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THE QUASI ISBELL TOPOLOGY ON FUNCTION SPACES

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Abstract. In this paper, on the family $\mathcal{O}(Y)$ of all open subsets of a space Y we define the so called quasi Scott topology, denoted by τ_{qSc}. This topology defines in a standard way, on the set $C(Y,Z)$ of all continuous maps of the space Y to a space Z, a topology t_{qIs} called the quasi Isbell topology. The latter topology is always larger than or equal to the Isbell topology, and smaller than or equal to the strong Isbell topology. Results and problems concerning the topology t_{qIs} are given.

1. Preliminaries. For every topological space Y we denote by $\mathcal{O}(Y)$ the set of all open subsets of Y. Recall the definitions of some topologies on $\mathcal{O}(Y)$.

The *Scott topology* τ_{Sc} on $\mathcal{O}(Y)$ (see, for example, [13]) is the family of all subsets $\mathbb{H}$ of $\mathcal{O}(Y)$ such that:

(a) The conditions $U \in \mathbb{H}$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in \mathbb{H}$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $\bigcup\{U_i : i \in I\} \in \mathbb{H}$, there exists a finite subset J of I such that $\bigcup\{U_i : i \in J\} \in \mathbb{H}$.

The *strong Scott topology* τ_{sSc} on $\mathcal{O}(Y)$ (see [18]) is the family of all subsets $\mathbb{H}$ of $\mathcal{O}(Y)$ such that:

(a) The conditions $U \in \mathbb{H}$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in \mathbb{H}$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $\bigcup\{U_i : i \in I\} = Y$, there exists a finite subset J of I such that $\bigcup\{U_i : i \in J\} \in \mathbb{H}$.

Let Y, Z be topological spaces and $C(Y,Z)$ the set of all continuous maps of Y into Z.

If τ_{Sc} is the Scott topology on $\mathcal{O}(Y)$, then the *Isbell topology* t_{Is} on $C(Y,Z)$ (see, for example, [13]) is the topology for which the family of all sets of the form

$$(\mathbb{H}, U) = \{f \in C(Y,Z) : f^{-1}(U) \in \mathbb{H}\},$$

where $\mathbb{H} \in \tau_{\mathrm{Sc}}$ and $U \in \mathcal{O}(Z)$, is a subbasis.

If τ_{sSc} is the strong Scott topology on $\mathcal{O}(Y)$, then the *strong Isbell topology* t_{sIs} on $C(Y,Z)$ (see [18]) is the topology for which the family of all sets

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of the form

$$(\mathbb{H}, U) = \{f \in C(Y,Z) : f^{-1}(U) \in \mathbb{H}\},$$

where $\mathbb{H} \in \tau_{\rm sSc}$ and $U \in \mathcal{O}(Z)$, is a subbasis.

The *compact-open topology* $t_{\rm co}$ on $C(Y,Z)$ (see [7]) is the topology for which the family of all sets of the form

$$(K, U) = \{f \in C(Y,Z) : f(K) \subseteq U\},$$

where K is a compact subset of Y and $U \in \mathcal{O}(Z)$, is a subbasis.

It is known that $t_{\rm co} \subseteq t_{\rm Is} \subseteq t_{\rm sIs}$ (see, for example, [18] and [20]).

In what follows, if t is a topology on the set $C(Y,Z)$, then the corresponding topological space is denoted by $C_t(Y,Z)$.

Let $F : X \times Y \to Z$ be a continuous map and $x \in X$. We denote by F_x the continuous map of Y into Z defined by $F_x(y) = F(x,y)$ for every $y \in Y$. Also, $\widehat{F}$ denotes the map of X into $C(Y,Z)$ defined by $\widehat{F}(x) = F_x$ for every $x \in X$. Let G be a map of X into $C(Y,Z)$. We denote by $\widetilde{G}$ the map of $X \times Y$ into Z given by $\widetilde{G}(x,y) = G(x)(y)$ for every $(x,y) \in X \times Y$.

A topology t on $C(Y,Z)$ is called *splitting* if for every space X, the continuity of a map $F : X \times Y \to Z$ implies that of $\widehat{F} : X \to C_t(Y,Z)$. A topology t on $C(Y,Z)$ is called *admissible* if for every space X, the continuity of a map $G : X \to C_t(Y,Z)$ implies that of $\widetilde{G} : X \times Y \to Z$ (see [2]). Let $\mathcal{A}$ be a fixed family of topological spaces. If in the above definitions it is assumed that the space X belongs to $\mathcal{A}$, then the topology t is called $\mathcal{A}$-*splitting* (respectively, $\mathcal{A}$-*admissible*) (see [12]).

A subset B of a space Y is called *bounded* if for every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $Y = \bigcup\{U_i : i \in I\}$ there exists a finite subset J of I such that $B \subseteq \bigcup\{U_i : i \in J\}$. A topological space Y is called *locally bounded* if each of its points has an open neighborhood that is bounded.

A topological space Y is called *core-compact* if for every $y \in Y$ and for every open neighborhood U of y there exists an open neighborhood V of y such that the set V is bounded in the space U. We observe that a topological space Y is core-compact if and only if for every $y \in Y$ and for every open neighborhood U of y there exists an open neighborhood V of y satisfying the following conditions:

(a) $V \subseteq U$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $U \subseteq \bigcup\{U_i : i \in I\}$ there exists a finite subset J of I such that $V \subseteq \bigcup\{U_i : i \in J\}$.

We recall the following results:

(1) Each splitting topology is contained in each admissible topology (see [2]).

(2) A topology which is larger than an admissible topology is also admissible (see [2]).
(3) A topology which is smaller than a splitting topology is also splitting (see [2]).
(4) The function space $C(Y, Z)$ can have at most one topology that is both admissible and splitting. Such a topology is necessarily the largest splitting topology and the smallest admissible topology (see, for example, [6]).
(5) The compact open topology and the Isbell topology on $C(Y, Z)$ are always splitting (see, for example, [2], [7], [20]).
(6) A topology t on $C(Y, Z)$ is admissible if the *evaluation map* $e : C_t(Y, Z) \times Y \to Z$, defined by $e(f, y) = f(y)$ for $(f, y) \in C(Y, Z) \times Y$, is continuous.
(7) The compact-open topology on $C(Y, Z)$ is admissible if Y is a regular locally compact space (see [2]).
(8) The Isbell topology on $C(Y, Z)$ is admissible if Y is a core-compact space. In this case the Isbell topology is also the greatest splitting topology (see, for example, [18] and [24]).
(9) The strong Isbell topology on $C(Y, Z)$ is admissible if Y is locally bounded (see [18]).
(10) If Y is a T_i-space, where $i = 0, 1, 2$, then $C_{t_{\mathrm{co}}}(Y, Z)$ is a T_i-space.

For a summary of all the above results and some open problems on function spaces see [11]. In the past years, there has been a great deal of progress in the field of function spaces. In particular, there are several papers about the Isbell topology (see, for example, [4], [5], [15], [19], [22]).

2. The quasi Scott topology on $\mathcal{O}(Y)$, core-compactness, and local boundedness

Definition 2.1. Let Y be a topological space. The *quasi Scott topology* τ_{qSc} on $\mathcal{O}(Y)$ is the family of all subsets $\mathbb{H}$ of $\mathcal{O}(Y)$ such that:

(a) The conditions $U \in \mathbb{H}$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in \mathbb{H}$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $\bigcup\{U_i : i \in I\}$ is a dense subset of Y and $\bigcup\{U_i : i \in I\} \in \mathbb{H}$, there exists a finite subset J of I such that $\bigcup\{U_i : i \in J\} \in \mathbb{H}$.

Remark 2.2. We have $\tau_{\mathrm{Sc}} \subseteq \tau_{\mathrm{qSc}} \subseteq \tau_{\mathrm{sSc}}$.

Example 2.3. We set $\mathbb{N} = \{1, 2, \ldots\}$. Let Y be the set of real numbers with the usual topology. Consider the set $\mathcal{O}(Y)$ of all open subsets of Y with the inclusion as order. We set

$$\mathbb{H} = O(Y) \setminus \{U \in \mathcal{O}(Y) : U \subsetneq (0, 1)\}.$$

Obviously, the conditions $U \in \mathbb{H}$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in \mathbb{H}$. We observe that

$$\bigcup\{(1/n, 1) : n \in \mathbb{N}\} = (0, 1) \in \mathbb{H},$$

but there does not exist a finite subset N of $\mathbb{N}$ such that $\bigcup\{(1/n, 1) : n \in N\} \in \mathbb{H}$. Therefore, $\mathbb{H} \notin \tau_{\mathrm{Sc}}$. Now, we consider a family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $\bigcup\{U_i : i \in I\}$ is a dense subset of Y and $\bigcup\{U_i : i \in I\} \in \mathbb{H}$. Then there exists $i_0 \in I$ such that $(2, 3) \cap U_{i_0} \neq \emptyset$. It follows that $U_{i_0} \in \mathbb{H}$. Hence, $\mathbb{H} \in \tau_{\mathrm{qSc}}$. By the above we have $\tau_{\mathrm{Sc}} \neq \tau_{\mathrm{qSc}}$.

Example 2.4. We set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Equip $Y = \{-1, 0, 1, 2, \ldots\}$ with the topology

$$\mathcal{O}(Y) = \big\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots, \{0, 1, 2, \ldots\}, Y\big\}.$$

We consider the subset $\mathbb{H} = \{\{0, 1, 2, \ldots\}, Y\}$ of $\mathcal{O}(Y)$. Obviously, the conditions $U \in \mathbb{H}$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in \mathbb{H}$. We observe that

$$\bigcup\{\{0, \ldots, n\} : n \in \mathbb{N}_0\} = \{0, 1, 2, \ldots\} \in \mathbb{H}$$

and $\{0, 1, 2, \ldots\}$ is a dense subset of Y. But for every finite subset N of $\mathbb{N}_0$,

$$\bigcup\{\{0, \ldots, n\} : n \in N\} = \{0, \ldots, \max(N)\} \notin \mathbb{H}.$$

Therefore, $\mathbb{H} \notin \tau_{\mathrm{qSc}}$. Now, we consider a family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $\bigcup\{U_i : i \in I\} = Y$. Then there exists $i_0 \in I$ such that $-1 \in U_{i_0}$. Hence, $U_{i_0} = Y \in \mathbb{H}$. It follows that $\mathbb{H} \in \tau_{\mathrm{sSc}}$. By the above we have $\tau_{\mathrm{qSc}} \neq \tau_{\mathrm{sSc}}$.

Definition 2.5. Let Y be a topological space. The complete lattice $(\mathcal{O}(Y), \subseteq)$ is called *q-continuous* if for every $y \in Y$ and for every open neighborhood U of y there exists an open neighborhood V of y satisfying the following conditions:

(a) $V \subseteq U$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $U \subseteq \bigcup\{U_i : i \in I\}$ and $\bigcup\{U_i : i \in I\}$ is a dense subset of Y there exists a finite subset J of I such that $V \subseteq \bigcup\{U_i : i \in J\}$.

In what follows we give some new characterizations of the notions of core-compactness and local boundedness.

Proposition 2.6. *Let Y be a topological space. The complete lattice $\mathcal{O}(Y)$ is q-continuous if and only if the space Y is core-compact.*

Proof. If Y is core-compact, then obviously $\mathcal{O}(Y)$ is q-continuous. Conversely, suppose that $\mathcal{O}(Y)$ is q-continuous. Let $y \in Y$ and U be an open neighborhood of y. Then there exists an open neighborhood V of y satisfying conditions (a) and (b) of Definition 2.5. Let $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ be such that $U \subseteq \bigcup\{U_i : i \in I\}$. Consider the family

$$\{U_i : i \in I\} \cup \left\{Y \setminus \mathrm{Cl}_Y\Big(\bigcup\{U_i : i \in I\}\Big)\right\} \subseteq \mathcal{O}(Y).$$

The union of this family is a dense subset of Y. Therefore, there exists a finite subset J of I such that $V \subseteq \bigcup\{U_i : i \in J\}$. ∎

Proposition 2.7. *A space Y is locally bounded if and only if for every $y \in Y$ there exists an open neighborhood U of y satisfying the following condition: For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $\mathrm{Cl}_Y(U) \subseteq \bigcup\{U_i : i \in I\}$ and $\bigcup\{U_i : i \in I\}$ is a dense subset of Y, there exists a finite subset J of I such that $U \subseteq \bigcup\{U_i : i \in J\}$.*

Proof. If the condition is satisfied, then obviously Y is locally bounded. Conversely, suppose that Y is locally bounded. Let $y \in Y$. Then there exists an open neighborhood U of y satisfying the following condition: For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $Y = \bigcup\{U_i : i \in I\}$ there exists a finite subset J of I such that $U \subseteq \bigcup\{U_i : i \in J\}$. Let $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ be such that $\mathrm{Cl}_Y(U) \subseteq \bigcup\{U_i : i \in I\}$ and $\bigcup\{U_i : i \in I\}$ is a dense subset of Y. Consider the family

$$\{U_i : i \in I\} \cup \{Y \setminus \mathrm{Cl}_Y(U)\} \subseteq \mathcal{O}(Y).$$

Then

$$\bigcup\{U_i : i \in I\} \cup \{Y \setminus \mathrm{Cl}_Y(U)\} = Y.$$

Therefore, there exists a finite subset J of I such that $U \subseteq \bigcup\{U_i : i \in J\}$. ∎

3. The quasi Isbell topology

Definition 3.1. Let Y and Z be two topological spaces and τ_{qSc} the quasi Scott topology on $\mathcal{O}(Y)$. The *quasi Isbell topology* t_{qIs} on $C(Y, Z)$ is the topology for which the family of all sets of the form

$$(\mathbb{H}, U) = \{f \in C(Y, Z) : f^{-1}(U) \in \mathbb{H}\},$$

where $\mathbb{H} \in \tau_{\mathrm{qSc}}$ and $U \in \mathcal{O}(Z)$, is a subbasis.

The following proposition can be easily proved.

Proposition 3.2. *The following statements are true:*

(1) $t_{\mathrm{co}} \subseteq t_{\mathrm{Is}} \subseteq t_{\mathrm{qIs}} \subseteq \tau_{\mathrm{sIs}}$.
(2) *If Y is a T_i-space, where $i = 0, 1, 2$, then the space $C_{t_{\mathrm{qIs}}}(Y, Z)$ is a T_i-space.*

Remark 3.3. Let $\mathbf{2}$ be the Sierpiński space, that is, $\mathbf{2} = \{0, 1\}$ with the topology $\{\emptyset, \{1\}, \{0, 1\}\}$. If Y is another topological space, then

$$C(Y, \mathbf{2}) = \{\mathcal{X}_U : U \in \mathcal{O}(Y)\},$$

where $\mathcal{X}_U : Y \to \mathbf{2}$ denotes the characteristic function of U,

$$\mathcal{X}_U(y) = \begin{cases} 1 & \text{if } y \in U, \\ 0 & \text{if } y \in Y \setminus U. \end{cases}$$

We note that:

(1) The spaces $C_{t_{\mathrm{Is}}}(Y, \mathbf{2})$ and $(\mathcal{O}(Y), \tau_{\mathrm{Sc}})$ are homeomorphic (see [13]).
(2) The spaces $C_{t_{\mathrm{sIs}}}(Y, \mathbf{2})$ and $(\mathcal{O}(Y), \tau_{\mathrm{sSc}})$ are homeomorphic (see [18]).

Proposition 3.4. *The spaces $C_{t_{\mathrm{qIs}}}(Y, \mathbf{2})$ and $(\mathcal{O}(Y), \tau_{\mathrm{qSc}})$ are homeomorphic for any space Y.*

Proof. We consider the map $h : C_{t_{\mathrm{qIs}}}(Y, \mathbf{2}) \to (\mathcal{O}(Y), \tau_{\mathrm{qSc}})$ defined by $h(f) = f^{-1}(\{1\})$ for every $f \in C_{t_{\mathrm{qIs}}}(Y, \mathbf{2})$. Obviously, h is one-to-one and onto. We observe that f is continuous. Indeed, let $\mathbb{H} \in \tau_{\mathrm{qSc}}$. Then

$$h^{-1}(\mathbb{H}) = \{f \in C_{t_{\mathrm{qIs}}}(Y, \mathbf{2}) : f^{-1}(\{1\}) \in \mathbb{H}\} = (\mathbb{H}, \{1\}).$$

Also, f is open. Indeed, let $(\mathbb{H}, \{1\})$ be a subbasic open set in $C_{\mathrm{qIs}}(Y, \mathbf{2})$. Then $h((\mathbb{H}, \{1\})) = \mathbb{H}$. Therefore, the map h is a homeomorphism. ∎

Example 3.5. Let Y be the space given in Example 2.3. As $\tau_{\mathrm{Sc}} \neq \tau_{\mathrm{qSc}}$, we have $t_{\mathrm{Is}} \neq t_{\mathrm{qIs}}$ on $C(Y, \mathbf{2})$. Moreover, since Y is core-compact, the topology t_{Is} on $C(Y, \mathbf{2})$ is admissible, and therefore the topology t_{Is} is the greatest splitting topology. Since $t_{\mathrm{Is}} \subseteq t_{\mathrm{qIs}}$, the topology t_{qIs} on $C(Y, \mathbf{2})$ is admissible. It follows that the topology t_{qIs} on $C(Y, \mathbf{2})$ is not splitting.

Example 3.6. Let Y be the space given in Example 2.4. As $\tau_{\mathrm{qSc}} \neq \tau_{\mathrm{sSc}}$, we have $t_{\mathrm{qIs}} \neq t_{\mathrm{sIs}}$ on $C(Y, \mathbf{2})$.

Definition 3.7 (see [3]). A topological space is called *irreducible* if every non-empty open subset is dense.

Proposition 3.8. *If the space Y is irreducible, then the quasi Isbell topology t_{qIs} and the Isbell topology t_{Is} coincide on $C(Y, Z)$.*

Proof. The proof is a straightforward verification of the fact that if the space Y is irreducible, then the quasi Scott topology τ_{qSc} and the Scott topology τ_{Sc} coincide on $\mathcal{O}(Y)$. ∎

Definition 3.9 (see [1]). A topological space is called *Alexandroff* if the intersection of every family of open sets is open.

Proposition 3.10. *Let $\mathcal{A}$ be the family of Alexandroff spaces. Then the topology t_{qIs} on $C(Y, Z)$ is $\mathcal{A}$-splitting.*

Proof. Let X be an Alexandroff space and $F : X \times Y \to Z$ be a continuous map. We prove that the map $\widehat{F} : X \to C_{\mathrm{qIs}}(Y, Z)$ is continuous. Let $x \in X$. Then the map $\widehat{F}(x) = F_x : Y \to Z$ is continuous. Let $(\mathbb{H}, U)$ be a subbasic open set in $C_{\mathrm{qIs}}(Y, Z)$ containing F_x. Then $F_x^{-1}(U) \in \mathbb{H}$. For each $y \in F_x^{-1}(U)$ we have $F(x, y) = F_x(y) \in U$. Hence there exist open sets V_y containing x and W_y containing y such that $F(V_y \times W_y) \subseteq U$. Since $F_x^{-1}(U) \subseteq \bigcup_{y \in F_x^{-1}(U)} W_y$ and $F_x^{-1}(U) \in \mathbb{H}$, we have $\bigcup_{y \in F_x^{-1}(U)} W_y \in \mathbb{H}$. Let $V = \bigcap_{y \in F_x^{-1}(U)} V_y$ and $W = \bigcup_{y \in F_x^{-1}(U)} W_y$. Then $x \in V$. We prove that $\widehat{F}(V) \subseteq (\mathbb{H}, U)$. Let $v \in V$. We prove that $\widehat{F}(v) \in (\mathbb{H}, U)$ or equivalently $F_v^{-1}(U) \in \mathbb{H}$. Since $W \in \mathbb{H}$, it suffices to prove that $W \subseteq F_v^{-1}(U)$. Indeed, let $w \in W$. Then $w \in W_{y_0}$ for some $y_0 \in F_x^{-1}(U)$. Moreover, $v \in V_{y_0}$. Thus, $F(v, w) \in F(V_{y_0} \times W_{y_0}) \subseteq U$, and therefore $F_v(w) \in U$ or equivalently $w \in F_v^{-1}(U)$. ∎

Proposition 3.11. *The following statements are true:*

(1) *If Y is a regular locally compact space, then the topology t_{qIs} on $C(Y, Z)$ is admissible.*
(2) *If Y is a core-compact space, then the topology t_{qIs} on $C(Y, Z)$ is admissible.*

Proof. The proof is a straightforward verification of the fact that a topology which is larger than an admissible topology is also admissible. ∎

Proposition 3.12. *For every space Y the following statements are equivalent:*

(1) *Y is core-compact.*
(2) *For every space Z the evaluation map $e : C_{t_{\mathrm{qIs}}}(Y, Z) \times Y \to Z$ is continuous.*
(3) *The evaluation map $e : C_{t_{\mathrm{qIs}}}(Y, \mathbf{2}) \times Y \to \mathbf{2}$ is continuous, where $\mathbf{2}$ is the Sierpiński space.*
(4) *The set $\{(U, y) \in \mathcal{O}(Y) \times Y : y \in U\}$ is open in $(\mathcal{O}(Y), \tau_{\mathrm{qSc}}) \times Y$.*
(5) *For every open neighborhood U of a point y of Y there is an open set $\mathbb{H} \in \tau_{\mathrm{qSc}}$ such that $U \in \mathbb{H}$ and the set $\bigcap\{W : W \in \mathbb{H}\}$ is a neighborhood of y in Y.*

Proof. (1) implies (2). Follows by Proposition 3.11(2).

(2) implies (3). It is obvious.

(3) implies (4). For every $U \in \mathcal{O}(Y)$ we have $\mathcal{X}_U \in C_{t_{\mathrm{qIs}}}(Y, \mathbf{2})$ and $U = \mathcal{X}_U^{-1}(\{1\})$. Moreover, $y \in U$ if and only if $e(\mathcal{X}_U, y) = \mathcal{X}_U(y) = 1$. By Proposition 3.4 we have

$$e^{-1}(\{1\}) \cong \{(U, y) \in \mathcal{O}(Y) \times Y : y \in U\},$$

and hence $\{(U, y) \in \mathcal{O}(Y) \times Y : y \in U\}$ is open in $(\mathcal{O}(Y), \tau_{\mathrm{qSc}}) \times Y$.

(4) implies (5). Let U be an open neighborhood of a point y of Y. By (4) there exist $\mathbb{H} \in \tau_{\mathrm{qSc}}$ and an open neighborhood V of y in Y such that

$$(U, y) \in \mathbb{H} \times V \subseteq \{(U, u) \in \mathcal{O}(Y) \times Y : u \in U\}.$$

We prove that $V \subseteq \bigcap\{W : W \in \mathbb{H}\}$. Let $v \in V$ and $W \in \mathbb{H}$. Then

$$(W, v) \in \mathbb{H} \times V \subseteq \{(U, u) \in \mathcal{O}(Y) \times Y : u \in U\},$$

and hence $v \in W$. It follows that $\bigcap\{W : W \in \mathbb{H}\}$ is a neighborhood of y in Y.

(5) implies (1). By Proposition 2.6 we have to show that for every $y \in Y$ and for every open neighborhood U of y there exists an open neighborhood V of y satisfying the following conditions:

(a) $V \subseteq U$.
(b) For every family $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ such that $U \subseteq \bigcup\{U_i : i \in I\}$ and $\bigcup\{U_i : i \in I\}$ is a dense subset of Y there exists a finite subset J of I such that $V \subseteq \bigcup\{U_i : i \in J\}$.

Let U be an open neighborhood of y. By (5) there exists an open set $\mathbb{H} \in \tau_{\mathrm{qSc}}$ such that $U \in \mathbb{H}$ and the set $\bigcap\{W : W \in \mathbb{H}\}$ is a neighborhood of y in Y. Therefore, there exists an open neighborhood V of y such that

$$V \subseteq \bigcap\{W : W \in \mathbb{H}\}.$$

Since $U \in \mathbb{H}$, we have $V \subseteq U$. Let $\{U_i : i \in I\} \subseteq \mathcal{O}(Y)$ be a family such that $U \subseteq \bigcup\{U_i : i \in I\}$ and $\bigcup\{U_i : i \in I\}$ is a dense subset of Y. Since $U \in \mathbb{H}$ and $U \subseteq \bigcup\{U_i : i \in I\}$, we have $\bigcup\{U_i : i \in I\} \in \mathbb{H}$. Hence, there exists a finite subset J of I such that $\bigcup\{U_i : i \in J\} \in \mathbb{H}$. It follows that $V \subseteq \bigcup\{U_i : i \in J\}$. ∎

Corollary 3.13. *Let Y be a locally bounded space. If the space Y is not core-compact, then $t_{\mathrm{qIs}} \neq t_{\mathrm{sIs}}$ on $C(Y, \mathbf{2})$.*

Proof. By Proposition 3.12, the topology t_{qIs} on $C(Y, \mathbf{2})$ is not admissible. However, the topology t_{sIs} on $C(Y, \mathbf{2})$ is admissible. Thus, they are different. ∎

Example 3.14 (see [17]). Let Y consist of points $\alpha, \beta, \gamma_i, \alpha_{ij}, b_{ij}, i, j \in \mathbb{N}$, and let $\mathcal{O}(Y)$ be the topology on Y defined by the neighborhood basis $\mathcal{B}(y)$ of each point $y \in Y$ as follows:

$$\mathcal{B}(\alpha) = \{V^n(\alpha) = \{\alpha\} \cup \{\alpha_{ij} : i \geq n,\, j \in \mathbb{N}\} : n \in \mathbb{N}\},$$
$$\mathcal{B}(\beta) = \{V^n(\beta) = \{\beta\} \cup \{\beta_{ij} : i \geq n,\, j \in \mathbb{N}\} : n \in \mathbb{N}\},$$
$$\mathcal{B}(\alpha_{ij}) = \{\{\alpha_{ij}\}\}, \quad i, j \in \mathbb{N},$$
$$\mathcal{B}(\beta_{ij}) = \{\{\beta_{ij}\}\}, \quad i, j \in \mathbb{N},$$
$$\mathcal{B}(\gamma_i) = \{V^n(\gamma_i) = \{\gamma_i\} \cup \{\alpha_{ij} : j \geq n\} \cup \{\beta_{ij} : j \geq n\} : n \in \mathbb{N}\}, \quad i \in \mathbb{N}.$$

The space Y is locally bounded. However, this space is not core-compact. By Corollary 3.13, $t_{\mathrm{qIs}} \neq t_{\mathrm{sIs}}$ on $C(Y, \mathbf{2})$.

Example 3.15. Let Y consist of the set of points of the plane $\mathbb{R}^2$. Neighborhoods of points other than the origin $(0,0)$ are the usual open sets of $\mathbb{R}^2 \setminus \{(0,0)\}$. As a neighborhood basis of $(0,0)$, we take

$$\mathcal{B}(0,0) = \{V^n(0,0) = \{(\alpha, \beta) \in Y : \alpha^2 + \beta^2 < 1/n^2,\, \beta > 0\} \cup \{(0,0)\} : n \in \mathbb{N}\}.$$

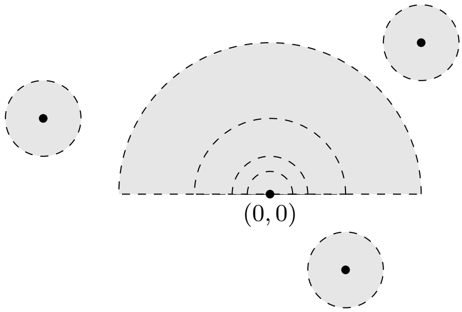

Fig. 1

Since the plane $\mathbb{R}^2$ is locally bounded, each point $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$ has an open neighborhood that is bounded. Moreover, the open neighborhood $V^1(0,0) = \{(\alpha, \beta) \in Y : \alpha^2 + \beta^2 < 1,\ \beta > 0\} \cup \{(0,0)\}$ of $(0,0)$ is bounded. Hence, Y is locally bounded. We show that Y is not core-compact. Indeed, consider the point $(0,0)$ of Y, the open neighborhood $V^1(0,0)$ of $(0,0)$, and an arbitrary open neighborhood V of $(0,0)$ such that $V \subseteq V^1(0,0)$. Without loss of generality we can suppose that $V = V^n(0,0)$ for some $n \in \mathbb{N}$. Let $(\beta_i)_{i=1}^{\infty}$ be a strictly decreasing sequence of numbers in $(0,1)$ such that $\lim_{i\to\infty} \beta_i = 0$. We set

$$I = \mathbb{N} \cup \{0\}, \qquad U_0 = V^{2n}(0,0),$$
$$U_i = \{(\alpha, \beta) \in Y : -1 < \alpha < 1,\ \beta_i < \beta < 1\}, \qquad i \in \mathbb{N},$$

and consider the family $\{U_i : i \in I\}$. Note that $V^1(0,0) \subseteq \bigcup\{U_i : i \in I\}$ and there does not exist a finite subset J of I such that $V \subseteq \bigcup\{U_i : i \in J\}$. Therefore, Y is not core-compact. By Corollary 3.13, $t_{\mathrm{qIs}} \neq t_{\mathrm{sIs}}$ on $C(Y, \mathbf{2})$.

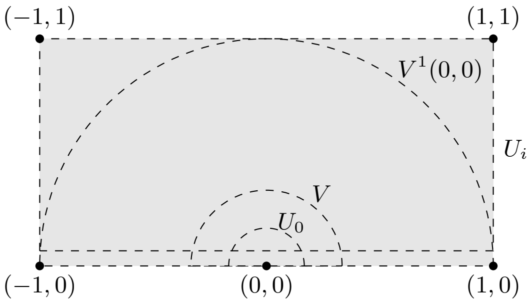

Fig. 2

Proposition 3.16. *Let $\mathcal{A}$ be a family of topological spaces such that the topology t_{qIs} on $C(Y, Z)$ is $\mathcal{A}$-splitting. If Y is core-compact, then for every space $X \in \mathcal{A}$ the map*

$$E : C(X \times Y, Z) \to C(X, C_{t_{\mathrm{qIs}}}(Y, Z))$$

defined by $E(F) = \widehat{F}$ for every $F \in C(X \times Y, Z)$ is a bijection.

Proof. First, we note that since the topology t_{qIs} on $C(Y,Z)$ is $\mathcal{A}$-splitting, the map E is well defined. Suppose that Y is core-compact. By Proposition 3.11(2) the topology t_{qIs} on $C(Y,Z)$ is admissible. It follows that for every space X, the continuity of a map $G : X \to C_{\mathrm{qIs}}(Y,Z)$ implies that of the map $\widetilde{G} : X \times Y \to Z$. We consider the map

$$R : C(X, C_{t_{\mathrm{qIs}}}(Y,Z)) \to C(X \times Y, Z)$$

defined by $R(G) = \widetilde{G}$ for every $G \in C(X, C_{t_{\mathrm{qIs}}}(Y,Z))$. We observe that

$$E \circ R = \mathrm{id}_{C(X, C_{t_{\mathrm{qIs}}}(Y,Z))} \quad \text{and} \quad R \circ E = \mathrm{id}_{C(X\times Y,Z)},$$

where id_A denotes the identity map on A. Therefore, E is a bijection. ∎

4. Problems. In this section we give some problems on quasi Isbell topology on function spaces.

PROBLEM 4.1. *Is the space $C_{t_{\mathrm{qIs}}}(Y,Z)$ regular (respectively, Tychonoff) when Z is regular (respectively, Tychonoff)?*

By $w(X)$ we denote the weight of an arbitrary topological space X. It is known (see, for example, [21]) that if Y and Z are arbitrary topological spaces, then $w(C_{t_{\mathrm{Is}}}(Y,Z)) \leq w(Y)w(Z)$.

PROBLEM 4.2. *Is it true that*

$$w(C_{t_{\mathrm{qIs}}}(Y,Z)) \leq w(Y)w(Z)$$

for any topological spaces Y and Z?

A family $\mathcal{F} \subseteq C(Y,Z)$ is called *evenly continuous* at a point $y \in Y$ (see [16]) if for every $z \in Z$ and every open neighborhood P of z in Z there are open neighborhoods W and R, of y and z respectively such that $f(W) \subseteq P$ whenever $f \in \mathcal{F}$ and $f(y) \in R$. The subset $\mathcal{F}$ of $C(Y,Z)$ is called *evenly continuous* if it is evenly continuous at each point $y \in Y$.

THEOREM 4.3 (see [23]). *Let Y be an arbitrary topological space, Z a regular topological space and $\mathcal{F} \subseteq C(Y,Z)$. Then $\mathcal{F}$ is compact in $C_{t_{\mathrm{Is}}}(Y,Z)$ if the following conditions are satisfied:*

(1) *$\mathcal{F}$ is closed.*
(2) $\mathrm{Cl}(\{g(y) : g \in \mathcal{F}\})$ *is a compact subset of Z for every $y \in Y$.*
(3) *$\mathcal{F}$ is evenly continuous.*

PROBLEM 4.4. *Is the above Ascoli type theorem true for the topology t_{qIs}?*

A family $\mathcal{F} \subseteq C(Y,Z)$ *satisfies the condition G_2* (see [8] and [14]) if for each open set U of Z and each $G \subseteq \mathcal{F}$ such that $G = \mathrm{Cl}(G) \cap \mathcal{F}$ the set $\bigcap\{g^{-1}(U) : g \in G\}$ is open in Y (where $\mathrm{Cl}(G)$ denotes the pointwise closure of G).

THEOREM 4.5 (see [22]). *Let Y be an arbitrary topological space, Z a regular topological space and $\mathcal{F} \subseteq C(Y,Z)$. Then $\mathcal{F}$ is compact in $C_{t_{\mathrm{Is}}}(Y,Z)$ if the following conditions are satisfied:*

(1) *$\mathcal{F}$ is closed.*
(2) *$\{g(y) : g \in \mathcal{F}\}$ is a compact subset of Z for every $y \in Y$.*
(3) *$\mathcal{F}$ satisfies the condition G_2.*

PROBLEM 4.6. *Is the above Ascoli type theorem true for the topology t_{qIs}?*

PROBLEM 4.7. *Compare the quasi Isbell and strong Isbell topologies with the fine Isbell topology of Jordan* (*see* [15]).

REMARK 4.8. For some other open problems on function spaces see [11], [10], and [9].

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