or [BS] for more details about local cohomology.

For an R-module M, the *cohomological dimension of M with respect to I* is defined as

$$\operatorname{cd}(I, M) := \max\{i \in \mathbb{Z} : H^i_I(M) \neq 0\}.$$

The cohomological dimension has been studied by several authors; see, for example, Faltings [F], Hartshorne [Ha2], Huneke–Lyubeznik [HL], Divaani-Aazar, Naghipour and Tousi [DNT], Hellus [He], Hellus–Stückrad [HS] and Mehrvarz, Bahmanpour and Naghipour [MBN].

Our aim in this paper is to provide some bounds for the cohomological dimensions of finitely generated R-modules over Noetherian rings.

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Throughout this paper, for any R-module N, we use the notation $E_R(N)$ for the injective envelope of the R-module N.

2. The results. The following two lemmata will be useful in this section.

LEMMA 2.1. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of a Noetherian ring R (not necessarily local) and M be an R-module such that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M)) = 0$ for all i and j (respectively, for $i \leq n$ and all j). Then $\operatorname{Ext}^i_R(R/\mathfrak{b}, M) = 0$ for all i (respectively, for all $i \leq n$).*

Proof. The case $n = 0$ is clear, so let $n > 0$ and argue by induction on n. We first reduce to the case $\Gamma_{\mathfrak{a}}(M) = 0$. This is possible, since if we let $\overline{M} = M/\Gamma_{\mathfrak{a}}(M)$, we have the long exact sequence

$$\begin{aligned} \cdots \to \operatorname{Ext}^{i-1}_R(R/\mathfrak{b}, \overline{M}) \\ \to \operatorname{Ext}^i_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^i_R(R/\mathfrak{b}, M) \to \operatorname{Ext}^i_R(R/\mathfrak{b}, \overline{M}) \\ \to \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)) \to \cdots, \end{aligned}$$

and the isomorphisms $H^0_{\mathfrak{a}}(\overline{M}) = 0$ and $H^i_{\mathfrak{a}}(\overline{M}) \cong H^i_{\mathfrak{a}}(M)$ for each $i \geq 1$. So let us assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be an injective hull of M and set $L = E/M$. Then also $\Gamma_{\mathfrak{a}}(E) = 0$ and $\operatorname{Hom}_R(R/\mathfrak{b}, E) = 0$, and therefore we get isomorphisms $H^i_{\mathfrak{a}}(L) \cong H^{i+1}_{\mathfrak{a}}(M)$ and $\operatorname{Ext}^i_R(R/\mathfrak{b}, L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{b}, M)$ for all $i \geq 0$. Now the assertion follows easily by applying the inductive hypothesis to the R-module L. ∎

LEMMA 2.2. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of a Noetherian ring R (not necessarily local) and M be an R-module such that $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M)) = 0$ for all i and j. Then $H^i_{\mathfrak{b}}(M) = 0$ for all i.*

Proof. Since $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M)) = 0$ for all i and j, [Me, Proposition 3.9] yields $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M)) = 0$ for all i and j. Hence by Lemma 2.1, $\operatorname{Ext}^i_R(R/\mathfrak{b}, M) = 0$ for all i. But, in this case, it follows from the method of the proof of [K, Lemma 1] that $\operatorname{Ext}^i_R(R/\mathfrak{b}^n, M) = 0$ for all i and j and $n \in \mathbb{N}$. Because $\operatorname{Supp}(R/\mathfrak{b}^n) \subseteq \operatorname{Supp}(R/\mathfrak{b})$ for each $n \in \mathbb{N}$ it follows from the definition of local cohomology modules that $H^i_{\mathfrak{b}}(M) = 0$ for each $i \geq 0$, as required. ∎

COROLLARY 2.3. *If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals of a Noetherian ring R (not necessarily local) and M is an R-module such that $\operatorname{cd}(\mathfrak{b}, M) \geq 0$, then*

$$\operatorname{cd}\Big(\mathfrak{b}, \bigoplus_{i=0}^{\operatorname{cd}(\mathfrak{a},M)} H^i_{\mathfrak{a}}(M)\Big) = \sup\{\operatorname{cd}(\mathfrak{b}, H^i_{\mathfrak{a}}(M)) : i \in \mathbb{N}_0\} \geq 0.$$

In particular, $\operatorname{cd}(\mathfrak{a}, M) \geq 0$.

Proof. The assertion is clear by Lemma 2.2. ∎

The following theorem is one of the main results of this section.

THEOREM 2.4. *Let R be a Noetherian ring (not necessarily local) and $I \subseteq J$ be ideals of R. Then, for any R-module $M \neq 0$ with* $\operatorname{cd}(J, M) \geq 0$, *we have*

$$\begin{aligned}\operatorname{cd}(J, M) &\leq \operatorname{cd}(I, M) + \operatorname{cd}\Big(J, \bigoplus_{i=0}^{\operatorname{cd}(I,M)} H_I^i(M)\Big) \\ &= \operatorname{cd}(I, M) + \sup\{\operatorname{cd}(J, H_I^i(M)) : i \in \mathbb{N}_0\}.\end{aligned}$$

Proof. Note that by Corollary 2.3 we have

$$\sup\{\operatorname{cd}(J, H_I^i(M)) : i \in \mathbb{N}_0\} \geq 0 \quad \text{and} \quad \operatorname{cd}(I, M) \geq 0.$$

Now, we use induction on $t := \operatorname{cd}(I, M)$. In the case $t = 0$ we have $H_I^i(M/\Gamma_I(M)) = 0$ for each $i \geq 0$, and hence by [Me, Proposition 3.9], we have $\operatorname{Ext}_R^i(R/I, M/\Gamma_I(M)) = 0$ for each $i \geq 0$. But in this case, as $\operatorname{Supp}(R/J^n) \subseteq \operatorname{Supp}(R/I)$ for each $n \in \mathbb{N}$, it follows that

$$\operatorname{Ext}_R^i(R/J^n, M/\Gamma_I(M)) = 0$$

for each $i \geq 0$ and $n \in \mathbb{N}$. Therefore it follows from the definition of local cohomology modules that $H_J^i(M/\Gamma_I(M)) = 0$ for each $i \geq 0$. Hence the exact sequence

$$0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$$

implies that $\operatorname{cd}(J, M) = \operatorname{cd}(J, \Gamma_I(M)) \leq \operatorname{cd}(J, \bigoplus_{i=0}^{\operatorname{cd}(I,M)} H_I^i(M))$, as required.

Now, let $t > 0$ and suppose the result holds for $t - 1$. Then it follows from the exact sequence

$$0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$$

that

$$\operatorname{cd}(J, M) \leq \sup\{\operatorname{cd}(J, \Gamma_I(M)), \operatorname{cd}(J, M/\Gamma_I(M))\}.$$

Now if $\operatorname{cd}(J, M) \leq t + \operatorname{cd}(J, \Gamma_I(M))$, then there is nothing to prove. Therefore we may assume that $\operatorname{cd}(J, M) > t + \operatorname{cd}(J, \Gamma_I(M))$. Then we have

$$\sup\{\operatorname{cd}(J, \Gamma_I(M)), \operatorname{cd}(J, M/\Gamma_I(M))\} = \operatorname{cd}(J, M/\Gamma_I(M)),$$

and hence $\operatorname{cd}(J, M) \leq \operatorname{cd}(J, M/\Gamma_I(M))$. Let $N := M/\Gamma_I(M)$. Then from the exact sequence

$$0 \to N \to E_R(N) \to E_R(N)/N \to 0$$

it follows that $\operatorname{cd}(I, E_R(N)/N) = t - 1$, and hence from inductive hypothesis,

$$\operatorname{cd}(J, E_R(N)/N) \leq t - 1 + \sup\{\operatorname{cd}(J, H_I^i(E_R(N)/N)) : i \in \mathbb{N}_0\},$$

thus

$$\operatorname{cd}(J, E_R(N)/N) \leq t - 1 + \sup\{\operatorname{cd}(J, H_I^i(N)) : i \in \mathbb{N}\},$$

and therefore

$$\operatorname{cd}(J, E_R(N)/N) \leq t - 1 + \sup\{\operatorname{cd}(J, H_I^i(M)) : i \in \mathbb{N}\}.$$

But $\operatorname{cd}(J, E_R(N)/N) = \operatorname{cd}(J, N) - 1$ implies

$$\operatorname{cd}(J, M) \leq \operatorname{cd}(J, M/\Gamma_I(M)) \leq \operatorname{cd}(I, M) + \sup\{\operatorname{cd}(J, H_I^i(M)) : i \in \mathbb{N}\}.$$

This completes the inductive step and the proof. ∎

We are now ready to state and prove the second main result of this paper.

THEOREM 2.5. *If R is a Noetherian ring* (*not necessarily local*), *$I \subseteq J$ are ideals of R, and M is a non-zero finitely generated R-module, then*

$$\operatorname{cd}(J, M) \leq \operatorname{cd}(I, M) + \operatorname{cd}(J, M/IM).$$

Proof. Without loss of generality we may assume that $\operatorname{cd}(J, M) \geq 0$. Hence by Lemma 2.3, $\operatorname{cd}(I, M) \geq 0$. Let $t := \operatorname{cd}(I, M)$. Then, in view of Theorem 2.4, we have

$$\operatorname{cd}(J, M) \leq t + \operatorname{cd}\Big(J, \bigoplus_{i=0}^{t} H_I^i(M)\Big).$$

Now let $k := \operatorname{cd}(J, \bigoplus_{i=0}^{t} H_I^i(M))$. Then by definition we have

$$H_J^k\Big(\bigoplus_{i=0}^{t} H_I^i(M)\Big) \neq 0.$$

But as the local cohomology functor commutes with direct limits and each R-module is the direct limit of the family of all finitely generated submodules, it follows that there exists a finitely generated submodule L of the R-module $\bigoplus_{i=0}^{t} H_I^i(M)$ such that $H_J^k(L) \neq 0$. But

$$\operatorname{Supp}(L) \subseteq \operatorname{Supp}\Big(\bigoplus_{i=0}^{t} H_I^i(M)\Big) \subseteq \operatorname{Supp}(M/IM).$$

Therefore, it follows from [DNT, Theorem 2.2] that

$$\operatorname{cd}(J, M/IM) \geq \operatorname{cd}(J, L) \geq k.$$

Consequently,

$$\begin{aligned}\operatorname{cd}(I, M) + \operatorname{cd}(J, M/IM) &\geq \operatorname{cd}(I, M) + \operatorname{cd}(J, L)\\ &\geq t + \operatorname{cd}\Big(J, \bigoplus_{i=0}^{t} H_I^i(M)\Big)\\ &\geq \operatorname{cd}(J, M).\end{aligned}$$ ∎

The following corollary is a generalization of [MBN, Lemma 2.10].

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R, and M a non-zero finitely generated R-module. Then*

$$\operatorname{cd}(I, M) \geq \dim(M) - \dim(M/IM).$$

Proof. If we set $J = \mathfrak{m}$ in Theorem 2.5, then the assertion follows immediately from [BS, Theorems 7.3.2 and 6.1.2]. ∎

As an immediate consequence of Corollary 2.6, we get the following result.

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R, and M a non-zero finitely generated R-module such that*

$$\operatorname{grade}(I, M) + \dim(M/IM) < \dim(M).$$

Then $\operatorname{cd}(I, M) > \operatorname{grade}(I, M)$.

We are now ready to state and prove the next main result of this paper.

Theorem 2.8. *Let R be a Noetherian ring* (*not necessarily local*), *let I and J be ideals of R, and let M be a non-zero finitely generated R-module such that $(I + J)M \neq M$. Then*

$$\operatorname{cd}(I, M) \leq \operatorname{cd}(IJ, M) + \operatorname{cd}(I, M/JM).$$

Proof. Since $IJ \subseteq I$, by Theorem 2.5 we have

$$\operatorname{cd}(I, M) \leq \operatorname{cd}(IJ, M) + \operatorname{cd}(I, M/IJM).$$

On the other hand, as $\operatorname{cd}(I, JM/IJM) = 0$, the exact sequence

$$0 \to JM/IJM \to M/IJM \to M/JM \to 0$$

yields $\operatorname{cd}(I, M/IJM) = \operatorname{cd}(I, M/JM)$. ∎

The following corollary is a consequence of Theorem 2.8.

Corollary 2.9. *Under the assumptions of Theorem* 2.8,

$$\operatorname{cd}(IJ, M) \geq \tfrac{1}{2}\big(\operatorname{cd}(I, M) + \operatorname{cd}(J, M) - \operatorname{cd}(I, M/JM) - \operatorname{cd}(J, M/IM)\big).$$

Proof. By Theorem 2.8, we have

$$\operatorname{cd}(I, M) \leq \operatorname{cd}(IJ, M) + \operatorname{cd}(I, M/JM),$$

and

$$\operatorname{cd}(J, M) \leq \operatorname{cd}(IJ, M) + \operatorname{cd}(J, M/IM).$$

Hence, the corollary follows. ∎

The following theorem is another main result of this paper.

Theorem 2.10. *Under the assumptions of Theorem* 2.8,

$$\operatorname{cd}(I + J, M) \leq \operatorname{cd}(I, M) + \operatorname{cd}(J, M/IM).$$

Proof. Apply Theorem 2.5 and the equality

$$\operatorname{cd}(I+J, M/IM) = \operatorname{cd}(J, M/IM). \blacksquare$$

As a consequence of Theorem 2.10, we get the following result.

COROLLARY 2.11. *Under the assumptions of Theorem* 2.8,

$$\operatorname{cd}(I+J, M) \leq \tfrac{1}{2}\big(\operatorname{cd}(I,M) + \operatorname{cd}(J,M) + \operatorname{cd}(I, M/JM) + \operatorname{cd}(J, M/IM)\big).$$

Proof. By Theorem 2.10, we have

$$\operatorname{cd}(I+J, M) \leq \operatorname{cd}(I,M) + \operatorname{cd}(J, M/IM),$$

and

$$\operatorname{cd}(I+J, M) \leq \operatorname{cd}(J,M) + \operatorname{cd}(I, M/JM).$$

Hence, the corollary follows. $\blacksquare$

Using Corollaries 2.9 and 2.11 we get the following proposition.

PROPOSITION 2.12. *Under the assumptions of Theorem* 2.8,

$$\operatorname{cd}(I+J, M) - \operatorname{cd}(IJ, M) \leq \operatorname{cd}(I, M/JM) + \operatorname{cd}(J, M/IM).$$

Proof. Apply Corollaries 2.9 and 2.11. $\blacksquare$

The following proposition is a consequence of Theorem 2.10.

PROPOSITION 2.13. *Under the assumptions of Theorem* 2.8,

$$\operatorname{cd}(I+J, M) \leq \operatorname{cd}(I,M) + \operatorname{cd}(J,M).$$

Proof. Since $\operatorname{Supp}(M/IM) \subseteq \operatorname{Supp}(M)$, [DNT, Theorem 2.2] yields $\operatorname{cd}(J, M/IM) \leq \operatorname{cd}(J, M)$. Hence, the proposition follows by applying Theorem 2.10. $\blacksquare$

The following propositions are immediate consequences of Proposition 2.13.

PROPOSITION 2.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I, J be a pair of proper ideals of R. If M is a finitely generated non-zero R-module of dimension $d \geq 0$ such that $M/(I+J)M$ is of dimension* 0, *then*

$$\operatorname{cd}(I,M) + \operatorname{cd}(J,M) \geq d.$$

Proof. Apply Proposition 2.13 and use the well-known Grothendieck vanishing and non-vanishing theorems (see [BS, Theorems 6.1.2 and 6.1.4]). Note that in this situation we have $\operatorname{cd}(I+J, M) = d$. $\blacksquare$

PROPOSITION 2.15. *Under the assumptions of Theorem* 2.8,

$$\operatorname{cd}(IJ, M) \leq \operatorname{cd}(I,M) + \operatorname{cd}(J,M) - 1.$$

Proof. Apply Proposition 2.13 and [BS, Theorem 3.2.3]. $\blacksquare$

The following example shows that the bounds given in Propositions 2.13 and 2.15 are optimal.

EXAMPLE 2.16. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$, and let s and t be positive integers such that $s + t \leq d$. Let $x_1, \ldots, x_{s+t}$ be a part of a system of parameters for R. Let $L := (x_1, \ldots, x_s)$, $K := (x_{s+1}, \ldots, x_{s+t})$ and $I := LK$. Then using [BN, Proposition 3.2] and [BS, Theorem 3.2.3], it is easy to see that $\operatorname{cd}(L, R) = s$, $\operatorname{cd}(K, R) = t$ and

$$\operatorname{cd}(L + K, R) = s + t = \operatorname{cd}(L, R) + \operatorname{cd}(K, R),$$
$$\operatorname{cd}(I, R) = s + t - 1 = \operatorname{cd}(L, R) + \operatorname{cd}(K, R) - 1. \blacksquare$$

The following result is a consequence of Proposition 2.15.

PROPOSITION 2.17. *Let R be a Noetherian ring (not necessarily local), $n \geq 2$ and $I_1, \ldots, I_n$ be proper ideals of R with* $\operatorname{cd}(I_j, R) = 1$ *for each $1 \leq j \leq n$. Then* $\operatorname{cd}(\bigcap_{j=1}^{n} I_j, R) \leq 1$.

Proof. Apply Proposition 2.15 and induction on $n \geq 2$. $\blacksquare$

COROLLARY 2.18. *Let R be a Noetherian ring (not necessarily local) and I be a proper ideal of R with* $\operatorname{height}(I) = 1$. *If* $\operatorname{cd}(\mathfrak{p}, R) = 1$ *for each minimal prime ideal $\mathfrak{p}$ of I, then* $\operatorname{cd}(I, R) = 1$.

Proof. It follows from Proposition 2.17 that $\operatorname{cd}(I, R) \leq 1$. On the other hand as $\operatorname{height}(I) = 1$, it follows from [BS, Theorems 7.3.2 and 4.3.2] that $\operatorname{cd}(I, R) \geq 1$. $\blacksquare$

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